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GENERALIZED DUHAME'S PRINCIPLE FOR SOME SEMI-LINEAR HYPERBOLIC TYPE OF EQUATIONS

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Abstract. In this paper, we shall generalize Duhamel's principle in order to represent solutions to some semi-linear hyperbolic type of equations. We also give some examples which will be useful in the study of the life span or the singularity.

1. Introduction

We are concerned with the Cauchy problem on $[0,T]\times\mathbf{R}^n_x$

$$\begin{cases} \partial_t^2 u - p(\partial_x)^2 u = f\Big(t,x,\partial_t u - p(\partial_x)u\Big), \\ u(0,x)=\varphi(x),\ \partial_t u(0,x)=\psi(x), \end{cases} \tag{1.1}$$

where $p(\partial_x)$ is a differential operator such that

$$p^*(\partial_x) = -p(\partial_x). \tag{1.2}$$

Through this article, we do not assume smoothness nor growth order for $\varphi(x)$, $\psi(x)$ and $f(t,x,\alpha)$. In particular, when $n=1$, the equation (1.1) with $p(\partial_x)\equiv \pm\partial_x$ is just a semi-linear wave equation. The equation (1.1) with $p(\partial_x)\equiv \pm i\Delta_x$ is a semi-linear plate equation (Timoshenko type equation) which can be regarded as a sort of hyperbolic type (see [1], [10], etc.)

By Fourier transform, one can show an exact representation formula for the linear equation with $f\equiv 0$. So we may suppose that the solution v to the

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following Cauchy problem on $[0,T]\times\mathbf{R}^n_x$ is known:

$$\begin{cases} \partial_t^2 v - p(\partial_x)^2 v = 0, \\ v(0,x) = \varphi(x),\ \partial_t v(0,x) = \psi(x). \end{cases} \tag{1.3}$$

Moreover, the existences of some typical non-linear equations have been already known. In particular, we shall suppose that the solution α to the following Cauchy problem on $[0,T]\times\mathbf{R}^n_x$ is known:

$$\begin{cases} \partial_t \alpha + p(\partial_x)\alpha = f(t,x,\alpha), \\ \alpha(0,x) = \psi(x) - p(\partial_x)\varphi(x). \end{cases} \tag{1.4}$$

Actually, when $n=1$, the equation (1.4) with $p(\partial_x)\equiv\pm\partial_x$ is the first order non-linear equation which can be reduced to an ordinary evolution equation by changes of variables. Not only the existence but also the exact representation formula is well-known classically. The equation (1.4) with $p(\partial_x)\equiv\pm i\Delta_x$ is a non-linear Schrödinger equation for which the existence has been studied by many authors (see [3], [9], etc.).

Our purpose is to represent the solution u to the semi-linear equation (1.1) with the solution v to the linear equation (1.3) and the solution α to the non-linear equation (1.4). So the exactly solvable model (1.1) is a new category of non-linear equations. We can prove the following:

Theorem 1.1. *Let us assume that v is the solution to (1.3) and α is the solution to (1.4). Then the solution to (1.1) is represented by*

$$u(t,x) = v(t,x) + \int_0^t w(t-s,x;s)ds, \tag{1.5}$$

where $w(t,x;s)$ is the solution to the following Cauchy problem on $[0,T]\times\mathbf{R}^n_x$:

$$\begin{cases} \partial_t^2 w - p(\partial_x)^2 w = 0, \\ w(0,x;s) = 0,\ \partial_t w(0,x;s) = f\Big(s,x,\alpha(s,x)\Big). \end{cases} \tag{1.6}$$

Remark 1.2. *When $n=1$ and $p(\partial_x)\equiv\pm\partial_x$, (1.1) is a linear inhomogeneous equation if $f(t,x,\alpha)$ is independent of α. Then (1.3) is a linear homogeneous equation and (1.6) is just an auxiliary equation for the Duhamel's principle.*

Our theorem gives a reduction method from a higher order equation (1.1) to a lower order equation (1.4) which inherits a nonlinearity from (1.1) (see §2.1). In general, it would be difficult to find an example for the general semi-linear wave equation

$$\partial_t^2 u - \Delta u = f\Big(t,x,u,\partial_t u,\partial_x u\Big). \tag{1.7}$$

Our theorem has good possibilities to construct useful examples as a special case (1.1) of (1.7) and to know the structure of the solution (see §2.2).

We shall also introduce more simpler cases of 1-dimensional semi-linear equations. We are concerned with the Cauchy problem on $[0,T]\times \mathbf{R}^1_x$

$$\begin{cases} \partial_t^2 u - a(t)^2\partial_x^2 u = a'(t)\partial_x u + f\Big(t,x,\partial_t u - a(t)\partial_x u\Big), \\ u(0,x)=\varphi(x),\ \partial_t u(0,x)=\psi(x), \end{cases} \tag{1.8}$$

where $a(t)$ is a real-valued differentiable function on $[0,T]$. Here we remark that $a(t)$ may possibly take zero. Thus, the equation (1.8) is a *weakly* hyperbolic equation with a variable coefficient. Linear weakly hyperbolic equations have been studied (see [4] and [5]) and applied to non-linear weakly hyperbolic equations (see [2], [6] and [7]).

We consider the following Cauchy problem on $[0,T]\times \mathbf{R}^1_x$ corresponding to (1.4):

$$\begin{cases} \partial_t\alpha + a(t)\partial_x\alpha = f(t,x,\alpha), \\ \alpha(0,x)=\psi(x)-a(0)\partial_x\varphi(x). \end{cases} \tag{1.9}$$

We shall also give the another representation of the solution u to the semi-linear equation (1.8) with the solution α to the non-linear equation (1.9), but without the Cauchy problem corresponding to (1.3).

