Nonlinear Functional Analysis and Applications
Vol. 24, No. 3 (2019), pp. 457-475
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

NONLOCAL CAUCHY PROBLEM FOR SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS AND FRACTIONAL DIFFERENTIAL EQUATIONS

K. Balachandran[1], L. Byszewski[2] and J. K. Kim[3]

[1]Department of Mathematics
Bharathiar University, Coimbatore 641 046, India
e-mail: kb.maths.bu@gmail.com

[2]Institute of Mathematics
Cracow University of Technology,ul. Warszawska 24, 31-155 Cracow, Poland
e-mail: lbyszews@pk.edu.pl

[3]Department of Mathematics
Kyungnam University, Changwon, Gyeongnam 51767, Korea
e-mail: jongkyuk@kyungnam.ac.kr

Abstract. The aim of this paper is to prove the existence and uniqueness of mild and classical solutions of a second order evolution equation with functional dependence on the solutions and on derivatives of the solutions. The theory of strongly continuous cosine families of linear operators in a Banach space is applied. Further we discuss the existence of solutions of nonlinear fractional differential equations in abstract spaces.

1. Introduction

In this paper, we consider the abstract nonlocal second order semilinear functional-differential equation of the form:

$$u''(t) = Au(t) + f(t, u(t), u(a_1(t)), u'(t), u'(a_2(t))),\ t \in (0, T], \tag{1.1}$$

[0]Received November 28, 2018. Revised March 5, 2019.
[0]2010 Mathematics Subject Classification: 34K05, 34K30, 34A08.
[0]Keywords: Nonlocal, Cauchy problem, second order, functional dependence, fractional derivatives, Banach spaces.
[0]Corresponding author: J.K. Kim(jongkyuk@kyungnam.ac.kr).

$$u(0) = x_0, \tag{1.2}$$

$$u'(0) + \sum_{i=1}^{p} h_i u(t_i) = x_1, \tag{1.3}$$

where A is a linear operator from a real Banach apace X into itself, $u : [0, T] \to X$, $f : [0, T] \times X^4 \to X$, $a_i : [0, T] \to [0, T]$ $(i = 1, 2)$, $x_0, x_1 \in X$, $h_i \in \mathbb{R}$ $(i = 1, 2, ..., p)$ and $0 < t_1 < t_2 < ... < t_p \leq T$.

We prove the existence and uniqueness of mild and classical solutions of problem (1.1) - (1.3). For this purpose, we apply the theory of strongly continuous cosine families of linear operators in a Banach space. We also apply the Banach contraction theorem and the Bochenek theorem (see Theorem 1.1 in [10]).

Assumption $(\mathbf{A_1})$. Operator A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators from X into itself.

Recall that the infinitesimal generator of a strongly continuous cosine family $C(t)$ is the operator $A : X \supset D(A) \to X$ defined by [24, 26]

$$Ax := \frac{d^2}{dt^2} C(t)x \,|_{t=0},\ x \in D(A),$$

where

$$D(A) := \{x \in X : \ C(t)x \text{ is of class } C^2 \text{ with respect to } t\}.$$

Let

$$E := \{x \in X : \ C(t)x \text{ is of class } C^1 \text{ with respect to } t\}.$$

The associated sine family $\{S(t) : t \in \mathbb{R}\}$ is defined by

$$S(t)x := \int_0^t C(s)x\, ds, \quad x \in X,\ t \in \mathbb{R}.$$

From Assumption (A_1) it follows (see [29,30]) that there are constants $M \geq 1$ and $\omega \geq 0$ such that

$$\|C(t)\| \leq Me^{\omega|t|} \quad \text{and} \quad \|S(t)\| \leq Me^{\omega|t|} \quad \text{for} \quad t \in \mathbb{R}.$$

We will also use the following assumption:

Assumption $(\mathbf{A_2})$. The adjoint operator A^* is densly defined in X^*; that is, $\overline{D(A^*)} = X^*$.

The paper is based on the publications [1-4, 6-9, 13-22, 27-28, 30] and is a generalization of papers [11] and [12].

