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# COMMON FIXED POINT THEOREMS FOR FOUR WEAKLY COMPATIBLE MAPPINGS IN $S^*$-METRIC SPACES

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**Abstract.** In this paper, we give some definitions of $S^*$-metric spaces and we prove a common fixed point theorem for four mappings under the condition of weakly compatible mappings in complete $S^*$-metric spaces. We get some improved versions of several fixed point theorems in complete $S^*$-metric spaces.

## 1. Introduction

Metrical fixed point theory became one of the most interesting area of research in the last fifty years. A lot of fixed and common fixed point results have been obtained by several authors in various types of spaces, such as metric spaces, fuzzy metric spaces, uniform spaces and others. One of the most interesting are partial metric spaces, which were defined by Matthews in the following way.

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**Definition 1.1.** ([4]) A partial metric on a nonempty set $X$ is a function $p : X \times X \to [0, +\infty)$ such that, for all $x, y, z \in X$:

($p_1$) $x = y \iff p(x,x) = p(x,y) = p(y,y)$,
($p_2$) $p(x,x) \leq p(x,y)$,
($p_3$) $p(x,y) = p(y,x)$,
($p_4$) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

In this case, the pair $(X, p)$ is called a partial metric space (see also [5]).

On the other hand, $S$-metric space were initiated by Sedghi, Shobe and Aliouche in [11] (see also [2, 7, 8, 12] and references cited therein).

**Definition 1.2.** ([11]) An $S$-metric on a nonempty set $X$ is a function $S : X \times X \times X \to [0, +\infty)$ such that for all $x, y, z, a \in X$, the following conditions are satisfied:

($s_1$) $S(x,y,z) = 0 \iff x = y = z$,
($s_2$) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.

In this case, the pair $(X, S)$ is called an $S$-metric space.

It is easy to see that in an $S$-metric space $(X, S)$ we always have $S(x,x,y) = S(y,y,x)$, $x, y \in X$.

In this paper, combining these two concepts, we introduce the notion of partial $S$-metric space and prove a common fixed point theorem for weakly increasing mappings in ordered spaces of this kind.

We recall some notions and properties in $S$-metric spaces.

**Definition 1.3.** ([9]) Let $(X, S)$ be an $S$-metric space and $\{x_n\}$ be a sequence in $X$.

(a) The sequence $\{x_n\}$ is convergent to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.
(b) $\{x_n\}$ is said to be a Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $S(x_n, x_n, x_m) < \epsilon$ for all $n, m \geq n_0$.
(c) The space $(X, S)$ is said to be complete if every Cauchy sequence is convergent in $X$.

**Lemma 1.4.** ([9]) *Let $(X, S)$ be an $S$-metric space. If $\{x_n\}$ and $\{y_n\}$ are sequences such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then*

$$\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y).$$

## 2. Partial $S$-metric spaces

In this section, we introduce partial $S$-metric spaces and investigate some of their simple properties.

**Definition 2.1.** A partial $S$-metric on a nonempty set $X$ is a function $S^* : X \times X \times X \to [0, +\infty)$ such that for all $x, y, z, a \in X$:

($s_{p1}$) $x = y = z \iff S^*(x, y, z) = S^*(x, x, x) = S^*(y, y, y) = S^*(z, z, z)$,
($s_{p2}$) $S^*(x, x, x) \le S^*(x, y, z)$,
($s_{p3}$) $S^*(x, y, z) \le S^*(x, x, a) + S^*(y, y, a) + S^*(z, z, a) - 2S^*(a, a, a)$.

The pair $(X, S^*)$ is then called a partial $S$-metric space or $S^*$-metric space.

Each $S$-metric space is also a partial $S$-metric space. The converse is not true, as shown by the following example.

**Example 2.2.** Let $X = [0, +\infty)$ and let $S^* : X \times X \times X \to [0, +\infty)$ be defined by $S^*(x, y, z) = \max\{x, y, z\}$. Then, it is easy to check that $(X, S^*)$ is a partial $S$-metric space. Obviously, $(X, S^*)$ is not an $S$-metric space.

**Lemma 2.3.** *For a partial $S$-metric $S^*$ on $X$, we have, for all $x, y \in X$:*

(a) $S^*(x, x, y) = S^*(y, y, x)$,
(b) *if* $S^*(x, x, y) = 0$ *then* $x = y$.

*Proof.* (a) By the condition ($s_{p3}$), we have

$$\begin{aligned} S^*(x, x, y) &\le S^*(x, x, x) + S^*(x, x, x) + S^*(y, y, x) - 2S^*(x, x, x) \\ &= S^*(y, y, x) \end{aligned}$$

and

$$\begin{aligned} S^*(y, y, x) &\le S^*(y, y, y) + S^*(y, y, y) + S^*(x, x, y) - 2S^*(y, y, y) \\ &= S^*(x, x, y). \end{aligned}$$

Hence, we get $S^*(x, x, y) = S^*(y, y, x)$.
(b) By the condition ($s_{p2}$), we have

$$S^*(x, x, x) \le S^*(x, x, y) = 0,$$

and similarly by relation (a), we also have

$$S^*(y, y, y) \le S^*(y, y, x) = S^*(x, x, y) = 0.$$

Therefore, we get $S^*(x, x, y) = S^*(x, x, x) = S^*(y, y, y) = 0$, which, by the condition ($s_{p1}$) implies that $x = y$. □

**Remark 2.4.** Dung, Hieu and Radojević noted in [3, Examples 2.1 and 2.2] that the class of $S$-metric spaces is incomparable with the the class of $G$-metric spaces, in the sense of Mustafa and Sims [6]. The same examples show that the class of partial $S$-metric spaces is incomparable with the class of $GP$-metric spaces, in the sense of Zand and Nezhad [14].

**Definition 2.5.** Let $(X, S^*)$ be a partial $S$-metric space and $\{x_n\}$ be a sequence in $X$.

(a) The sequence $\{x_n\}$ is convergent to $x \in X$ (denoted as $x_n \to x$ as $n \to \infty$) if
$$\lim_{n\to\infty} S^*(x_n, x_n, x) = \lim_{n\to\infty} S^*(x_n, x_n, x_n) = S^*(x, x, x).$$

(b) $\{x_n\}$ is said to be a Cauchy sequence if there exists (finite) $\lim_{n,m\to\infty} S^*(x_n, x_n, x_m)$.

(c) The space $(X, S^*)$ is complete if every Cauchy sequence in $X$ is convergent.