Then we can prove the following:

Corollary 1.3. *Let us assume that α is the solution to (1.9). Then the solution to (1.8) is represented by*

$$u(t,x)=\varphi\Big(x+\int_0^t a(\tau)d\tau\Big)+\int_0^t \alpha\Big(s,x+\int_s^t a(\tau)d\tau\Big)ds, \tag{1.10}$$

in particular, if $a(t)\equiv a$,

$$u(t,x)=\varphi(x+at)+\int_0^t \alpha\Big(s,x+a(t-s)\Big)ds. \tag{1.11}$$

Remark 1.4. *The result (1.11) for (1.8) with $a(t)\equiv a$ should coincide with the result (1.5) for (1.1) with $p(\partial_x)\equiv a\partial_x$ $(n=1)$. One can check this fact after a long computation (see §6).*

2. Applications

In this section we shall introduce some examples to apply our theorems.

2.1. n-dimensional semi-linear plate equations.

We shall consider (1.1) with $p(\partial_x)\equiv \pm i\Delta_x$. From the following proposition, we can get the solution v to the Cauchy problems (1.3) and (1.6):

Proposition 2.1. *Let $\varphi, \psi \in L^1(\mathbf{R}^n_x)$. The solution to the Cauchy problem on $[0,T] \times \mathbf{R}^n_x$*

$$\begin{cases} \partial_t^2 v + \Delta_x^2 v = 0, \\ v(0,x) = \varphi(x), \ \partial_t v(0,x) = \psi(x), \end{cases}$$

is represented by

$$v(t,x) = \frac{1}{\sqrt{4\pi}^n} \int_{\mathbf{R}^n_y} \Big\{ \varphi(x - \sqrt{t}y) + \int_0^t \psi(x - \sqrt{\tau}y) d\tau \Big\} \cos \Big\{ \frac{|y|^2 - n\pi}{4} \Big\} dy.$$

For the proof of Proposition 2.1, see §5.
Hence, by Theorem 1.1 we find that

$$\begin{aligned} u(t,x) &= \frac{1}{\sqrt{4\pi}^n} \int_{\mathbf{R}^n_y} \Big\{ \varphi(x - \sqrt{t}y) + \int_0^t \psi(x - \sqrt{\tau}y) d\tau \Big\} \cos \Big\{ \frac{|y|^2 - n\pi}{4} \Big\} dy \\ &+ \frac{1}{\sqrt{4\pi}^n} \int_{\mathbf{R}^n_y} \int_0^t \int_0^{t-s} f\Big(s, x - \sqrt{\tau}y, \alpha(s, x - \sqrt{\tau}y)\Big) \\ &\times \cos \Big\{ \frac{|y|^2 - n\pi}{4} \Big\} d\tau ds dy \end{aligned}$$

solves the Cauchy problem on $[0,T] \times \mathbf{R}^n_x$

$$\begin{cases} \partial_t^2 u + \Delta_x^2 u = f\Big(t, x, \partial_t u \mp i\Delta_x u\Big), \\ u(0,x) = \varphi(x), \ \partial_t u(0,x) = \psi(x). \end{cases}$$

Here, α is given by the solution to the Cauchy problem on $[0,T] \times \mathbf{R}^n_x$

$$\begin{cases} \partial_t \alpha \mp i\Delta_x \alpha = f(t,x,\alpha), \\ \alpha(0,x) = \psi(x) \pm i\Delta_x \varphi(x). \end{cases} \tag{2.1}$$

The existence of the solution α to the non-linear Schrödinger equation depends on its non-linearity (see [3], [9], etc.).

2.2. n-dimensional semilinear wave equations.

Let $q > 1$. We shall consider the Cauchy problem on $[0,T] \times \mathbf{R}^n_x$

$$\begin{cases} \partial_t^2 u - \Delta_x u = \dfrac{1}{1-q} \Big(\partial_t u - \dfrac{1}{\sqrt{n}} \mathrm{div}_x u \Big)^q, \\ u(0,x) = \varphi(x), \ \partial_t u(0,x) = \psi(x). \end{cases} \tag{2.2}$$

Putting $X = \sum_{i=1}^n x_i$, we assume that φ and ψ satisfy

$$\varphi(x) \equiv \Phi(X), \quad \psi(x) \equiv \Psi(X).$$

Then, we will know the fact that u also satisfies $u(t,x) \equiv U(t,X)$ from the later representation. Therefore, we shall use this fact in advance. Since $\Delta_x =$

$(\mathrm{div}_x)^2/n + \sum_{1\le i<j\le n}(\partial_{x_i} - \partial_{x_j})^2/n$ and $(\partial_{x_i} - \partial_{x_j})u = (\partial_{x_i} - \partial_{x_j})U = 0$, the Cauchy problem (2.2) is changed into the Cauchy problem on $[0,T] \times \mathbf{R}^n_x$

$$\begin{cases} \partial_t^2 u - \dfrac{1}{n}(\mathrm{div}_x)^2 u = \dfrac{1}{1-q}\Big(\partial_t u - \dfrac{1}{\sqrt{n}}\mathrm{div}_x u\Big)^q, \\ u(0,x) = \Phi(X),\ \partial_t u(0,x) = \Psi(X). \end{cases}$$

We can find that

$$v(t,x) = \frac{1}{2}\Big\{\Phi(X+\sqrt{n}t) + \Phi(X-\sqrt{n}t)\Big\} + \frac{1}{2}\int_{-t}^{t}\Psi(X+\sqrt{n}\tau)d\tau$$

solves the Cauchy problem on $[0,T] \times \mathbf{R}^n_x$

$$\begin{cases} \partial_t^2 v - \dfrac{1}{n}(\mathrm{div}_x)^2 v = 0, \\ v(0,x) = \Phi(X),\ \partial_t v(0,x) = \Psi(X), \end{cases}$$

and that

$$\alpha(t,x) = \Big\{t + \Big\{\Psi(X-\sqrt{n}t) - \sqrt{n}\Phi'(X-\sqrt{n}t)\Big\}^{1-q}\Big\}^{\frac{1}{1-q}}$$

solves the Cauchy problem on $[0,T] \times \mathbf{R}^n_x$

$$\begin{cases} \partial_t \alpha + \dfrac{1}{\sqrt{n}}\mathrm{div}_x \alpha = \dfrac{1}{1-q}\alpha^q, \\ \alpha(0,x) = \Psi(X) - \dfrac{1}{\sqrt{n}}\mathrm{div}_x \Phi(X). \end{cases}$$