2. Mild solutions

A function u belonging to $C^1([0,T],X)$ and satisfying the integral equation

$$\begin{aligned} u(t) &= C(t)x_0 + S(t)x_1 - S(t)\Big(\sum_{i=1}^{p} h_i u(t_i)\Big) \\ &\quad + \int_0^t S(t-s) f(s,u(s),u(a_1(s)),u'(s),u'(a_2(s)))ds, \quad t \in [0,T], \end{aligned}$$

is said to be a mild solution of the nonlocal Cauchy problem (1.1) - (1.3).

Theorem 2.1. *Suppose that:*

(i) *Assumption* (A_1) *is satisfied,*
(ii) $a_i : [0,T] \to [0,T], (i = 1,2)$ *are of class* C^1 *on* $[0,T]$, $f : [0,T] \times X^4 \to X$ *is continuous with respect to the first variable* $t \in [0,T]$ *and there exists a positive constant* L_1 *such that*

$$\|f(s,z_1,z_2,z_3,z_4) - f(s,\tilde{z}_1,\tilde{z}_2,\tilde{z}_3,\tilde{z}_4)\| \le L_1 \sum_{i=1}^{4} \|z_i - \tilde{z}_i\|$$
$$\text{for } s \in [0,T], \quad z_i, \tilde{z}_i \in X \quad (i = 1,2,3,4),$$

(iii) $2C(2TL_1 + \sum_{i=1}^{p} | h_i |) < 1$, *where* $C := \sup\{\|C(t)\| + \|S(t)\| + \|S'(t)\| : t \in [0,T]\}$,
(iv) $x_0 \in E$ *and* $x_1 \in X$.

Then the nonlocal Cauchy problem (1.1) - (1.3) *has a unique mild solution.*

Proof. Let the operator $F : C^1([0,T],X) \to C^1([0,T],X)$ be given by

$$\begin{aligned} (Fu)(t) &= C(t)x_0 + S(t)x_1 - S(t)\Big(\sum_{i=1}^{p} h_i u(t_i)\Big) \\ &\quad + \int_0^t S(t-s) f(s,u(s),u(a_1(s)),u'(s),u'(a_2(s)))ds, \quad t \in [0,T]. \end{aligned}$$

Now we shall show that F is a contraction on the Banach space $C^1([0,T],X)$ equipped with the norm

$$\|w\|_1 := \sup\{\|w(t)\| + \|w'(t)\| : \quad t \in [0,T]\}.$$

To do this, observe that

$$\begin{aligned}\|(Fw)(t)-(F\tilde{w})(t)\| = \|S(t)\Big(\sum_{i=1}^{p} h_i(\tilde{w}(t_i)-w(t_i))\Big)\\ + \int_0^t S(t-s)(f(s,w(s),w(a_1(s)),w'(s),w'(a_2(s))))\\ - f(s,\tilde{w}(s),\tilde{w}(a_1(s)),\tilde{w}'(s),\tilde{w}'(a_2(s))))ds\|,\end{aligned}$$

$$\begin{aligned}\|(Fw)(t)-(F\tilde{w})(t)\| &\le C\Big(\sum_{i=1}^{p} \mid h_i \mid\Big)\|w-\tilde{w}\|_1\\ &\quad + \int_0^t \|S(t-s)\|L_1(\|w(s)-\tilde{w}(s)\|\\ &\quad + \|w(a_1(s))-\tilde{w}(a_1(s))\| + \|w'(s)-\tilde{w}'(s)\|\\ &\quad + \|w'(a_2(s))-\tilde{w}'(a_2(s))\|)ds\\ &\le C\Big(2TL_1+\sum_{i=1}^{p} \mid h_i \mid\Big)\|w-\tilde{w}\|_1\end{aligned}$$