Note that if $x_n \to x$ as $n \to \infty$, then for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that
$$|S^*(x_n, x_n, x) - S^*(x, x, x)| < \epsilon, \quad \forall n \geq n_0, \tag{2.1}$$
and
$$|S^*(x_n, x_n, x_n) - S^*(x, x, x)| < \epsilon, \quad \forall n \geq n_0. \tag{2.2}$$
Hence, for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that
$$|S^*(x_n, x_n, x_n) - S^*(x_n, x_n, x)| < \epsilon, \quad \forall n \geq n_0. \tag{2.3}$$

**Lemma 2.6.** *Let $(X, S^*)$ be a partial $S$-metric space. If a sequence $\{x_n\}$ in $X$ converges to $x \in X$, then $x$ is unique.*

*Proof.* Let $\{x_n\}$ converges to $x$ and $y$. Then we have
$$\lim_{n\to\infty} S^*(x_n, x_n, x_n) = \lim_{n\to\infty} S^*(x_n, x_n, x) = S^*(x, x, x) \tag{2.4}$$
and
$$\lim_{n\to\infty} S^*(x_n, x_n, x_n) = \lim_{n\to\infty} S^*(x_n, x_n, y) = S^*(y, y, y).$$
Then, by the condition ($\mathrm{s}_{p3}$), relation (2.4) and Lemma 2.3, we have
$$\begin{aligned} S^*(x, x, y) &\leq 2S^*(x, x, x_n) + S^*(y, y, x_n) - 2S^*(x_n, x_n, x_n) \\ &= 2(S^*(x_n, x_n, x) - S^*(x_n, x_n, x_n)) + S^*(x_n, x_n, y) \\ &\quad - S^*(y, y, y) + S^*(y, y, y). \end{aligned}$$
By taking the limit as $n \to \infty$, we get $S^*(x, x, y) \leq S^*(y, y, y)$.

Also, by the condition ($\mathrm{s}_{p2}$), we have
$$S^*(y, y, y) \leq S^*(y, y, x) = S^*(x, x, y).$$
Hence, we get
$$S^*(x, x, y) = S^*(y, y, y).$$
Similarly, we have
$$S^*(x, x, y) = S^*(x, x, x).$$
Hence, by the condition, ($\mathrm{s}_{p1}$) it follows that $x = y$. $\square$

**Lemma 2.7.** *Let $(X, S^*)$ be a partial $S$-metric space. Then every convergent sequence $\{x_n\}$ in $X$ is a Cauchy sequence.*

*Proof.* Let $\{x_n\}$ converges to $x$, that is for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that inequalities (2.1), (2.2) and (2.3) hold for all $n \geq n_0$. Then, by the condition ($\mathrm{s}_{p3}$) and these inequalities, we have, for $m, n \geq n_0$,

$$\begin{aligned} S^*(x_n, x_n, x_m) &\leq S^*(x_n, x_n, x) + S^*(x_n, x_n, x) \\ &\quad + S^*(x_m, x_m, x) - 2S^*(x, x, x) \\ &\leq 2(S^*(x_n, x_n, x) - S^*(x, x, x)) \\ &\quad + S^*(x_m, x_m, x) - S^*(x, x, x) + S^*(x, x, x) \\ &< 2\epsilon + \epsilon + S^*(x, x, x). \end{aligned} \tag{2.5}$$

Similarly, by the condition ($\mathrm{s}_{p3}$) and Lemma 2.6,

$$\begin{aligned} S^*(x, x, x) &\leq S^*(x, x, x_n) + S^*(x, x, x_n) \\ &\quad + S^*(x, x, x_n) - 2S^*(x_n, x_n, x_n) \\ &= 2(S^*(x_n, x_n, x) - S^*(x_n, x_n, x_n)) + S^*(x, x, x_n) \\ &\leq 2(S^*(x_n, x_n, x) - S^*(x_n, x_n, x_n)) + 2S^*(x, x, x_m) \\ &\quad + S^*(x_n, x_n, x_m) - 2S^*(x_m, x_m, x_m). \\ &< 2\epsilon + 2\epsilon + S^*(x_n, x_n, x_m). \end{aligned} \tag{2.6}$$

Hence, by (2.5) and (2.6), we have

$$|S^*(x_n, x_n, x_m) - S^*(x, x, x)| < 4\epsilon$$

for $m, n \geq n_0$. Thus, $\lim_{n,m\to\infty} S^*(x_n, x_n, x_m) = S^*(x, x, x)$, and the sequence $\{x_n\}$ is Cauchy. □

The notion of $S_b$-metric spaces was introduced independently in [10] and [13].

**Definition 2.8.** Let $X$ be a nonempty set and $b \geq 1$ a given real number. An $S_b$-metric on $X$, with parameter $b$, is a function $S_b : X \times X \times X \to [0, +\infty)$ such that for all $x, y, z, a \in X$, the following conditions are satisfied:

($\mathrm{s}_{b1}$) $S_b(x, y, z) = 0 \iff x = y = z$,
($\mathrm{s}_{b2}$) $S_b(x, x, y) = S_b(y, y, x)$,
($\mathrm{s}_{b3}$) $S_b(x, y, z) \leq b(S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a))$.

In this case, the pair $(X, S_b)$ is called an $S_b$-metric space.

A connection between partial $S$-metric and $S_b$-metric is given by the following lemma.

**Lemma 2.9.** *If $(X, S^*)$ is a partial $S$-metric space, then $S^s : X \times X \times X \to [0, +\infty)$, given by*

$$\begin{aligned} S^s(x,y,z) = S^*(x,x,y) &+ S^*(y,y,z) \\ &+ S^*(z,z,x) - S^*(x,x,x) \\ &- S^*(y,y,y) - S^*(z,z,z), \end{aligned}$$

*is an $S_b$-metric on $X$, with parameter $b = 2$.*

*Proof.* First of all, by the condition ($s_{p2}$) and the definition of $S^s$, we have $S^s(x,y,z) \geq 0$. Further, we check that the conditions of Definition 2.8 are fulfilled.

($s_{b1}$) If $S^s(x,y,z) = 0$ then it follows that

$$S^*(x,y,z) = S^*(x,x,x) = S^*(y,y,y) = S^*(z,z,z).$$

That is, $x = y = z$. Conversely, if $x = y = z$, then we have $S^s(x,y,z) = 0$.