Moreover, writing $\alpha(t,x) \equiv A(t,X)$, we can also find that

$$w(t,x;s) = \frac{1}{2(1-q)}\int_{-t}^{t} A(s, X+\sqrt{n}\tau)^q d\tau$$

solves the Cauchy problem on $[0,T] \times \mathbf{R}^n_x$

$$\begin{cases} \partial_t^2 w - \dfrac{1}{n}(\mathrm{div}_x)^2 w = 0, \\ w(0,x;s) = 0,\ \partial_t w(0,x;s) = \dfrac{1}{1-q}\alpha(s,x)^q \Big(\equiv \dfrac{1}{1-q}A(s,X)^q\Big). \end{cases}$$

Thus, by Theorem 1.1 we get the following:

Theorem 2.2. *Let* $X = \sum_{i=1}^n x_i$. *Then the solution to the Cauchy problem on* $[0,T] \times \mathbf{R}^n_x$

$$\begin{cases} \partial_t^2 u - \Delta_x u = \dfrac{1}{1-q}\Big(\partial_t u - \dfrac{1}{\sqrt{n}}\mathrm{div}_x u\Big)^q, \\ u(0,x) = \Phi(X),\ \partial_t u(0,x) = \Psi(X), \end{cases}$$

is represented by

$$u(t,x) = \frac{1}{2}\Big\{\Phi(X+\sqrt{n}t)+\Phi(X-\sqrt{n}t)\Big\} + \frac{1}{2}\int_{-t}^{t}\Psi(X+\sqrt{n}\tau)d\tau$$
$$+\frac{1}{2(1-q)}\int_0^t\int_{-(t-s)}^{t-s}\Big\{s+\Big\{\Psi\Big(X+\sqrt{n}(\tau-s)\Big)-\sqrt{n}\Phi'\Big(X+\sqrt{n}(\tau-s)\Big)\Big\}^{1-q}\Big\}^{\frac{q}{1-q}} d\tau ds.$$

For instance, taking $\Phi(X) \equiv 0$, $\Psi(X) = 1/X$ and $q = 2$, we have for sufficiently large $X > 0$

$$\begin{aligned} u(t,x) &= \frac{1}{2}\int_{-t}^{t}\frac{1}{X+\sqrt{n}\tau}d\tau - \frac{1}{2}\int_0^t\int_{-(t-s)}^{t-s}\Big\{s+X+\sqrt{n}(\tau-s)\Big\}^{-2}d\tau ds \\ &= \frac{2\sqrt{n}+1}{4n-1}\log\frac{X+\sqrt{n}t}{X+(1-\sqrt{n})t}. \end{aligned}$$

Remark 2.3. *In the computations of the above formula, we need the integrability. In fact, formal computations give for all* $(x_1,\cdots,x_n) \in \mathbf{R}_x^n$

$$u(t,x) = \frac{2\sqrt{n}+1}{4n-1}\log\Big|\frac{X+\sqrt{n}t}{X+(1-\sqrt{n})t}\Big|.$$

Hence, we see that the solution is singular at $X+\sqrt{n}t = 0$ *and* $X+(1-\sqrt{n})t = 0$.

2.3. 3-dimensional semilinear wave equation.

We shall consider the Cauchy problem on $[0,T]\times\mathbf{R}_x^3$

$$\begin{cases} \partial_t^2 u - \Delta_x u = f(t,x,|x|\partial_t u - x\cdot\nabla_x u - u), \\ u(0,x) = \varphi(x),\ \partial_t u(0,x) = \psi(x). \end{cases} \tag{2.3}$$

We assume that φ and ψ are radially symmetric, i.e.,

$$\varphi(x) \equiv \Phi(|x|), \quad \psi(x) \equiv \Psi(|x|).$$

Then, we know that u is radially symmetric, i.e., $u(t,x) \equiv U(t,|x|)$. Since $\Delta_x u = \partial_r^2 U + \frac{2}{r}\partial_r U$ and $x\cdot\nabla u = r\partial_r U$, the Cauchy problem (2.2) is changed into

$$\begin{cases} \partial_t^2 U - \partial_r^2 U - \dfrac{2}{r}\partial_r U = f(t,x,r\partial_t U - r\partial_r U - U), \\ U(0,r) = \Phi(r),\ \partial_t U(0,r) = \Psi(r). \end{cases}$$

Moreover, putting $U = r^{-1}V$, we have

$$\begin{cases} \partial_t^2 V - \partial_r^2 V = rf(t,x,\partial_t V - r\partial_r V), \\ V(0,r) = r\Phi(r),\ \partial_t V(0,r) = r\Psi(r). \end{cases}$$

Thus, by Corollary 1.2 it follows that

$$V(t,r) = r\Phi(r+t) + \int_0^t \alpha(s,r+t-s)ds.$$

Here, α is given by the solution to the Cauchy problem on $[0,T]\times\mathbf{R}^1_x$

$$\begin{cases} \partial_t\alpha+\partial_r\alpha=rf(t,r,\alpha),\\ \alpha(0,r)=r\Psi(r)-r\partial_r\Phi(r)-\Phi(r). \end{cases}$$