and

$$\begin{aligned}\|(Fw)'(t)-(F\tilde{w})'(t)\| &= \|S'(t)\Big(\sum_{i=1}^{p} h_i(\tilde{w}(t_i)-w(t_i))\Big)\\ &\quad + \int_0^t C(t-s)(f(s,w(s)),w(a_1(s)),w'(s),w'(a_2(s)))\\ &\quad - f(s,\tilde{w}(s),\tilde{w}(a_1(s)),\tilde{w}'(s),\tilde{w}'(a_2(s))))ds\|\\ &\le C\Big(\sum_{i=1}^{p} \mid h_i \mid\Big)\|w-\tilde{w}\|_1\\ &\quad + \int_0^t \|C(t-s)\|L_1(\|w(s)-\tilde{w}(s)\|\\ &\quad + \|w(a_1(s))-\tilde{w}(a_1(s))\| + \|w'(s)-\tilde{w}'(s)\|\\ &\quad + \|w'(a_2(s))-\tilde{w}'(a_2(s))\|)ds\\ &\le C\Big(2TL_1+\sum_{i=1}^{p} \mid h_i \mid\Big)\|w-\tilde{w}\|_1, \quad t\in[0,T].\end{aligned}$$

Consequently

$$\|Fw-F\tilde{w}\|_1 \le 2C\Big(2TL_1+\sum_{i=1}^{p} \mid h_i \mid\Big)\|w-\tilde{w}\|_1 \text{ for } \ w,\tilde{w}\in C^1([0,T],X).$$

Therefore, in the space $C^1([0,T],X)$ there is only one fixed point of F and this point is the mild solution of the nonlocal Cauchy problem (1.1) - (1.3). So the proof of Theorem 2.1 is complete. □

3. Classical solutions

A function $u : [0,T] \longrightarrow X$ is said to be a classical solution to problem (1.1) - (1.3) if

$$u \in C^1([0,T],X) \cap C^2((0,T],X),$$

$$u(0) = x_0, \quad u'(0) + \sum_{i=1}^{p} h_i u(t_i) = x_1,$$

and

$$u''(t) = Au(t) + f(t, u(t), u(a_1(t)), u'(t), u'(a_2(t))) \text{ for } t \in [0,T].$$

Theorem 3.1. *Suppose that:*

(i) *Assumptions* (A_1) *and* (A_2) *are satisfied and* $a_i : [0,T] \to [0,T]$ $(i = 1,2)$ *are of class* C^1 *on* $[0,T]$.

(ii) *there exists a positive constant* L_2 *such that*

$$\|f(s, z_1, z_2, z_3, z_4) - f(\tilde{s}, \tilde{z_1}, \tilde{z_2}, \tilde{z_3}, \tilde{z_4})\| \leq L_2(| s - \tilde{s} | + \sum_{i=1}^{4} \|z_i - \tilde{z_i}\|)$$

for $s, \ \tilde{s} \in [0,T], \ \ z_i, \tilde{z_i} \in X \ \ (i = 1,2,3,4)$.

(iii) $2C\Big(2TL_2 + \sum_{i=1}^{p} | h_i | \Big) < 1$.

(iv) $x_0 \in E$ *and* $x_1 \in X$.

Then the nonlocal Cauchy problem (1.1) - (1.3) *has a unique mild solution* u. *Moreover, if* $x_0 \in D(A), \ x_1 \in E$ *and* $u(t_i) \in E \ \ (i = 1,2,...,p)$, *and there exist positive constants* c_i $(i = 1,2)$ *such that*

$$\|u(a_1(s)) - u(a_1(\tilde{s}))\| \leq c_1 \|u(s) - u(\tilde{s})\| \text{ for } s, \ \tilde{s} \in [0,T]$$

and

$$\|u'(a_2(s)) - u'(a_2(\tilde{s}))\| \leq c_2 \|u'(s) - u'(\tilde{s})\| \text{ for } s, \ \tilde{s} \in [0,T],$$

then u *is the unique classical solution of nonlocal problem* (1.1) - (1.3).