($s_{b2}$) By the definition of $S^s$ and Lemma 2.3, we have

$$\begin{aligned} S^s(x,x,y) &= S^*(x,x,x) + S^*(x,x,y) \\ &\quad + S^*(y,y,x) - S^*(x,x,x) \\ &\quad - S^*(x,x,x) - S^*(y,y,y) \\ &= S^*(x,x,x) + S^*(x,x,y) \\ &\quad + S^*(x,x,y) - S^*(x,x,x) \\ &\quad - S^*(x,x,x) - S^*(y,y,y) \\ &= 2S^*(x,x,y) - S^*(x,x,x) - S^*(y,y,y). \end{aligned}$$

Similarly, we can show that

$$S^s(y,y,x) = 2S^*(x,x,y) - S^*(x,x,x) - S^*(y,y,y).$$

Therefore, $S^s(x,x,y) = S^s(y,y,x)$. Also, we have always that

$$S^*(x,x,y) - S^*(x,x,x) \leq S^s(x,x,y).$$

($s_{b3}$) By the condition ($s_{p3}$) and Lemma 2.3, we have

$$
\begin{aligned}
S^s(x,y,z) &= S^*(x,x,y) + S^*(y,y,z) + S^*(z,z,x) - S^*(x,x,x) \\
&\quad - S^*(y,y,y) - S^*(z,z,z) \\
&\leq 2S^*(x,x,a) - 2S^*(a,a,a) + S^*(y,y,a) \\
&\quad + 2S^*(y,y,a) - 2S^*(a,a,a) + S^*(z,z,a) \\
&\quad + 2S^*(z,z,a) - 2S^*(a,a,a) + S^*(x,x,a) \\
&\quad - S^*(x,x,x) - S^*(y,y,y) - S^*(z,z,z) \\
&= 3S^*(a,a,x) - 2S^*(a,a,a) - S^*(x,x,x) \\
&\quad + S^*(a,a,x) - S^*(x,x,x) \\
&\quad + 3S^*(a,a,y) - 2S^*(a,a,a) - S^*(y,y,y) \\
&\quad + S^*(a,a,y) - S^*(y,y,y) \\
&\quad + 3S^*(a,a,z) - 2S^*(a,a,a) - S^*(z,z,z) \\
&\quad + S^*(a,a,z) - S^*(z,z,z) \\
&= 2[S^s(x,x,a) + S^s(y,y,a) + S^s(z,z,a)].
\end{aligned}
$$

□

**Lemma 2.10.** *Let $(X, S^*)$ be a partial $S$-metric space and $S^s$ the respective $S_b$-metric introduced in Lemma 2.9. Then, we have the following statement:*

(a) *A sequence $\{x_n\}$ in $X$ is a Cauchy sequence in $(X, S^*)$ if and only if it is a Cauchy sequence in $(X, S^s)$.*
(b) *The space $(X, S^*)$ is complete if and only if the space $(X, S^s)$ is complete. Furthermore, $\lim_{n\to\infty} S^s(x_n, x_n, x) = 0$ if and only if*

$$S^*(x,x,x) = \lim_{n\to\infty} S^*(x_n, x_n, x) = \lim_{n,m\to\infty} S^*(x_n, x_n, x_m).$$

*Proof.* (a) Let $\{x_n\}$ be a Cauchy sequence in $(X, S^*)$. Then there exists (finite) $\lim_{n,m\to\infty} S^*(x_n, x_n, x_m) = \lim_{n\to\infty} S^*(x_n, x_n, x_n)$. Since

$$S^s(x_n, x_n, x_m) = 2S^*(x_n, x_n, x_m) - S^*(x_n, x_n, x_n) - S^*(x_m, x_m, x_m),$$

we have

$$
\begin{aligned}
\lim_{n,m\to\infty} S^s(x_n, x_n, x_m) &= 2 \lim_{n,m\to\infty} S^*(x_n, x_n, x_m) \\
&\quad - \lim_{n\to\infty} S^*(x_n, x_n, x_n) - \lim_{m\to\infty} S^*(x_m, x_m, x_m) \\
&= 0.
\end{aligned}
$$

We conclude that $\{x_n\}$ is a Cauchy sequence in $(X, S^s)$.

(b) Next we prove that completeness of $(X, S^s)$ implies completeness of $(X, S^*)$. Indeed, if $\{x_n\}$ is a Cauchy sequence in $(X, S^*)$ then it is also a Cauchy sequence in $(X, S^s)$. Since the space $(X, S^s)$ is complete, we deduce that there exists $y \in X$ such that $\lim_{n\to\infty} S^s(x_n, x_n, y) = 0$, since

$$S^s(x_n, x_n, y) = 2S^*(x_n, x_n, y) - S^*(y, y, y) - S^*(x_n, x_n, x_n).$$

Also, we know that

$$0 \leq S^*(x_n, x_n, y) - S^*(y, y, y) < S^s(x_n, x_n, y)$$

and

$$0 \leq S^*(x_n, x_n, y) - S^*(x_n, x_n, x_n) < S^s(x_n, x_n, y).$$

Therefore, we have

$$\lim_{n\to\infty} S^*(x_n, x_n, y) = \lim_{n\to\infty} S^*(x_n, x_n, x_n) = \lim_{n\to\infty} S^*(y, y, y).$$

Hence, we deduce that $\{x_n\}$ is a convergent sequence in $(X, S^*)$. Now we prove that every Cauchy sequence $\{x_n\}$ in $(X, S^s)$ is a Cauchy sequence in $(X, S^*)$. Let $\epsilon = \frac{1}{2}$. Then there exists $n_0 \in \mathbb{N}$ such that $S^s(x_n, x_n, x_m) < \frac{1}{2}$ for all $n, m \geq n_0$. Since

$$\begin{aligned} S^*(x_n, x_n, x_n) &\leq 4S^*(x_{n_0}, x_{n_0}, x_n) - 3S^*(x_n, x_n, x_n) \\ &\quad - S^*(x_{n_0}, x_{n_0}, x_{n_0}) + S^*(x_n, x_n, x_n) \\ &\leq 2S^s(x_n, x_n, x_{n_0}) + S^*(x_{n_0}, x_{n_0}, x_{n_0}), \end{aligned}$$

we have

$$\begin{aligned} S^*(x_n, x_n, x_n) &\leq 2S^s(x_n, x_n, x_{n_0}) + S^*(x_{n_0}, x_{n_0}, x_{n_0}) \\ &\leq 1 + S^*(x_{n_0}, x_{n_0}, x_{n_0}). \end{aligned}$$

Consequently, the sequence $\{S^*(x_n, x_n, x_n)\}$ is bounded in $\mathbb{R}$, and so there exists an $\alpha \in \mathbb{R}$ such that a subsequence $\{S^*(x_{n_k}, x_{n_k}, x_{n_k})\}$ is convergent to $\alpha$, that is, $\lim_{k\to\infty} S^*(x_{n_k}, x_{n_k}, x_{n_k}) = \alpha$.