In conclusion, the solution to (2.3) is represented by

$$u(t,x)=\Phi(|x|+t)+|x|^{-1}\int_0^t\alpha(s,|x|+t-s)ds.$$

2.4. **1-dimensional semilinear wave equations.**

Let $F(\alpha)$ and $G(t)$ be differentiable functions such that $F'(\alpha)\neq 0$ and $G(0)=0$. We shall consider (1.8) with $a(t)\equiv 1$ and $f(t,x,\alpha)$ defined by

$$f(t,x,\alpha)\equiv\frac{G'(t)}{F'(\alpha)}.$$

Since $F'(\alpha)\neq 0$, there exists an inverse function $F^{-1}(\alpha)$. By the reduction to an ordinary equation and the method of separation of variables we can solve the Cauchy problem (1.9) on $[0,T]\times\mathbf{R}^1_x$ and get

$$\alpha(t,x)=F^{-1}\Big(G(t)+F\Big(\psi(x-t)-\partial_x\varphi(x-t)\Big)\Big).$$

Thus, by (1.11) in Corollary1.2 we have

$$u(t,x)=\varphi(x+t)+\int_0^t F^{-1}\Big(G(s)+F\Big(\psi(x+t-2s)-\partial_x\varphi(x+t-2s)\Big)\Big)ds. \tag{2.4}$$

Hence we observe that the regularity of f with respect to α (the non-linearity of f) has influence on the regularity of the solution u with respect to t and x. For instance, we solve the Cauchy problem with special initial data $\varphi\equiv 0$, $\psi=F^{-1}(x)$ and $G(t)=2t$, and get $u(t,x)=tF^{-1}(x+t)$. When f belongs to a Gevrey class with respect to α , this simple case shows that the solution u belongs to the same Gevrey class with respect to x as f(see [2], [6] and [7]).

(i) Taking $F(\alpha)=\tan^{-1}\alpha$ and $G(t)=\tan^{-1}t$, we find that

$$u(t,x)=\varphi(x+t)+\int_0^t\tan\Big(\tan^{-1}s+\tan^{-1}\Big(\psi(x+t-2s)-\partial_x\varphi(x+t-2s)\Big)\Big)ds$$

$$\Big(=\varphi(x+t)+\int_0^t\frac{s+\psi(x+t-2s)-\partial_x\varphi(x+t-2s)}{1-s\{\psi(x+t-2s)-\partial_x\varphi(x+t-2s)\}}ds\Big)$$

solves the Cauchy problem on $[0,T]\times\mathbf{R}^1_x$

$$\begin{cases} \partial_t^2u-\partial_x^2u=\dfrac{(\partial_tu-\partial_xu)^2}{t^2+1}+\dfrac{1}{t^2+1},\\ u(0,x)=\varphi(x),\ \partial_tu(0,x)=\psi(x). \end{cases}$$

This solution suggests that the initial data must be small for the global solvability, since for $\varphi \equiv \varepsilon$ and $\psi \equiv \varepsilon$

$$u(t,x) = \varepsilon + \int_0^t \frac{s+\varepsilon}{1-s\varepsilon} ds = \varepsilon - \frac{t}{\varepsilon} - \Big(1 + \frac{1}{\varepsilon^2}\Big) \log(1 - t\varepsilon).$$

The lifespan T_ε tends to infinity as ε tends to zero, i.e., $T_\varepsilon < \frac{1}{\varepsilon}$.

Remark 2.4. *In general, if the equation has an inhomogeneous term, one can expect only the local solvability (see* [12]*). But, in the above we get the global solvability due to the inhomogeneous term* $1/(t^2+1)$ *degenerating at infinity.*

(ii) Taking $F(\alpha) = \tan^{-1}\alpha$ and $G(t) = t$, we find that

$$u(t,x) = \varphi(x+t) + \int_0^t \tan\Big(s + \tan^{-1}\Big(\psi(x+t-2s) - \partial_x\varphi(x+t-2s)\Big)\Big) ds$$

solves the Cauchy problem on $[0,T] \times \mathbf{R}^1_x$

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = (\partial_t u - \partial_x u)^2 + 1, \\ u(0,x) = \varphi(x), \ \partial_t u(0,x) = \psi(x). \end{cases}$$

This solution suggests the local solvability due to the inhomogeneous term even if the initial data are small, since for $\varphi \equiv \varepsilon$ and $\psi \equiv \varepsilon$

$$u(t,x) = \varepsilon - \log\Big|\frac{\cos(t + \tan^{-1}\varepsilon)}{\cos(\tan^{-1}\varepsilon)}\Big| = \varepsilon - \log|\cos t - \varepsilon \sin t|.$$

The lifespan T_ε is bounded, i.e., $T_\varepsilon < \tan^{-1}\frac{1}{\varepsilon} < \frac{\pi}{2}$.

(iii) Taking $F(\alpha) = -1/\alpha$ and $G(t) = t$, we find that

$$u(t,x) = \varphi(x+t) + \int_0^t \frac{\psi(x+t-2s) - \partial_x\varphi(x+t-2s)}{1 - s\Big\{\psi(x+t-2s) - \partial_x\varphi(x+t-2s)\Big\}} ds$$

solves the Cauchy problem on $[0,T] \times \mathbf{R}^1_x$

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = (\partial_t u - \partial_x u)^2, \\ u(0,x) = \varphi(x), \ \partial_t u(0,x) = \psi(x). \end{cases}$$

This solution suggests that the initial data must be small for the global solvability, since for $\varphi \equiv \varepsilon$ and $\psi \equiv \varepsilon$

$$u(t,x) = \varepsilon - \log(1 - t\varepsilon). \tag{2.5}$$

The lifespan T_ε tends to infinity as ε tends to zero, i.e., $T_\varepsilon < \frac{1}{\varepsilon}$.