Proof. Since the assumptions of Theorem 2.1 are satisfied, the nonlocal Cauchy problem (1.1) - (1.3) possesses a unique mild solution which is denoted by u. Now we shall show that u is the classical solution of problem (1.1) - (1.3). First we shall prove that $u, u(a_1(\cdot), u'$ and $u'(a_2(\cdot))$ satisfy the Lipschitz condition on $[0,T]$. Let t and $t+h$ be any two points belonging to $[0,T]$. Observe that

$$\begin{aligned} u(t+h)-u(t) = {} & C(t+h)x_0 + S(t+h)x_1 - S(t+h)\Big(\sum_{i=1}^{p} h_i u(t_i)\Big) \\ & + \int_0^{t+h} S(t+h-s)f(s,u(s),u(a_1(s)),u'(s),u'(a_2(s)))ds \\ & - C(t)x_0 - S(t)x_1 + S(t)\Big(\sum_{i=1}^{p} h_i u(t_i)\Big) \\ & - \int_0^{t} S(t-s)f(s,u(s),u(a_1(s)),u'(s),u'(a_2(s)))ds. \end{aligned}$$

Since

$$C(t)x_0 + S(t)\Big(x_1 - \sum_{i=1}^{p} h_i u(t_i)\Big)$$

is of class C^2 in $[0,T]$, there are constants $C_1 > 0$ and $C_2 > 0$ such that

$$\|(C(t+h)-C(t))x_0 + (S(t+h)-S(t))\Big(x_1 - \sum_{i=1}^{p} h_i u(t_i))\Big)\| \le C_1 \mid h \mid$$

and

$$\|((C(t+h)-C(t))x_0)' + ((S(t+h)-S(t))\Big(x_1 - \sum_{i=1}^{p} h_i u(t_i))\Big)'\| \le C_2 \mid h \mid .$$

Hence

$$\begin{aligned} & \|u(t+h)-u(t)\| \\ & \le C_1 \mid h \mid + \| \int_0^t S(s)(f(t+h-s, u(t+h-s), \\ & \qquad\qquad u(a_1(t+h-s), u'(t+h-s), u'(a_2(t+h-s))) \\ & \quad - f(t-s, u(t-s), u(a_1(t-s)), u'(t-s), u'(a_2(t-s))))ds\| \\ & \quad + \| \int_t^{t+h} S(s)f(t+h-s, u(t+h-s), u(a_1(t+h-s)), \\ & \qquad\qquad u'(t+h-s), u'(a_2(t+h-s)))ds\| \\ & \le C_1 \mid h \mid + \int_0^t Me^{\omega T} L_2(\mid h \mid + \|u(t+h-s)-u(t-s)\| \\ & \quad + \|u(a_1(t+h-s)) - u(a_1(t-s))\| + \|u'(t+h-s)-u'(t-s)\| \\ & \quad + \|u'(a_2(t+h-s) - u'(a_2(t-s))\| + Me^{\omega T} N \mid h \mid, \end{aligned}$$

where

$$N := \sup\{\|f(s,u(s),u(a_1(s)),u'(s),u'(a_2(s)))\| : \quad s \in [0,T]\}.$$

From this, we obtain

$$\begin{aligned} \|u(t+h)-u(t)\| \le C_3 \mid h \mid + C_4 \int_0^t (\|u(s+h)-u(s)\| \\ + \|u'(s+h)-u'(s)\|)ds. \end{aligned} \tag{3.1}$$

Moreover we have

$$\begin{aligned} u'(t) = (C(t)x_0 + S(t)\Big(x_1 - \sum_{i=1}^{p} h_i u(t_i))\Big)' \\ + \int_0^t C(t-s) f(s,u(s),u(a_1(s)),u'(s),u'(a_2(s)))ds. \end{aligned}$$

From the above formula, we obtain analogously

$$\begin{aligned} \|u'(t+h)-u'(t)\| \le C_5 \mid h \mid + C_6 \int_0^t (\|u(s+h)-u(s)\| \\ + \|u'(s+h)-u'(s)\|)ds. \end{aligned} \tag{3.2}$$

By inequalities (3.1) and (3.2), we get

$$\begin{aligned} &\|u(t+h)-u(t)\| + \|u'(t+h)-u'(t)\| \\ &\le C_* \mid h \mid + C_{**} \int_0^t (\|u(s+h)-u(s)\| + \|u'(s+h)-u'(s)\|)ds. \end{aligned}$$

From Gronwall's inequality, we have

$$\|u(t+h)-u(t)\| + \|u'(t+h)-u'(t)\| \le \tilde{C} \mid h \mid, \tag{3.3}$$

where $\tilde{C}$ is a positive constant.

