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QUADRATIC ρ-FUNCTIONAL EQUATIONS IN β-HOMOGENEOUS NORMED SPACES

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Abstract. In this paper, we solve the quadratic ρ-functional equations:

$$\begin{aligned} & f(x+y)+f(x-y)-2f(x)-2f(y) \\ & = \rho\left(4f\left(\frac{x+y}{2}\right)+f\left(x-y\right)-2f(x)-2f(y)\right), \end{aligned} \tag{1}$$

and

$$\begin{aligned} & 4f\left(\frac{x+y}{2}\right)+f\left(x-y\right)-2f(x)-2f(y) \\ & = \rho(f(x+y)+f(x-y)-2f(x)-2f(y)), \end{aligned} \tag{2}$$

where ρ is a fixed complex number with $\rho \neq 1$.

Using the direct method, we prove the Hyers-Ulam stability of the quadratic ρ-functional equations (1) and (2) in β-homogeneous complex Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [19] concerning the stability of group homomorphisms. The functional equation $f(x+y)=f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [9] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [2] for additive mappings and by Rassias [16] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta

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[8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [18] for mappings $f: E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 3, 4, 6, 7, 10, 11, 12, 13, 14, 15]).

Definition 1.1. Let X be a linear space. A nonnegative valued function $\|\cdot\|$ is an F-norm if it satisfies the following conditions:

(FN_1) $\|x\|=0$ if and only if $x=0$;
(FN_2) $\|\lambda x\|=\|x\|$ for all $x\in X$ and all λ with $|\lambda|=1$;
(FN_3) $\|x+y\|\le\|x\|+\|y\|$ for all $x,y\in X$;
(FN_4) $\|\lambda_n x\|\to 0$ provided $\lambda_n\to 0$;
(FN_5) $\|\lambda x_n\|\to 0$ provided $\|x_n\|\to 0$.

Then $(X,\|\cdot\|)$ is called an F^*-space. An F-space is a complete F^*-space.

An F-norm is called β-homogeneous ($\beta>0$) if $\|tx\|=|t|^\beta\|x\|$ for all $x\in X$ and all $t\in\mathbb{C}$ (see [17]).

In this paper, we solve the quadratic ρ-functional equations (1) and (2) and prove the Hyers-Ulam stability of the quadratic ρ-functional equations (1) and (2) in β_2-homogeneous complex Banach space.

Throughout this paper, let β_1,β_2 be positive real numbers with $\beta_1\le 1$ and $\beta_2\le 1$. Assume that X is a β_1-homogeneous real or complex normed space with norm $\|\cdot\|$ and that Y is a β_2-homogeneous complex Banach space with norm $\|\cdot\|$. Assume that ρ is a complex number with $\rho\ne 1$.

2. Quadratic ρ-functional equation (1) in β-homogeneous complex Banach spaces

We solve and investigate the quadratic ρ-functional equation (1) in complex normed spaces.

Lemma 2.1. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\begin{aligned} & f(x+y) + f(x-y) - 2f(x) - 2f(y) \\ & = \rho\left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right) \end{aligned} \tag{2.1}$$

for all $x, y \in X$, then $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (2.1). Letting $y = x$ in (2.1), we get $f(2x) - 4f(x) = 0$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} & f(x+y) + f(x-y) - 2f(x) - 2f(y) \\ & = |\rho\left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right) \\ & = \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$. □

Now, we prove the Hyers-Ulam stability of the quadratic ρ-functional equation (2.1) in β-homogeneous complex Banach spaces.

Theorem 2.2. *Let $r > \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} & \|f(x+y) + f(x-y) - 2f(x) - 2f(y) \\ & -\rho\left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right)\Bigg\| \le \theta(\|x\|^r + \|y\|^r) \end{aligned} \tag{2.3}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{2\theta}{2^{\beta_1 r} - 4^{\beta_2}}\|x\|^r \tag{2.4}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.3), we get

$$\|f(2x) - 4f(x)\| \le 2\theta\|x\|^r \tag{2.5}$$

for all $x \in X$. So

$$\left\| f(x) - 4f\left(\frac{x}{2}\right)\right\| \le \frac{2\theta}{2^{\beta_1 r}}\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1}\left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \frac{2}{2^{\beta_1 r}}\sum_{j=l}^{m-1}\frac{4^{\beta_2 j}}{2^{\beta_1 r j}}\theta\|x\|^r \end{aligned} \tag{2.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.6) that the sequence $\{4^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{4^k f(\frac{x}{2^k})\}$ is convergent. So we can define a mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} 4^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.4).