Remark 2.5. *The equation satisfying the null condition can be solved by putting* $v = 1 - \exp[-u]$ *(see* [12]*). The following due to Nirenberg is very well-known:*

$$u(t,x) = -\log\Big\{\frac{\exp[-\varphi(x+t)] + \exp[-\varphi(x-t)]}{2} - \frac{1}{2}\int_{x-t}^{x+t} \psi(s)\exp[-\varphi(s)] ds\Big\}$$

solves

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = (\partial_t u)^2 - (\partial_x u)^2, \\ u(0,x) = \varphi(x), \ \partial_t u(0,x) = \psi(x). \end{cases}$$

This solution suggests that the initial data must be small for the global solvability, since for $\varphi \equiv \varepsilon$ and $\psi \equiv \varepsilon$

$$u(t,x) = \varepsilon - \log(1 - t\varepsilon),$$

which is quite same as (2.5). So its lifespan T_ε is also same.

2.5. **1-dimensional semilinear hyperbolic equations.**

Under the same situation as Example 4, we shall consider (1.8) with $a(t) \equiv t^k$ $(k > 0)$. In this case, the equation (1.8) is a *weakly* hyperbolic non-linear equations, more precisely Tricomi-type equations (see [11] and [13]). Similarly we can solve the Cauchy problem (1.9) on $[0,T] \times \mathbf{R}_x^1$ and get

$$\alpha(t,x) = F^{-1}\Big(G(t) + F\Big(\psi\Big(x - \frac{t^{k+1}}{k+1}\Big)\Big)\Big).$$

Thus, by (1.11) in Corollary 1.2 we have

$$u(t,x) = \varphi\Big(x + \frac{t^{k+1}}{k+1}\Big) + \int_0^t F^{-1}\Big(G(s) + F\Big(\psi\Big(x + \frac{t^{k+1} - 2s^{k+1}}{k+1}\Big)\Big)\Big) ds.$$

In the same way as §2.4, we get the following:

(iii)′ Taking $F(\alpha) = -1/\alpha$ and $G(t) = t$, we see that

$$u(t,x) = \varphi\Big(x + \frac{t^{k+1}}{k+1}\Big) + \int_0^t \frac{\psi\Big(x + \frac{t^{k+1}-2s^{k+1}}{k+1}\Big)}{1 - s\psi\Big(x + \frac{t^{k+1}-2s^{k+1}}{k+1}\Big)} ds$$

solves the Cauchy problem on $[0,T] \times \mathbf{R}_x^1$

$$\begin{cases} \partial_t^2 u - t^{2k}\partial_x^2 u = kt^{k-1}\partial_x u + (\partial_t u - t^k \partial_x u)^2, \\ u(0,x) = \varphi(x), \ \partial_t u(0,x) = \psi(x). \end{cases}$$

This solution suggests that the initial data must be small for the global solvability, since for $\varphi \equiv \varepsilon$ and $\psi \equiv \varepsilon$

$$u(t,x) = \varepsilon - \log(1 - t\varepsilon).$$

The lifespan T_ε tends to infinity as ε tends to zero, i.e., $T_\varepsilon < \frac{1}{\varepsilon}$.

Remark 2.6. *We know C^∞ well-posedness for the linear equation $\partial_t^2 u - t^{2k}\partial_x^2 u = kt^{k-1}\partial_x u$ (see [8]).*

3. Proof of Theorem 1.1

By (1.3) we easily see that

$$u(0,x) = v(0,x) = \varphi(x).$$

Differentiating u in t, by (1.6) we have

$$\begin{aligned} \partial_t u(t,x) &= \partial_t v(t,x) + w(0,x;t) + \int_0^t \partial_t w(t-s,x;s)ds \\ &= \partial_t v(t,x) + \int_0^t \partial_t w(t-s,x;s)ds. \end{aligned} \tag{3.1}$$

Hence, by (1.3) we easily see that

$$\partial_t u(0,x) = \partial_t v(0,x) = \psi(x).$$

Moreover, differentiating $\partial_t u$ in t, by (1.6) and (3.1) we have

$$\begin{aligned} \partial_t^2 u(t,x) &= \partial_t^2 v(t,x) + (\partial_t w)(0,x;t) + \int_0^t \partial_t^2 w(t-s,x;s)ds \\ &= \partial_t^2 v(t,x) + f\Big(t,x,\alpha(t,x)\Big) + \int_0^t \partial_t^2 w(t-s,x;s)ds. \end{aligned} \tag{3.2}$$

While, we also get

$$p(\partial_x)u(t,x) = p(\partial_x)v(t,x) + \int_0^t p(\partial_x)w(t-s,x;s)ds, \tag{3.3}$$

$$p(\partial_x)^2 u(t,x) = p(\partial_x)^2 v(t,x) + \int_0^t p(\partial_x)^2 w(t-s,x;s)ds. \tag{3.4}$$

Thus, by (1.3), (1.6), (3.2) and (3.4) it follows that

$$\begin{aligned} \partial_t^2 u(t,x) - p(\partial_x)^2 u(t,x) &= \partial_t^2 v(t,x) - p(\partial_x)^2 v(t,x) + f\Big(t,x,\alpha(t,x)\Big) \\ &\quad + \int_0^t \Big\{\partial_t^2 w(t-s,x;s) - p(\partial_x)^2 w(t-s,x;s)\Big\}ds \\ &= f\Big(t,x,\alpha(t,x)\Big). \end{aligned} \tag{3.5}$$

Let $\tilde{w}(t,x;s)$ be the solution to the Cauchy problem on $[0,T]\times\mathbf{R}_x^n$

$$\begin{cases} \partial_t^2 \tilde{w} - p(\partial_x)^2 \tilde{w} = 0, \\ \tilde{w}(0,x;s) = 0,\ \partial_t \tilde{w}(0,x;s) = \alpha(s,x). \end{cases} \tag{3.6}$$

Hence, we also find that

$$\begin{cases} \partial_t^2 \partial_s \tilde{w} - p(\partial_x)^2 \partial_s \tilde{w} = 0, \\ \partial_s \tilde{w}(0,x;s) = 0,\ \partial_t \partial_s \tilde{w}(0,x;s) = \partial_s \alpha(s,x), \end{cases}$$

and

$$\begin{cases} \partial_t^2 p(\partial_x)\tilde{w} - p(\partial_x)^2 p(\partial_x)\tilde{w} = 0, \\ p(\partial_x)\tilde{w}(0,x;s) = 0,\ \partial_t p(\partial_x)\tilde{w}(0,x;s) = p(\partial_x)\alpha(s,x). \end{cases}$$