By (3.3), it follows that $u, u(a_1(\cdot)), u'$ and $u'(a_2(\cdot))$ satisfy the Lipschitz condition on $[0,T]$ with a positive constant. This implies that the mapping

$$[0,T] \ni t \to f(t,u(t),u(a_1(t)),u'(t),u'(a_2(t))) \in X$$

also satisfies the Lipschitz condition.

The above property of f together with the assumptions of Theorem 3.1 imply, by Theorem 1.1 in [10] and by Theorem (2.1), that the linear Cauchy problem

$$\begin{aligned} v''(t) &= Av(t) + f(t, u(t), u(a_1(t)), u'(t), u'(a_2(t))), \quad t \in [0, T], \\ v(0) &= x_0, \\ v'(0) &= x_1 - \sum_{i=1}^{p} h_i u(t_i) \end{aligned}$$

has a unique classical solution v such that

$$\begin{aligned} v(t) = C(t)x_0 + S(t)\Big(x_1 - \sum_{i=1}^{p} h_i u(t_i)\Big) \\ + \int_0^t S(t-s) f(s, u(s), u(a_1(s)), u'(s)), u'(a_2(s)))ds, \quad t \in [0, T]. \end{aligned}$$

Consequently u is the unique classical solution of the semilinear Cauchy problem (1.1) - (1.3) and, therefore, the proof of Theorem 3.1 is complete. □

4. Abstract fractional differential equations

Fractional differential equations appear more frequently in different areas of science and engineering. In fact, real world processes generally or most likely result in fractional order systems. The main reason for using the integer order models was the absence of solution methods for fractional differential equations. The most important advantage of using fractional differential equations is their non-local property. It is well known that the integer order differential operator is a local operator but the fractional order differential operator is nonlocal. This means that the next state of a system depends not only upon its current state but also upon all its past states.

Many real world systems are better characterized by using a non-integer order dynamic model based on fractional calculus. Recently, due to increasing applications of fractional calculus, several papers on the existence of solutions of fractional differential equations have appeared.

In this section, we discuss the existence of solutions of nonlinear fractional order differential equations in Banach spaces.

Let X be a Banach space and $J = [0, T]$. Then $Y = C(J, X)$ is the Banach space of all real-valued continuous functions defined on the compact interval J, endowed with the maximum norm. The space of linear bounded operators on X is denoted by $\mathfrak{L}(X)$. We use the symbol I to denote the identity operator.

The fractional integral of a function $f \in Y$ is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s) ds,$$

for any order $n-1 < \alpha \leq n$, $n \in \mathbb{N}$ and the Caputo derivative of f is ${}^{C}\!D^{\alpha} f = I^{n-\alpha} f^{(n)}$, $f^{(n)} \in Y$.

The Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \alpha > 0, \beta > 0.$$

and, when $\beta = 1$, we denote $E_{\alpha,1}(z) = E_{\alpha}(z)$. The reader may refer the book [23] for more information about the facts on fractional calculus.

Lemma 4.1. ([23]) *Let $\alpha > 0$, $t \in J$, $x \in Y$. Then*

$$I^{\alpha} D^{\alpha} x(t) = x(t) + \sum_{k=0}^{n-1} c_k t^k, \ c_k \in \mathbb{R}.$$

Lemma 4.2. ([25, Theorem 7.3.1]) *Suppose that A is a linear bounded operator defined on a Banach space X and assume that $\|A\| < 1$. Then $(I-A)^{-1}$ is linear and bounded. Also*

$$(I-A)^{-1} = \sum_{k=0}^{\infty} A^k,$$

the convergence of the series being in the operator norm and

$$\left\|(I-A)^{-1}\right\| \leq (1 - \|A\|)^{-1}.$$

Using Lemmas 4.1 and 4.2, the following lemmas have been established for the solutions representation of some linear fractional differential equations in [5].