It remains to prove that $\{S^*(x_n, x_n, x_n)\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\{x_n\}$ is a Cauchy sequence in $(X, S^s)$, for given $\epsilon > 0$, there exists $n_\epsilon$ such that $S^s(x_n, x_n, x_m) < \frac{\epsilon}{2}$ for all $n, m \geq n_\epsilon$. Thus, for all $n, m \geq n_\epsilon$,

$$\begin{aligned} |S^*(x_n, x_n, x_n) - S^*(x_m, x_m, x_m)| &\leq 4S^*(x_n, x_n, x_m) - 3S^*(x_n, x_n, x_n) \\ &\quad - S^*(x_m, x_m, x_m) + S^*(x_n, x_n, x_n) \\ &\quad - S^*(x_m, x_m, x_m) \\ &\leq 2S^s(x_n, x_n, x_m) \\ &< \epsilon. \end{aligned}$$

On the other hand,

$$\begin{aligned} |S^*(x_n, x_n, x_n) - \alpha| &\leq |S^*(x_n, x_n, x_n) - S^*(x_{n_k}, x_{n_k}, x_{n_k})| \\ &\quad + |S^*(x_{n_k}, x_{n_k}, x_{n_k}) - \alpha| \\ &< \epsilon + \epsilon = 2\epsilon, \end{aligned}$$

for all $n, n_k \geq n_\epsilon$. Hence $\lim_{n\to\infty} S^*(x_n, x_n, x_n) = \alpha$. Now,

$$\begin{aligned}
&|2S^*(x_n, x_n, x_m) - 2\alpha| \\
&= |S^s(x_n, x_n, x_m) + S^*(x_n, x_n, x_n) - \alpha + S^*(x_m, x_m, x_m) - \alpha| \\
&\leq S^s(x_m, x_m, x_m) + |S^*(x_n, x_n, x_n) - \alpha| + |S^*(x_m, x_m, x_m) - \alpha| \\
&< \frac{\epsilon}{2} + 2\epsilon + 2\epsilon = \frac{9}{2}\epsilon.
\end{aligned}$$

Thus, $\{x_n\}$ is a Cauchy sequence in $(X, S^*)$.

In order to complete the proof, we have to prove that $(X, S^s)$ is complete if such is $(X, S^*)$. Let $\{x_n\}$ be a Cauchy sequence in $(X, S^s)$. Then $\{x_n\}$ is a Cauchy sequence in $(X, S^*)$, and so it is convergent to a point $y \in X$ with

$$\lim_{n,m\to\infty} S^*(x_n, x_n, x_m) = \lim_{n\to\infty} S^*(y, y, x_n) = S^*(y, y, y).$$

Thus, given $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that

$$|S^*(y, y, x_n) - S^*(y, y, y)| < \frac{\epsilon}{2} \quad \text{and} \quad |S^*(y, y, y) - S^*(x_n, x_n, x_n)| < \frac{\epsilon}{2},$$

whenever $n \geq n_\epsilon$. Hence, we have

$$\begin{aligned}
S^s(y, y, x_n) &= 2S^*(y, y, x_n) - S^*(x_n, x_n, x_n) - S^*(y, y, y) \\
&\leq |S^*(y, y, x_n) - S^*(y, y, y)| + |S^*(y, y, x_n) - S^*(x_n, x_n, x_n)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

whenever $n \geq n_\epsilon$. Therefore $(X, S^s)$ is complete. Finally, it is a simple matter to check that $\lim_{n\to\infty} S^s(a, a, x_n) = 0$ if and only if

$$S^*(a, a, a) = \lim_{n\to\infty} S^*(a, a, x_n) = \lim_{n,m\to\infty} S^*(x_n, x_n, x_m).$$

□

**Lemma 2.11.** *Let $\{x_n\}$ and $\{y_n\}$ be two convergent sequences to $x \in X$ and $y \in X$, respectively, in a partial $S$-metric space $(X, S^*)$. Then*

$$\lim_{n\to\infty} S^*(x_n, x_n, y_n) = S^*(x, x, y).$$

*In particular, $\lim_{n\to\infty} S^*(x_n, y_n, z) = S^*(x, y, z)$ for every $z \in X$.*

*Proof.* By the assumptions, for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}
&|S^*(x_n, x_n, x) - S^*(x, x, x)| < \frac{\epsilon}{4}, \qquad |S^*(y_n, y_n, y) - S^*(y, y, y)| < \frac{\epsilon}{4}, \\
&|S^*(x_n, x_n, x_n) - S^*(x, x, x)| < \frac{\epsilon}{4}, \qquad |S^*(y_n, y_n, y_n) - S^*(y, y, y)| < \frac{\epsilon}{4}, \\
&|S^*(x_n, x_n, x_n) - S^*(x_n, x_n, x)| < \frac{\epsilon}{4}, \qquad |S^*(y_n, y_n, y_n) - S^*(y_n, y_n, y)| < \frac{\epsilon}{4},
\end{aligned}$$

hold for all $n \geq n_0$. By the condition ($s_{p3}$), for $n \geq n_0$ we have

$$\begin{aligned} S^*(x_n, x_n, y_n) &\leq S^*(x_n, x_n, x) + S^*(x_n, x_n, x) + S^*(y_n, y_n, x) - 2S^*(x, x, x) \\ &\leq S^*(x_n, x_n, x) + S^*(x_n, x_n, x) + S^*(y_n, y_n, y) + S^*(y_n, y_n, y) \\ &\quad + S^*(x, x, y) - 2S^*(y, y, y) - 2S^*(x, x, x) \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + S^*(x, x, y), \end{aligned}$$

and so we obtain

$$S^*(x_n, x_n, y_n) - S^*(x, x, y) < \epsilon.$$

Also,

$$\begin{aligned} S^*(x, x, y) &\leq S^*(x, x, x_n) + S^*(x, x, x_n) + S^*(y, y, x_n) - 2S^*(x_n, x_n, x_n) \\ &\leq S^*(x, x, x_n) + S^*(x, x, x_n) + S^*(y, y, y_n) + S^*(y, y, y_n) \\ &\quad + S^*(x_n, x_n, y_n) - 2S^*(y_n, y_n, y_n) - 2S^*(x_n, x_n, x_n) \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + S^*(x_n, x_n, y_n). \end{aligned}$$

Thus,

$$S^*(x, x, y) - S^*(x_n, x_n, y_n) < \epsilon.$$

Hence for all $n \geq n_0$, we have $|S^*(x_n, x_n, y_n) - S^*(x, x, y)| < \epsilon$ and the result follows. □

**Lemma 2.12.** *If $(X, S^*)$ is a partial $S$-metric space, the $S_b$-metrics $S^s$ (defined in Lemma 2.9) and $S^m : X \times X \times X \to \mathbb{R}^+$ given by*

$$S^m(x, y, z) = \max \left\{ \begin{array}{l} 2S^*(x, x, y) - S^*(x, x, x) - S^*(y, y, y), \\ 2S^*(y, y, z) - S^*(y, y, y) - S^*(z, z, z), \\ 2S^*(z, z, x) - S^*(z, z, z) - S^*(x, x, x) \end{array} \right\}$$