Combining $\partial_s \tilde{w}$ with $p(\partial_x)\tilde{w}$ and noting that

$$\partial_t w(0,x;s) = f\Big(s,x,\alpha(s,x)\Big) = \partial_t \alpha(s,x) + p(\partial_x)\alpha(s,x),$$

we can write $w(t,x;s)$ as

$$w(t,x;s) = \Big(\partial_s + p(\partial_x)\tilde{w}\Big)(t,x;s).$$

Therefore, by (3.1), (3.3) and (3.6) we have

$$\begin{aligned}
&\partial_t u(t,x) - p(\partial_x)u(t,x) \\
&= \partial_t v(t,x) - p(\partial_x)v(t,x) + \int_0^t \Big\{\partial_t w(t-s,x;s) - p(\partial_x)w(t-s,x;s)\Big\}ds \\
&= \partial_t v(t,x) - p(\partial_x)v(t,x) + \int_0^t \Big(\partial_t - p(\partial_x)\Big)\Big\{\Big(\partial_s + p(\partial_x)\tilde{w}\Big)(t-s,x;s)\Big\}ds \\
&= \partial_t v(t,x) - p(\partial_x)v(t,x) + \int_0^t \Big(\partial_t - p(\partial_x)\Big)\Big(\partial_s + \partial_t + p(\partial_x)\Big)\tilde{w}(t-s,x;s)ds \\
&= \partial_t v(t,x) - p(\partial_x)v(t,x) + \int_0^t \Big(\partial_t^2 - p(\partial_x)^2\Big)\tilde{w}(t-s,x;s)ds \\
&\quad + \int_0^t \partial_s\Big\{\Big(\partial_t - p(\partial_x)\Big)\tilde{w}(t-s,x;s)\Big\}ds \\
&= \partial_t v(t,x) - p(\partial_x)v(t,x) + \Big(\partial_t - p(\partial_x)\tilde{w}\Big)(0,x;t) - \Big(\partial_t - p(\partial_x)\Big)\tilde{w}(t,x;0) \\
&= \partial_t\Big\{v(t,x) - \tilde{w}(t,x;0)\Big\} - p(\partial_x)\Big\{v(t,x) - \tilde{w}(t,x;0)\Big\} + \alpha(t,x).
\end{aligned}$$

Thus it follows that

$$\partial_t u(t,x) - p(\partial_x)u(t,x) = \partial_t \tilde{v}(t,x) - p(\partial_x)\tilde{v}(t,x) + \alpha(t,x), \tag{3.7}$$

where $\tilde{v}(t,x) \equiv v(t,x) - \tilde{w}(t,x;0)$. We remark that $\tilde{v}(t,x)$ is the solution to the Cauchy problem on $[0,T] \times \mathbf{R}_x^n$

$$\begin{cases} \partial_t^2 \tilde{v} - p(\partial_x)^2 \tilde{v} = 0, \\ \tilde{v}(0,x) = \varphi(x),\ \partial_t \tilde{v}(0,x) = \psi(x) - \alpha(0,x) = p(\partial_x)\varphi(x). \end{cases} \tag{3.8}$$

Lemma 3.1. *Let us assume that $\tilde{v}$ is the solution to (3.8). Then $\tilde{v}$ satisfies for all $(t,x) \in [0,T] \times \mathbf{R}_x^n$*

$$\partial_t \tilde{v}(t,x) \equiv p(\partial_x)\tilde{v}(t,x).$$

Proof. We put

$$E(t) = \int_{\mathbf{R}^n_x} \Big| \partial_t \tilde{v}(t,x) - p(\partial_x)\tilde{v}(t,x) \Big|^2 dx$$

Differentiating $E(t)$, by (1.7) and (3.8) we have

$$\begin{aligned} E'(t) &= 2\Re \int_{\mathbf{R}^n_x} \Big\{ \partial_t^2 \tilde{v}(t,x) - p(\partial_x)\partial_t \tilde{v}(t,x) \Big\} \overline{\Big\{ \partial_t \tilde{v}(t,x) - p(\partial_x)\tilde{v}(t,x) \Big\}} dx \\ &= -2\Re \int_{\mathbf{R}^n_x} p(\partial_x) \Big\{ \partial_t \tilde{v}(t,x) - p(\partial_x)\tilde{v}(t,x) \Big\} \overline{\Big\{ \partial_t \tilde{v}(t,x) - p(\partial_x)\tilde{v}(t,x) \Big\}} dx \\ &= -\Re \int_{\mathbf{R}^n_x} p(\partial_x) \Big\{ \partial_t \tilde{v}(t,x) - p(\partial_x)\tilde{v}(t,x) \Big\} \overline{\Big\{ \partial_t \tilde{v}(t,x) - p(\partial_x)\tilde{v}(t,x) \Big\}} dx \\ &\quad + \Re \int_{\mathbf{R}^n_x} \Big\{ \partial_t \tilde{v}(t,x) - p(\partial_x)\tilde{v}(t,x) \Big\} \overline{p(\partial_x) \Big\{ \partial_t \tilde{v}(t,x) - p(\partial_x)\tilde{v}(t,x) \Big\}} dx \\ &= 0. \end{aligned}$$

Therefore, we find that

$$E(t) = E(0) = \int_{\mathbf{R}^n_x} \Big| \partial_t \tilde{v}(0,x) - p(\partial_x)\tilde{v}(0,x) \Big|^2 dx = 0.$$

□

From Lemma 3.1 and (3.7) it follows that

$$\partial_t u(t,x) - p(\partial_x)u(t,x) = \alpha(t,x).$$

Hence by (3.5), we obtain

$$\partial_t^2 u(t,x) - p(\partial_x)^2 u(t,x) = f\Big(t, x, \partial_t u(t,x) - p(\partial_x)u(t,x)\Big).$$