Lemma 4.3. *The fractional differential equations*

$$\begin{cases} {}^{C}\!D^{\alpha} u(t) = Au(t) + f(t), \quad 0 < \alpha \leq 1, \\ u(0) = u_0 \end{cases}$$

has a solution

$$u(t) = E_{\alpha}(At^{\alpha}) u_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^{\alpha}) f(s) ds,$$

provided
(H) The operator $A \in \mathfrak{L}(X)$ commutes with the fractional integral operator I^{α}

on X *and* $\|A\| \leq \frac{\Gamma(\alpha+1)}{T^{\alpha}}$.

Next we introduce a linear differential equation with fractional order $1 < \alpha \leq 2$.

Lemma 4.4. *Let the condition* (H) *hold. Then the fractional differential equation*

$$\begin{cases} {}^{C}\!D^{\alpha}u(t) = Au(t) + f(t), & 1 < \alpha \leq 2, \\ u(0) = u_0 \in X, & u'(0) = v_0 \in X, \end{cases}$$

has a solution

$$\begin{aligned} u(t) &= E_{\alpha}(At^{\alpha})u_0 + tE_{\alpha,2}(At^{\alpha})v_0 + f(t) * t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha}) \\ &= \Phi_0(t)u_0 + \Phi_1(t)v_0 + \int_0^t \Phi(t-s)f(s)\mathrm{d}s, \end{aligned}$$

where $\Phi_0(t) = E_{\alpha}(At^{\alpha})$, $\Phi_1(t) = tE_{\alpha,2}(At^{\alpha})$ *and* $\Phi(t) = t^{\alpha-1}E_{\alpha,\alpha}(At^{\alpha})$.

In general, if the hypothesis (H) is satisfied, then the fractional differential equations

$$\begin{cases} {}^{C}\!D^{\alpha}u(t) = Au(t) + f(t), & n-1 < \alpha \leq n, \\ u^{(k)}(0) = v_k \in X, & k = 0, 1, ..., n-1 \end{cases}$$

has a solution of the form

$$u(t) = \sum_{k=0}^{n-1} t^k E_{\alpha,k+1}(At^{\alpha})v_k + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^{\alpha})f(s)ds.$$

Consider the nonlinear fractional differential equation of the form

$$\begin{cases} {}^{C}D^{\alpha}u(t) = Au(t) + f(t, u(t), {}^{C}D^{\beta}u(t)), \ t \in J, \\ u(0) = u_0, \ u'(0) = v_0, \end{cases} \tag{4.1}$$

with $1 < \alpha \leq 2$, $0 < \beta \leq 1$, A is a bounded linear operator and the nonlinear function $f : J \times X \times X \to X$ is continuous. The solution of (4.1) is given by

$$u(t) = \Phi_0(t)\, u_0 + \Phi_1(t)\, v_0 + \int_0^t \Phi(t-s) f(s, u(s), {}^{C}D^{\beta}u(s))\mathrm{d}s.$$

For brevity let us take

$$\begin{aligned} n_1 &= \sup\{\|\Phi_0(t)\|, t \in J\}; & n_2 &= \sup\{\|\Phi_1(t)\|, t \in J\}; \\ n_3 &= \sup\{\|\Phi(t-s)\|, t, s \in J\}; & n_4 &= \sup\{\|A\Phi(t)\|, t \in J\}; \\ n_5 &= \sup\{\|\Phi_2(t-s)\|, t, s \in J\}; & \Phi_2(t) &= t^{\alpha-1}E_{\alpha,\alpha-1}(At^{\alpha}); \\ n_6 &= n_4\|u_0\| + n_1\|v_0\|; & c &= n_1\|u_0\| + n_2\|v_0\|. \end{aligned}$$

Now we make the following assumptions to obtain the existence results for the equation:

(H1) For each $t \in J$, the function $f(t,\ \cdot,\ \cdot) : X \times X \to X$ is continuous and the function $f(\cdot,\ u,\ v) : J \to X$ is strongly measurable for each $u, v \in X$.