It follows from (2.3) that

$$\begin{aligned} &\|Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y) \\ &\quad -\rho\left(4Q\left(\frac{x+y}{2}\right) + Q(x-y) - 2Q(x) - 2Q(y)\right)\Bigg\| \\ &= \lim_{n\to\infty}\Bigg\|4^n\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right. \\ &\quad \left.-\rho\left(4f\left(\frac{x+y}{2^{n+1}}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right)\right)\Bigg\| \\ &\le \lim_{n\to\infty}\frac{4^{\beta_2 n}}{2^{\beta_1 r n}}\theta(\|x\|^r + \|y\|^r) = 0 \end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned} &Q\left(\frac{x+y}{2}\right) + Q\left(\frac{x-y}{2}\right) - 2Q(x) - 2Q(y) \\ &= \rho\left(4Q\left(\frac{x+y}{2}\right) + Q(x-y) - 2Q(x) - 2Q(y)\right) \end{aligned}$$

for all $x, y \in X$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (2.4). Then we have

$$\begin{aligned}\|Q(x) - T(x)\| &= \left\|4^q Q\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right)\right\| \\ &\leq \left\|4^q Q\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\| + \left\|4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\| \\ &\leq \frac{2\theta}{2^{\beta_1 r} - 4^{\beta_2}} \frac{4^{\beta_2 q}}{2^{\beta_1 q r}} \|x\|^r,\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q, as desired. □

Theorem 2.3. *Let $r < \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.3). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{4^{\beta_2} - 2^{\beta_1 r}} \|x\|^r \tag{2.7}$$

for all $x \in X$.

Proof. It follows from (2.5) that

$$\left\|f(x) - \frac{1}{4} f(2x)\right\| \leq \frac{2\theta}{4^{\beta_2}} \|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{4^l} f(2^l x) - \frac{1}{4^m} f(2^m x)\right\| &\leq \sum_{j=l}^{m-1} \left\|\frac{1}{4^j} f\left(2^j x\right) - \frac{1}{4^{j+1}} f\left(2^{j+1} x\right)\right\| \\ &\leq \frac{2\theta}{4^{\beta_2}} \sum_{j=l}^{m-1} \frac{2^{\beta_1 r}}{4^{\beta_2 j}} \|x\|^r \end{aligned} \tag{2.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.8) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence in Y. Since Y is complete, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ convergent. So we can define a mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.8), we get (2.7).

The rest of the proof is similar to the proof of Theorem 2.2. □

3. Quadratic ρ-functional equation (2) in β-homogeneous complex Banach spaces

We solve and investigate the quadratic ρ-functional equation (2) in β-homogeneous complex normed spaces.

Lemma 3.1. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\begin{aligned} &4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y) \\ &= \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \end{aligned} \tag{3.1}$$

for all $x, y \in X$, then $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get

$$4f\left(\frac{x}{2}\right) = f(x) \tag{3.2}$$

for all $x \in X$. It follows from (3.1) and (3.2) that

$$\begin{aligned} &f(x+y) + f(x-y) - 2f(x) - 2f(y) \\ &= 4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y) \\ &= \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$. □

Now, we prove the Hyers-Ulam stability of the quadratic ρ-functional equation (3.1) in β-homogeneous complex Banach spaces.

Theorem 3.2. *Let $r > \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} &\left\| 4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y) \right. \\ &\left. -\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \right\| \le \theta(\|x\|^r + \|y\|^r) \end{aligned} \tag{3.3}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{2^{\beta_1 r}\theta}{2^{\beta_1 r} - 4^{\beta_2}} \|x\|^r \tag{3.4}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.3), we get

$$\left\| f(x) - 4f\left(\frac{x}{2}\right) \right\| = \left\| 4f\left(\frac{x}{2}\right) - f(x) \right\| \leq \theta \|x\|^r \tag{3.5}$$

for all $x \in X$. So

$$\begin{aligned} \left\| 4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| &\leq \sum_{j=l}^{m-1} \left\| 4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \\ &\leq \sum_{j=l}^{m-1} \frac{4^{\beta_2 j}}{2^{\beta_1 r j}} \theta \|x\|^r \end{aligned} \tag{3.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.6) that the sequence $\{4^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{4^k f(\frac{x}{2^k})\}$ convergnt. So we can define a mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} 4^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.4).

The rest of the proof is similar to the proof of Theorem 2.2 □

Theorem 3.3. *Let $r < \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.3). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{2^{\beta_1 r}\theta}{4^{\beta_2} - 2^{\beta_1 r}} \|x\|^r \tag{3.7}$$

for all $x \in X$.

Proof. It follows from (3.5) that

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{2^{\beta_1 r}}{4^{\beta_2}} \theta \|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^m} f(2^m x) \right\| &\leq \sum_{j=l}^{m-1} \left\| \frac{1}{4^j} f\left(2^j x\right) - \frac{1}{4^{j+1}} f\left(2^{j+1} x\right) \right\| \\ &\leq \sum_{j=l+1}^{m} \frac{2^{\beta_1 r j}}{4^{\beta_2 j}} \theta \|x\|^r \end{aligned} \tag{3.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.8) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$.

Since Y is complete, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ convergent. So we can define a mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.8), we get (3.7).

The rest of the proof is similar to the proof of Theorem 2.2. □

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