4. Proof of Corollary 1.3

We obviously see that

$$u(0,x) = \varphi(x).$$

Differentiating u in t, then we have

$$\partial_t u(t,x) = a(t)\partial_x \varphi\Big(x + \int_0^t a(\tau)d\tau\Big) + \alpha(t,x) + a(t)\int_0^t \partial_x \alpha\Big(s, x + \int_s^t a(\tau)d\tau\Big) ds. \quad (4.1)$$

Hence, by (1.9) we easily see that

$$\partial_t u(0,x) = a(0)\partial_x \varphi(x) + \alpha(0,x) = \psi(x).$$

Moreover, differentiating $\partial_t u$ in t, by (1.9) we have

$$\begin{aligned}
&\partial_t^2 u(t,x) \qquad (4.2)\\
&= a'(t)\partial_x\varphi\Big(x+\int_0^t a(\tau)d\tau\Big) + \partial_t\alpha(t,x) + a'(t)\int_0^t \partial_x\alpha\Big(s,x+\int_s^t a(\tau)d\tau\Big)ds\\
&\quad + a(t)\partial_x\alpha(t,x) + a(t)^2\int_0^t \partial_x^2\alpha\Big(s,x+\int_s^t a(\tau)d\tau\Big)ds\\
&= a'(t)\partial_x\varphi\Big(x+\int_0^t a(\tau)d\tau\Big) + a(t)^2\partial_x^2\varphi\Big(x+\int_0^t a(\tau)d\tau\Big) + f\Big(t,x,\alpha(t,x)\Big)\\
&\quad + a'(t)\int_0^t \partial_x\alpha\Big(s,x+\int_s^t a(\tau)d\tau\Big)ds + a(t)^2\int_0^t \partial_x^2\alpha\Big(s,x+\int_s^t a(\tau)d\tau\Big)ds.
\end{aligned}$$

While, we also get

$$\partial_x u(t,x) = \partial_x\varphi\Big(x+\int_0^t a(\tau)d\tau\Big) + \int_0^t \partial_x\alpha\Big(s,x+\int_s^t a(\tau)d\tau\Big)ds, \quad (4.3)$$

$$\partial_x^2 u(t,x) = \partial_x^2\varphi\Big(x+\int_0^t a(\tau)d\tau\Big) + \int_0^t \partial_x^2\alpha\Big(s,x+\int_s^t a(\tau)d\tau\Big)ds. \quad (4.4)$$

Thus, by (**??**), (4.3) and (4.4) it follows that

$$\begin{aligned}
\partial_t^2 u(t,x) - a(t)^2\partial_x^2 u(t,x) &= a'(t)\partial_x\varphi\Big(x+\int_0^t a(\tau)d\tau\Big) + f\Big(t,x,\alpha(t,x)\Big)\\
&\quad + a'(t)\int_0^t \partial_x\alpha\Big(s,x+\int_s^t a(\tau)d\tau\Big)ds\\
&= a'(t)\partial_x u(t,x) + f\Big(t,x,\alpha(t,x)\Big). \qquad (4.5)
\end{aligned}$$

On the other hand, by (4.1) and (4.3) we immediately get

$$\partial_t u(t,x) - a(t)\partial_x u(t,x) = \alpha(t,x).$$

Hence by (4.5) we obtain

$$\partial_t^2 u(t,x) - a(t)^2\partial_x^2 u(t,x) = a'(t)\partial_x u(t,x) + f\Big(t,x,\partial_t u(t,x) - a(t)\partial_x u(t,x)\Big).$$

5. Proof of Proposition 2.1

It is sufficient to prove that the solution to the Cauchy problem on $[0,T]\times\mathbf{R}_x^n$

$$\begin{cases} \partial_t^2 v_1 + \Delta_x^2 v_1 = 0, \\ v_1(0,x) = \varphi(x), \ \partial_t v_1(0,x) = 0, \end{cases} \quad (5.1)$$

is represented by

$$v_1(t,x) = \frac{1}{\sqrt{4\pi}^n}\int_{\mathbf{R}_y^n} \varphi(x-\sqrt{t}y)\cos\Big\{\frac{|y|^2 - n\pi}{4}\Big\}dy.$$

Actually, we easily see that $v_2 = \int_0^t v_1(s)ds$ solves the Cauchy problem on $[0,T]\times \mathbf{R}_x^n$

$$\begin{cases} \partial_t^2 v_2 + \Delta_x^2 v_2 = 0, \\ v_2(0,x) = 0,\ \partial_t v_2(0,x) = \psi(x), \end{cases} \tag{5.2}$$

since $\partial_t^2 v_2 + \Delta_x^2 v_2 = \partial_t v_1 + \int_0^t \Delta_x^2 v_1(s)ds = \partial_t v_1 - \int_0^t \partial_t^2 v_1(s)ds = \partial_t v_1(0) = 0$. Thus, by (5.1) and (5.2) we find that $v = v_1 + v_2$.