*for all $x, y, z \in X$, are equivalent.*

*Proof.* It is easy to see that $S^m$ is an $S_b$-metric on $X$. Let $x, y, z \in X$. It is obvious that

$$S^m(x, y, z) \leq 2S^s(x, y, z).$$

On the other hand, since $a+b+c \leq 3\max\{a,b,c\}$, it follows that

$$\begin{aligned}
S^s(x,y,z) &= S^*(x,x,y)+S^*(y,y,z)+S^*(z,z,x)-S^*(x,x,x)\\
&\quad -S^*(y,y,y)-S^*(z,z,z)\\
&= \frac{1}{2}[2S^*(x,x,y)-S^*(x,x,x)-S^*(y,y,y)]\\
&\quad +\frac{1}{2}[2S^*(y,y,z)-S^*(y,y,y)-S^*(z,z,z)]\\
&\quad +\frac{1}{2}[2S^*(z,z,x)-S^*(z,z,z)-S^*(x,x,x)]\\
&\leq \frac{3}{2}\max\left\{\begin{array}{l} 2S^*(x,x,y)-S^*(x,x,x)-S^*(y,y,y),\\ 2S^*(y,y,z)-S^*(y,y,y)-S^*(z,z,z),\\ 2S^*(z,z,x)-S^*(z,z,z)-S^*(x,x,x)\end{array}\right\}\\
&= \frac{3}{2}S^m(x,y,z).
\end{aligned}$$

Thus, we have

$$\frac{1}{2}S^m(x,y,z) \leq S^s(x,y,z) \leq \frac{3}{2}S^m(x,y,z).$$

These inequalities imply that $S^s$ and $S^m$ are equivalent. $\square$

## 3. Main results

**A class of implicit relation:** Throughout this section $(X,S^*)$ denotes a partial $S$-metric space, that is, $S^*$-metric space and $\Phi$ denotes a family of mappings such that for each $\phi \in \Phi$, $\phi : (\mathbb{R}^+)^4 \longrightarrow \mathbb{R}^+$, is continuous and increasing in each co-ordinate variable. Also $\gamma(t) = \phi(t,t,t,t) \leq t$ for every $t \in \mathbb{R}^+$.

**Example 3.1.** Let $\phi : (\mathbb{R}^+)^4 \longrightarrow \mathbb{R}^+$,be defined by

$$\phi(t_1,t_2,t_3,t_4) = \frac{1}{5}(t_1+t_2+t_3+t_4).$$

Then $\phi \in \Phi$.

Our main result, for a complete $S^*$-metric space $X$, reads follows:

**Theorem 3.2.** *Let $A$, $T$, $C$ and $R$ be self-mappings of a complete $S^*$-metric space $(X,S^*)$ with:*

(i) *$A(X) \subseteq T(X)$, $C(X) \subseteq R(X)$ and $T(X)$ or $R(X)$ is a closed subset of $X$,*

(ii)

$$S^*(Ax, Ay, Cz) \quad \leq \quad q\phi \left( \begin{array}{l} S^*(Rx, Ry, Tz), S^*(Rx, Ry, Ay), \\ \frac{1}{2}S^*(Ry, Tz, Cz), \frac{1}{2}S^*(Tz, Rx, Ax) \end{array} \right),$$

*for every $x, y, z \in X$, some $0 < q < \frac{1}{2}$ and $\phi \in \Phi$,*

(iii) *the pair $(A, R)$ and $(T, C)$ are weak compatible.*

*Then $A$, $T$, $C$ and $R$ have a unique common fixed point in $X$.*

*Proof.* Let $x_0 \in X$ be an arbitrary point. By (i), there exists $x_1, x_2 \in X$ such that

$$Ax_0 = Tx_1 = y_0 \quad \text{and} \quad Cx_1 = Rx_2 = y_1.$$

Inductively, construct sequence $\{y_n\}$ in $X$ such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \quad \text{and} \quad y_{2n+1} = Cx_{2n+1} = Rx_{2n+2},$$

for $n = 0, 1, 2, \cdots$.

Now, we prove that $\{y_n\}$ is a Cauchy sequence. Let $S_m^* = S^*(y_m, y_m, y_{m+1})$. Then, we have

$$\begin{aligned} S_{2n}^* &= S^*(y_{2n}, y_{2n}, y_{2n+1}) \\ &= S^*(Ax_{2n}, Ax_{2n}, Cx_{2n+1}) \\ &\leq q\phi \left( \begin{array}{l} S^*(Rx_{2n}, Rx_{2n}, Tx_{2n+1}), S^*(Rx_{2n}, Rx_{2n}, Ax_{2n}), \\ \frac{1}{2}S^*(Rx_{2n}, Tx_{2n+1}, Cx_{2n+1}), \frac{1}{2}S^*(Tx_{2n+1}, Rx_{2n}, Ax_{2n}) \end{array} \right) \\ &= q\phi \left( \begin{array}{l} S^*(y_{2n-1}, y_{2n-1}, y_{2n}), S^*(y_{2n-1}, y_{2n-1}, y_{2n}), \\ \frac{1}{2}S^*(y_{2n-1}, y_{2n}, y_{2n+1}), \frac{1}{2}S^*(y_{2n}, y_{2n-1}, y_{2n}) \end{array} \right) \\ &= q\phi(S_{2n-1}^*, S_{2n-1}^*, \frac{1}{2}S^*(y_{2n-1}, y_{2n}, y_{2n+1}), \frac{1}{2}S^*(y_{2n}, y_{2n-1}, y_{2n})). \end{aligned} \tag{3.1}$$

Since

$$\begin{aligned} S^*(y_{2n-1}, y_{2n}, y_{2n+1}) \leq{}& S^*(y_{2n-1}, y_{2n-1}, y_{2n}) + S^*(y_{2n}, y_{2n}, y_{2n}) \\ &+ S^*(y_{2n+1}, y_{2n+1}, y_{2n}) - 2S^*(y_{2n}, y_{2n}, y_{2n}), \end{aligned}$$

that is,

$$S^*(y_{2n-1}, y_{2n}, y_{2n+1}) \leq S^*(y_{2n-1}, y_{2n-1}, y_{2n}) + S^*(y_{2n}, y_{2n}, y_{2n+1}),$$

also, since

$$\begin{aligned} S^*(y_{2n}, y_{2n-1}, y_{2n}) \leq{}& 2S^*(y_{2n}, y_{2n}, y_{2n-1}) + S^*(y_{2n-1}, y_{2n-1}, y_{2n-1}) \\ &- 2S^*(y_{2n-1}, y_{2n-1}, y_{2n-1}), \end{aligned}$$

that is,

$$S^*(y_{2n}, y_{2n-1}, y_{2n}) \quad \leq \quad 2S^*(y_{2n-1}, y_{2n-1}, y_{2n}),$$

we prove that $S^*_{2n} \leq S^*_{2n-1}$, for every $n \in \mathbb{N}$. If $S^*_{2n} > S^*_{2n-1}$ for some $n \in \mathbb{N}$, then we get

$$S^*(y_{2n-1}, y_{2n}, y_{2n+1}) < 2S^*(y_{2n}, y_{2n}, y_{2n+1}) = 2S^*_{2n}$$

and

$$S^*(y_{2n}, y_{2n-1}, y_{2n}) \leq 2S^*(y_{2n-1}, y_{2n-1}, y_{2n}) < 2S^*_{2n}.$$