(H2) For every positive constant k, there exists $h_k \in L^1(J)$ such that

$$\sup_{\|u\|,\ \|v\| \le k} \|f(t,u,v)\| \le h_k(t), \quad \text{for every } t \in J.$$

(H3) There exists a continuous function $m_1 : J \to [0,\infty)$ such that

$$\|f(t,u,v)\| \le m_1(t)\Omega\left(\|u\| + \|v\|\right),\ t \in J,\ u,v \in X$$

where $\Omega : (0,\infty) \to (0,\infty)$ is a continuous nondecreasing function.

(H4) There exists a constant $M > 0$ and a continuous function $m_2 : J \to [0,\infty)$ such that

$$\frac{n_6 t^{-\beta}}{\Gamma(1-\beta)} + \frac{n_5}{\Gamma(1-\beta)} \int_0^t (t-\tau)^{-\beta} m_1(\tau)\Omega(w(\tau))\mathrm{d}\tau \le M m_2(t)\Omega(w(t))$$

and

$$\int_0^T m(s)\mathrm{d}s < \int_c^\infty \frac{\mathrm{d}s}{\Omega(s)}.$$

where $m(t)$=max$\{n_3 m_1(t), M m_2(t)\}$.

Theorem 4.5. *Assume that the hypotheses* $(H1) - (H4)$ *hold. Then there exists a solution to the nonlinear equation* (4.1) *on* J.

Proof. Consider the Banach space $Z = \left\{u : u \in C(J,X) \text{ and } {}^C\!D^\beta u \in C(J,X)\right\}$ with norm $\|u\|^* = \max\{\|u\|, \|{}^C\!D^\beta u\|\}$. We now show that the nonlinear operator $F : Z \to Z$ defined by

$$(Fu)(t) = \Phi_0(t)\, u_0 + \Phi_1(t)\, v_0 + \int_0^t \Phi(t-s) f(s, u(s), {}^C\!D^\beta u(s))\mathrm{d}s$$

has a fixed point. This fixed point is then a solution to (4.1).
The first step is to obtain a priori bound of the set

$$\zeta(F) = \{u \in Z : u = \lambda F u \quad \text{for some} \quad \lambda \in (0,1)\}.$$

Let $u \in \zeta(F)$. Then $u = \lambda F u$ for some $0 < \lambda < 1$. Thus, for each $t \in J$, we have

$$u(t) \;=\; \lambda\Phi_0(t)\, u_0 + \lambda\Phi_1(t)\, v_0 + \lambda \int_0^t \Phi(t-s) f(s, u(s), {}^C\!D^\beta u(s))\mathrm{d}s.$$

Then

$$\|u(t)\| \leq n_1\|u_0\| + n_2\|v_0\| + n_3\int_0^t m_1(s)\Omega(\|u(s)\| + \|{}^C\!D^\beta u(s)\|)\mathrm{d}s$$

$$\equiv c + n_3\int_0^t m_1(s)\Omega(\|u(s)\| + \|{}^C\!D^\beta u(s)\|)\mathrm{d}s.$$

Denoting the right-hand side of the above inequality by $r_1(t)$, we have $r_1(0) = c$,

$$\|u(t)\| \leq r_1(t)$$

and

$$r_1'(t) = n_3 m_1(t)\Omega(\|u(t)\| + \|{}^C\!D^\beta u(t)\|).$$

Also, we have

$$u'(t) = \lambda A\Phi(t)\,u_0 + \lambda\Phi_0(t)\,v_0 + \lambda\int_0^t \Phi_2(t-s)f(s,u(s),{}^C\!D^\beta u(s))\mathrm{d}s.$$

and

$$\begin{aligned}\|u'(t)\| \quad &\leq \quad n_4\|u_0\| + n_1\|v_0\| + n_5\int_0^t m_1(s)\Omega(\|u(s)\| + \|{}^C\!D^\beta u(s)\|)\mathrm{d}s \\ &\equiv \; n_6 + n_5\int_0^t m_1(s)\Omega(\|u(s)\| + \|{}^C\!D^\beta u(s)\|)\mathrm{d}s.\end{aligned}$$