By Fourier transform, the Cauchy problem (5.1) is changed into

$$\begin{cases} \partial_t^2 \hat{v}_1 + |\xi|^4 \hat{v}_1 = 0, \\ \hat{v}_1(0,x) = \hat{\varphi}(x),\ \partial_t \hat{v}_1(0,x) = 0. \end{cases} \tag{5.3}$$

Solving the Cauchy problem (5.3) for the ordinary equation, we have

$$\hat{v}_1(t,\xi) = \hat{\varphi}(\xi)\cos(|\xi|^2 t).$$

Therefore, we get

$$\begin{aligned} v_1(t,x) &= \frac{1}{\sqrt{2\pi}^n}\int_{\mathbf{R}_\xi^n} e^{ix\xi}\hat{\varphi}(\xi)\cos(|\xi|^2 t)d\xi \\ &= \frac{1}{2\sqrt{2\pi}^n}\int_{\mathbf{R}_\xi^n} e^{ix\xi}\hat{\varphi}(\xi)\Big\{e^{i|\xi|^2 t} + e^{-i|\xi|^2 t}\Big\}d\xi \\ &= \frac{1}{2}\int_{\mathbf{R}_y^n} \varphi(x-y)\Big\{\frac{e^{-\frac{i|y|^2}{4t}}}{\sqrt{-4\pi i t}^n} + \frac{e^{\frac{i|y|^2}{4t}}}{\sqrt{4\pi i t}^n}\Big\}dy \\ &= \frac{1}{2\sqrt{4\pi t}^n}\int_{\mathbf{R}_y^n} \varphi(x-y)\Big\{e^{-i\frac{|y|^2 - tn\pi}{4t}} + e^{i\frac{|y|^2-tn\pi}{4t}}\Big\}dy \\ &= \frac{1}{\sqrt{4\pi t}^n}\int_{\mathbf{R}_y^n} \varphi(x-y)\cos\frac{|y|^2 - tn\pi}{4t}dy \\ &= \frac{1}{\sqrt{4\pi}^n}\int_{\mathbf{R}_y^n} \varphi(x-\sqrt{t}y)\cos\frac{|y|^2 - n\pi}{4}dy. \end{aligned}$$

Here we used the fundamental solutions $\frac{e^{\frac{\pm i|y|^2}{4t}}}{\sqrt{\pm 4\pi i t}^n}$ for Schrödinger equations $\partial_t u \mp i\Delta_x u = 0$.

6. Appendix

We shall show that the result (1.11) for (1.8) with $a(t)\equiv a$ coincides with the result (1.5) for (1.1) with $p(\partial_x)\equiv a\partial_x$ $(n=1)$. Solving (1.3) with $p(\partial_x)\equiv a\partial_x$, we have

$$v(t,x) = \frac{1}{2}\int_{-t}^{t}\psi(x+|a|y)dy + \frac{1}{2}\Big\{\varphi(x+|a|t) + \varphi(x-|a|t)\Big\}. \tag{6.1}$$

Solving (1.6) with $p(\partial_x) \equiv a\partial_x$, by (1.4) we have

$$
\begin{aligned}
w(t,x;s) &= \frac{1}{2|a|}\int_{-|a|t}^{|a|t} f\Big(s, x+y, \alpha(s,x+y)\Big)dy \\
&= \frac{1}{2}\int_{-t}^{t} f\Big(s, x+|a|y, \alpha(s,x+|a|y)\Big)dy \\
&= \frac{1}{2}\int_{-t}^{t} \partial_s\alpha(s,x+|a|y)dy + \frac{a}{2}\int_{-t}^{t}\partial_x\alpha(s,x+|a|y)dy \\
&= \frac{1}{2}\int_{-t}^{t} \partial_s\alpha(s,x+|a|y)dy + \frac{a}{2|a|}\Big\{\alpha(s,x+|a|t) - \alpha(s,x-|a|t)\Big\}.
\end{aligned}
$$

Changing the order of integration, by (1.4) we have

$$
\begin{aligned}
&\int_0^t w(t-s,x;s)ds \\
&= \frac{1}{2}\int_0^t\int_{-(t-s)}^{t-s}\partial_s\alpha(s,x+|a|y)dyds \\
&\quad + \frac{a}{2|a|}\int_0^t\Big\{\alpha\Big(s,x+|a|(t-s)\Big) - \alpha\Big(s,x-|a|(t-s)\Big)\Big\}ds \\
&= \frac{1}{2}\int_0^t\int_0^{t-y}\partial_s\alpha(s,x+|a|y)dsdy + \frac{1}{2}\int_{-t}^0\int_0^{t+y}\partial_s\alpha(s,x+|a|y)dsdy \\
&\quad + \frac{a}{2|a|}\int_0^t\Big\{\alpha\Big(s,x+|a|(t-s)\Big) - \alpha\Big(s,x-|a|(t-s)\Big)\Big\}ds \\
&= \frac{1}{2}\int_0^t\alpha(t-y,x+|a|y)dy + \frac{1}{2}\int_{-t}^0\alpha(t+y,x+|a|y)dy \\
&\quad - \frac{1}{2}\int_{-t}^t\psi(x+|a|y)dy + \frac{a}{2}\int_{-t}^t\partial_x\varphi(x+|a|y)dy \\
&\quad + \frac{a}{2|a|}\int_0^t\Big\{\alpha\Big(s,x+|a|(t-s)\Big) - \alpha\Big(s,x-|a|(t-s)\Big)\Big\}ds \\
&= \frac{1}{2}\int_0^t\alpha\Big(s,x+|a|(t-s)\Big)ds + \frac{1}{2}\int_0^t\alpha\Big(s,x-|a|(t-s)\Big)ds \\
&\quad - \frac{1}{2}\int_{-t}^t\psi(x+|a|y)dy + \frac{a}{2|a|}\Big\{\varphi(x+|a|t) - \varphi(x-|a|t)\Big\} \\
&\quad + \frac{a}{2|a|}\int_0^t\Big\{\alpha\Big(s,x+|a|(t-s)\Big) - \alpha\Big(s,x-|a|(t-s)\Big)\Big\}ds
\end{aligned}
$$

$$= \int_0^t \alpha\Big(s, x + a(t-s)\Big) ds$$
$$- \frac{1}{2}\int_{-t}^{t} \psi(x + |a|y) dy + \frac{a}{2|a|}\Big\{\varphi(x + |a|t) - \varphi(x - |a|t)\Big\}.$$

Hence, by (6.1) it follows that

$$u(t,x) = v(t,x) + \int_0^t w(t-s, x; s) ds = \varphi(x + at) + \int_0^t \alpha\Big(s, x + a(t-s)\Big) ds.$$

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