Hence by inequality (3.1) we have $S^*_{2n} < qS^*_{2n}$, is a contradiction. Now, if $m = 2n+1$, then

$$\begin{aligned}
S^*_{2n+1} &= S^*(y_{2n+1}, y_{2n+1}, y_{2n+2}) \\
&= S^*(y_{2n+2}, y_{2n+2}, y_{2n+1}) \\
&= S^*(Ax_{2n+2}, Ax_{2n+2}, Cx_{2n+1}) \\
&\leq q\phi\left(\begin{array}{l} S^*(Rx_{2n+2}, Rx_{2n+2}, Tx_{2n+1}), S^*(Rx_{2n+2}, Rx_{2n+2}, Ax_{2n+2}), \\ \frac{1}{2}S^*(Rx_{2n+2}, Tx_{2n+1}, Cx_{2n+1}), \frac{1}{2}S^*(Tx_{2n+1}, Rx_{2n+2}, Ax_{2n+2}) \end{array}\right) \\
&= q\phi\left(\begin{array}{l} S^*(y_{2n+1}, y_{2n+1}, y_{2n}), S^*(y_{2n+1}, y_{2n+1}, y_{2n+2}), \\ \frac{1}{2}S^*(y_{2n+1}, y_{2n}, y_{2n+1}), \frac{1}{2}S^*(y_{2n}, y_{2n+1}, y_{2n+2}) \end{array}\right) \\
&= q\phi(S^*_{2n}, S^*_{2n+1}, \frac{1}{2}S^*(y_{2n+1}, y_{2n}, y_{2n+1}), \frac{1}{2}S^*(y_{2n}, y_{2n+1}, y_{2n+2})).
\end{aligned}$$

Since

$$\begin{aligned}
S^*(y_{2n+1}, y_{2n}, y_{2n+1}) \leq 2S^*(y_{2n+1}, y_{2n+1}, y_{2n}) &+ S^*(y_{2n}, y_{2n}, y_{2n}) \\
&- 2S^*(y_{2n}, y_{2n}, y_{2n}), \qquad (3.2)
\end{aligned}$$

that is,

$$S^*(y_{2n+1}, y_{2n}, y_{2n+1}) \leq 2S^*(y_{2n}, y_{2n}, y_{2n+1}),$$

also, since

$$\begin{aligned}
S^*(y_{2n}, y_{2n+1}, y_{2n+2}) \leq S^*(y_{2n}, y_{2n}, y_{2n+1}) &+ S^*(y_{2n+1}, y_{2n+1}, y_{2n+1}) \\
+ S^*(y_{2n+2}, y_{2n+2}, y_{2n+1}) &- 2S^*(y_{2n+1}, y_{2n+1}, y_{2n+1}),
\end{aligned}$$

that is,

$$S^*(y_{2n}, y_{2n+1}, y_{2n+2}) \leq S^*(y_{2n}, y_{2n}, y_{2n+1}) + S^*(y_{2n+1}, y_{2n+1}, y_{2n+2}),$$

we prove that $S^*_{2n+1} \leq S^*_{2n}$, for every $n \in \mathbb{N}$. If $S^*_{2n+1} > S^*_{2n}$ for some $n \in \mathbb{N}$, then we get

$$S^*(y_{2n+1}, y_{2n}, y_{2n+1}) \leq 2S^*(y_{2n}, y_{2n}, y_{2n+1}) = 2S^*_{2n}$$

and

$$S^*(y_{2n}, y_{2n+1}, y_{2n+2}) < 2S^*(y_{2n+1}, y_{2n+1}, y_{2n+2}) = 2S^*_{2n+1}.$$

Hence, by inequality (3.2) we have $S^*_{2n+1} < qS^*_{2n+1}$ which is a contradiction. Hence for every $n \in \mathbb{N}$ we have $S^*_n \leq qS^*_{n-1}$. That is

$$S^*_n = S^*(y_n, y_n, y_{n+1}) \leq qS^*(y_{n-1}, y_{n-1}, y_n) \leq \cdots \leq q^n S^*(y_0, y_0, y_1).$$

Hence we get

$$S^*(y_n, y_n, y_{n+1}) \leq q^n S^*(y_0, y_0, y_1),$$

so that

$$\lim_{n\to\infty} S^*(y_n, y_n, y_{n+1}) = 0. \tag{3.3}$$

Since $S^s(y_n, y_n, y_{n+1}) \leq 2S^*(y_n, y_n, y_{n+1})$ we have

$$S^s(y_n, y_n, y_{n+1}) \leq 2S^*(y_n, y_n, y_{n+1}) \leq 2q^n S^*(y_0, y_0, y_1).$$

By the triangle inequality in $S_b-$ metric space, for $m > n$ we have

$$\begin{aligned} S^s(y_n, y_n, y_m) \leq\ & 2.2S^s(y_n, y_n, y_{n+1}) + 2.2^2 S^s(y_{n+1}, y_{n+1}, y_{n+2}) \\ & + \cdots + 2.2^{m-n} S^s(y_{m-1}, y_{m-1}, y_m), \end{aligned}$$

hence we get

$$\begin{aligned} S^s(y_n, y_n, y_m) &\leq 2^3 q^n S^*(y_0, y_0, y_1) + 2^4 q^{n+1} S^*(y_0, y_0, y_1) \\ &\quad + \cdots . + 2^{m-n+2} q^{m-1} S^*(y_0, y_0, y_1) \\ &\leq 2^3 q^n [1 + 2q + 2^2 q^2 + \cdots] S^*(y_0, y_0, y_1) \\ &\leq \frac{2^3 q^n}{1-2q} S^*(y_0, y_0, y_1) \\ &\longrightarrow 0. \end{aligned} \tag{3.4}$$

It follows that $\{y_n\}$ is a Cauchy sequence in the $S_b-$metric space $(X, S^s)$. Since $(X, S^*)$ is complete, then from Lemma 1.4 follows that the sequence $\{y_n\}$ converges to some $y$ in the $S_b-$metric space $(X, S^s)$. Hence

$$\lim_{n\to\infty} S^s(y_n, y_n, y) = 0.$$

Again, from Lemma 1.4 we have

$$S^*(y, y, y) = \lim_{n\to\infty} S^*(y_n, y_n, y) = \lim_{n,m\to\infty} S^*(y_n, y_n, y_m). \tag{3.5}$$

Since $\{y_n\}$ is a Cauchy sequence in the $S_b-$metric space $(X, S^s)$ and

$$S^s(y_n, y_n, y_m) = 2S^*(y_n, y_n, y_m) - S^*(y_n, y_n, y_n) - S^*(y_m, y_m, y_m),$$

we have

$$\lim_{n,m\to\infty} S^s(y_n, y_n, y_m) = 0,$$

and by (3.3), we have

$$\lim_{n\to\infty} S^*(y_n, y_n, y_n) = 0.$$

Thus by definition of $S^s$ we have

$$\lim_{n,m\to\infty} S^*(y_n, y_n, y_m) = 0.$$

Therefore by (3.5), we have

$$S^*(y,y,y) = \lim_{n\to\infty} S^*(y_n,y_n,y) = \lim_{n,m\to\infty} S^*(y_n,x_y,y_m) = 0.$$

That is,

$$\begin{aligned}\lim_{n\to\infty} y_n &= \lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Rx_{2n+2} \\ &= \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Cx_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = y.\end{aligned}$$

Let $R(X)$ be a closed subset of $X$, hence there exist $x \in X$ such that $Rx = y$. We prove that $Ax = y$. By the inequality (3.1), for $x = x, y = x$ and $z = x_{2n+1}$, then we have

$$\begin{aligned}S^*(Ax,Ax,y_{2n+1}) &= S^*(Ax,Ax,Cx_{2n+1}) \\ &\leq q\phi\begin{pmatrix} S^*(Rx,Rx,Tx_{2n+1}), S^*(Rx,Rx,Ax), \\ \frac{1}{2}S^*(Rx,Tx_{2n+1},Cx_{2n+1}), \frac{1}{2}S^*(Tx_{2n+1},Rx,Ax)\end{pmatrix} \\ &= q\phi\begin{pmatrix} S^*(y,y,y_{2n}), S^*(y,y,Ax), \\ \frac{1}{2}S^*(y,y_{2n},y_{2n+1}), \frac{1}{2}S^*(y_{2n},y,Ax)\end{pmatrix}.\end{aligned}$$

Since

$$S^*(Ax,Ax,y_{2n+1}) = S^*(y_{2n+1},y_{2n+1},Ax),$$

$$\begin{aligned}S^*(y,y_{2n},y_{2n+1}) &\leq S^*(y,y,y) + S^*(y_{2n},y_{2n},y) \\ &\quad + S^*(y_{2n+1},y_{2n+1},y) - 2S^*(y,y,y)\end{aligned}$$

and

$$S^*(y_{2n},y,Ax) \leq S^*(y_{2n},y_{2n},y) + S^*(y,y,y) + S^*(Ax,Ax,y) - 2S^*(y,y,y),$$

taking the limit as $n \to \infty$ we get

$$\begin{aligned}\limsup_{n\longrightarrow\infty} S^*(y,y_{2n},y_{2n+1}) &\leq S^*(y,y,y) + \limsup_{n\longrightarrow\infty} S^*(y_{2n},y_{2n},y) \\ &\quad + \limsup_{n\longrightarrow\infty} S^*(y_{2n+1},y_{2n+1},y) - 2S^*(y,y,y) \\ &= S^*(y,y,y)\end{aligned}$$

and

$$\begin{aligned}\limsup_{n\longrightarrow\infty} S^*(y_{2n},y,Ax) &\leq \limsup_{n\longrightarrow\infty} S^*(y_{2n},y_{2n},y) + S^*(y,y,y) \\ &\quad + S^*(Ax,Ax,y) - 2S^*(y,y,y) \\ &= S^*(Ax,Ax,y).\end{aligned}$$

Hence we have

$$\begin{aligned}
S^*(Ax,Ax,y) &= \lim_{n\to\infty} S^*(y_{2n+1},y_{2n+1},Ax)\\
&\le q\phi\begin{pmatrix} S^*(y,y,y), S^*(y,y,Ax),\\ \frac{1}{2}S^*(y,y,y), \frac{1}{2}S^*(Ax,Ax,y)\end{pmatrix}\\
&\leqq \phi\begin{pmatrix} S^*(Ax,Ax,y), S^*(Ax,Ax,y),\\ S^*(Ax,Ax,y), S^*(Ax,Ax,y)\end{pmatrix}\\
&\le qS^*(Ax,Ax,y)\\
&< (Ax,Ax,y).
\end{aligned}$$

If $S^*(Ax,Ax,y) > 0$, then we have $S^*(Ax,Ax,y) < S^*(Ax,Ax,y)$ which is a contradiction. Thus $Ax = y$. By the weak compatibility of the pair $(A,R)$ we have $ARx = RAx$. Hence $Ay = Ry$. We prove that $Ay = y$. If $Ay \neq y$ , then by the inequality (3.1), for $x = y, y = y$ and $z = x_{2n+1}$, we have

$$\begin{aligned}
S^*(Ay,Ay,y_{2n+1}) &= S^*(Ay,Ay,Cx_{2n+1})\\
&\le q\phi\begin{pmatrix} S^*(Ry,Ry,Tx_{2n+1}), S^*(Ry,Ry,Ay),\\ \frac{1}{2}S^*(Ry,Tx_{2n+1},Cx_{2n+1}), \frac{1}{2}S^*(Tx_{2n+1},Ry,Ay)\end{pmatrix}\\
&= q\phi\begin{pmatrix} S^*(Ay,Ay,y_{2n}), S^*(Ay,Ay,Ay),\\ \frac{1}{2}S^*(Ay,y_{2n},y_{2n+1}), \frac{1}{2}S^*(y_{2n},Ay,Ay)\end{pmatrix}.
\end{aligned}$$

Similarly, taking the limit as $n \to \infty$, we get

$$\begin{aligned}
S^*(Ay,Ay,y) &= \lim_{n\to\infty} S^*(Ay,Ay,y_{2n+1})\\
&\le q\phi\begin{pmatrix} S^*(Ay,Ay,y), S^*(Ay,Ay,Ay),\\ \frac{1}{2}S^*(Ay,Ay,y), \frac{1}{2}S^*(Ay,Ay,y)\end{pmatrix}\\
&< qS^*(Ay,Ay,y),
\end{aligned}$$

which is a contradiction. Therefore, $Ry = Ay = y$, that is, $y$ is a common fixed of $R$ and $A$.

Since $y = Ay \in A(X) \subseteq T(X)$, there exists $v \in X$ such that $Tv = y$. We prove that $Cv = y$. For

$$\begin{aligned}
S^*(Ay,Ay,Cv) &= S^*(y,y,Cv)\\
&\le q\phi\begin{pmatrix} S^*(Ry,Ry,Tv), S^*(Ry,Ry,Ay),\\ \frac{1}{2}S^*(Ry,Tv,Cv), \frac{1}{2}S^*(Tv,Ry,Ay)\end{pmatrix}\\
&= q\phi\begin{pmatrix} S^*(y,y,y), S^*(y,y,y),\\ \frac{1}{2}S^*(y,y,Cv), \frac{1}{2}S^*(y,y,y)\end{pmatrix}\\
&< qS^*(y,y,Cv).
\end{aligned}$$

Thus $Cv = y$. By the weak compatibility of the pair $(C, T)$ we have $TCv = CTv$. Hence $Cy = Ty$. We prove that $Cy = y$. If $Cy \neq y$, then

$$\begin{aligned} S^*(Ay, Ay, Cy) &= S^*(y, y, Cy) \\ &\leq q\phi \begin{pmatrix} S^*(Ry, Ry, Ty), S^*(Ry, Ry, Ay), \\ \frac{1}{2}S^*(Ry, Ty, Cy), \frac{1}{2}S^*(Ty, Ry, Ay) \end{pmatrix} \\ &= q\phi \begin{pmatrix} S^*(y, y, y), S^*(y, y, y), \\ \frac{1}{2}S^*(y, y, Cy), \frac{1}{2}S^*(y, y, y) \end{pmatrix} \\ &< qS^*(y, y, Cy), \end{aligned}$$

which is a contradiction. Therefore, $Cy = Ty = y$, that is, $y$ is a common fixed of $C$ and $T$. That is,

$$Cy = Ty = Ay = Ry = y$$

To prove uniqueness, let $v$ be another common fixed point of $A, C, R, T$. If $S^*(y, y, v) > 0$, then

$$\begin{aligned} S^*(y, y, v) &= S^*(Ay, Ay, Cv) \\ &\leq q\phi \begin{pmatrix} S^*(Ry, Ry, Tv), S^*(Ry, Ry, Ay), \\ \frac{1}{2}S^*(Ry, Tv, Cv), \frac{1}{2}S^*(Tv, Ry, Ay) \end{pmatrix} \\ &= q\phi \begin{pmatrix} S^*(y, y, v), S^*(y, y, y), \\ \frac{1}{2}S^*(y, v, v), \frac{1}{2}S^*(v, y, y) \end{pmatrix} \\ &< qS^*(y, y, v), \end{aligned}$$

which is a contradiction. Therefore, $y = v$. This means that $y$ is the unique common fixed point of self-maps $A, C, R, T$. $\square$

**Example 3.3.** Let $X = [0, \infty)$ be equipped with the partial $S-$ metric $S^*(x, y, z) = \max\{x, y, z\}$.

Consider the mappings $A, T, C$ and $R$ be self-mappings of a complete $S^*-$ metric space $(X, S^*)$ with:

$A(x) = \frac{x}{9}$, $T(x) = \frac{x}{2}$, $C(x) = \frac{x}{6}$ and $R(x) = \frac{x}{3}$. Choose $\phi \in \Phi$ as $\phi(t_1, t_2, t_3, t_4) = \max\{t_1, t_2, t_3, t_4\}$.

We will check that conditions of Theorem 3.2 are fulfilled.

First of all, since $A(X) = T(X) = C(X) = R(X) = X$ hence $A(X) \subseteq T(X), C(X) \subseteq R(X)$ holds for $x \in X$ and $T(X)$ or $R(X)$ is a closed subset of $X$ and the pair $(A, R)$ and $(T, C)$ are weak compatible. Since

$$S^*(Ax, Ay, Cz) = \max\{\frac{x}{9}, \frac{y}{9}, \frac{z}{6}\} = \frac{1}{3}\max\{\frac{x}{3}, \frac{y}{3}, \frac{z}{2}\}$$

and

$$S^*(Rx, Ry, Tz) = \max\{\frac{x}{3}, \frac{y}{3}, \frac{z}{2}\},$$

this reduces to

$$S^*(Ax, Ay, Cz) \quad \leq \quad q\phi \left( \begin{array}{l} S^*(Rx, Ry, Tz), S^*(Rx, Ry, Ay), \\ \frac{1}{2}S^*(Ry, Tz, Cz), \frac{1}{2}S^*(Tz, Rx, Ax) \end{array} \right),$$

for every $x, y, z \in X$ and $q = \frac{1}{3}$. By Theorem 3.2, the mappings $A, T, C$ and $R$ have a unique common fixed point 0 in $X$.

**Corollary 3.4.** *Let $T, R$ and $\{A_\alpha\}_{\alpha \in I}$ and $\{C_\gamma\}_{\gamma \in K}$ be the set of all self-mappings of a complete $S^*$-metric space $(X, S^*)$. Suppose that the following conditions are satisfied:*

(i) *there exists $\alpha_0 \in I$ and $\gamma_0 \in K$ such that $A_{\alpha_0}(X) \subseteq T(X)$ and $C_{\gamma_0}(X) \subseteq R(X)$,*

(ii) *$A_{\alpha_0}(X)$ or $C_{\gamma_0}(X)$ is a closed subset of $X$,*

(iii)

$$S^*(A_\alpha x, A_\alpha y, C_\gamma z) \leq q\phi \left( \begin{array}{l} S^*(Rx, Ry, Tz), S^*(Rx, Ry, A_\alpha y), \\ \frac{1}{2}S^*(Ry, Tz, C_\gamma z), \frac{1}{2}S^*(Tz, Rx, A_\alpha x) \end{array} \right)$$

*for every $x, y, z \in X$, some $0 < q < \frac{1}{2}$ and $\phi \in \Phi$, and every $\alpha \in I, \gamma \in K$,*

(iv) *the pair $(A_{\alpha_0}, R)$ or $(C_{\gamma_0}, T)$ is weak compatible.*

*Then for every $\lambda \in I$ and $\eta \in K$ $A_\lambda, C_\eta, R, T$ have a unique common fixed point in $X$.*

*Proof.* By Theorem 3.2 $R, T$ and $A_{\alpha_0}$ and $C_{\gamma_0}$ for some $\alpha_0 \in I, \gamma_0 \in K$, have a unique common fixed point in $X$. That is, there exist a unique $a \in X$ such that $R(a) = T(a) = A_{\alpha_0}(a) = C_{\gamma_0}(a) = a$. Suppose that there exist $\lambda \in I$ such that $\lambda \neq \alpha_0$ and $S^*(A_\lambda a, A_\lambda a, a) > 0$. Then we have

$$\begin{aligned} S^*(A_\lambda a, A_\lambda a, a) &= S^*(A_\lambda a, A_\lambda a, C_{\gamma_0} a) \\ &\leq q\phi \left( \begin{array}{l} S^*(Ra, Ra, Ta), S^*(Ra, Ra, A_\lambda a), \\ \frac{1}{2}S^*(Ra, Ta, C_{\gamma_0} a), \frac{1}{2}S^*(Ta, Ra, A_\lambda a) \end{array} \right) \\ &\leq q\phi \left( \begin{array}{l} S^*(a, a, a), S^*(a, a, A_\lambda a), \\ \frac{1}{2}S^*(a, a, a), \frac{1}{2}S^*(a, a, A_\lambda a) \end{array} \right) \\ &\leq qS^*(A_\lambda a, A_\lambda a, a) < S^*(A_\lambda a, A_\lambda a, a), \end{aligned}$$

which is a contradiction. Hence for every $\lambda \in I$ we have $A_\lambda(a) = a$. Similarly for every $\eta \in K$ we get $C_\eta(a) = a$. Therefore for every $\lambda \in I$ and $\eta \in K$ we have $A_\lambda(a) = C_\eta(a) = R(a) = T(a) = a$. This completes the proof. □

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