Hence it follows that

$$\begin{aligned}\|{}^C\!D^\beta u(t)\| &\leq \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\|u'(s)\|\mathrm{d}s \\ &\leq \frac{n_6}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\mathrm{d}s \\ &\quad + \frac{n_5}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\left(\int_0^s m_1(\tau)\Omega(\|u(\tau)\| + \|{}^C\!D^\beta u(\tau)\|)\mathrm{d}\tau\right)\mathrm{d}s \\ &\leq \frac{n_6}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\mathrm{d}s \\ &\quad + \frac{n_5}{\Gamma(1-\beta)}\int_0^t\int_\tau^t (t-s)^{-\beta}\mathrm{d}s\, m_1(\tau)\Omega(\|u(\tau)\| + \|{}^C\!D^\beta u(\tau)\|)\mathrm{d}\tau \\ &\leq \frac{n_6\ t^{1-\beta}}{\Gamma(2-\beta)} + \frac{n_5}{\Gamma(2-\beta)}\int_0^t (t-\tau)^{1-\beta}m_1(\tau)\Omega(\|u(\tau)\| + \|{}^C\!D^\beta u(\tau)\|)\mathrm{d}\tau.\end{aligned}$$

Denoting the right-hand side of the above inequality by $r_2(t)$, we have $r_2(0) = 0$,

$$\|{}^C\!D^\beta u(t)\| \leq r_2(t)$$

and

$$r_2'(t) = \frac{n_6\ t^{-\beta}}{\Gamma(1-\beta)} + \frac{n_5}{\Gamma(1-\beta)} \int_0^t (t-\tau)^{-\beta} m_1(\tau)\Omega(\|u(\tau)\| + \|{}^C\!D^\beta u(\tau)\|)\mathrm{d}\tau.$$

Let $w(t) = r_1(t) + r_2(t),\ t \in J$. Then $w(0) = r_1(0) + r_2(0) = c$ and

$$w'(t) = r_1'(t) + r_2'(t) \le m(t)\Omega(w(t))$$

which implies that for each $t \in J$,

$$\int_{w(0)}^{w(t)} \frac{\mathrm{d}s}{\Omega(s)} \le \int_0^T m(s)\mathrm{d}s < \int_c^\infty \frac{\mathrm{d}s}{\Omega(s)}.$$

From the above inequality, we see that there exists a constant K such that

$$w(t) = r_1(t) + r_2(t) \le K,\ t \in J.$$

Then $\|u(t)\| \le r_1(t)$ and $\|{}^C\!D^\beta u(t)\| \le r_2(t),\ t \in J$, and hence

$$\|u\|^* = \max\{\|u\|, \|{}^C\!D^\beta u\|\} \le K$$

and the set $\zeta(F)$ is bounded.

Next we prove that the operator $F : X \to X$ is completely continuous.

Let $B_q = \{u \in Z : \|u\|^* \le q\}$. We first show that F maps bounded sets into equicontinuous family in B_q. Let $u \in B_q$ and $t_1, t_2 \in J$. Then if $0 < t_1 < t_2 \le T$,

$$\begin{aligned}
\|(Fu)(t_2) - (Fu)(t_1)\| \le{}& \|\Phi_0(t_2) - \Phi_0(t_1)\|\|u_0\| + \|\Phi_1(t_2) - \Phi_1(t_1)\|\|y_0\| \\
&+ \left\| \int_0^{t_1} [\Phi(t_2 - s) - \Phi(t_1 - s)] f(s, u(s), {}^C\!D^\beta u(s))\mathrm{d}s \right\| \\
&+ \left\| \int_{t_1}^{t_2} \Phi(t_2 - s) f(s, u(s), {}^C\!D^\beta u(s))\mathrm{d}s \right\| \\
\le{}& \|\Phi_0(t_2) - \Phi_0(t_1)\|\|u_0\| + \|\Phi_1(t_2) - \Phi_1(t_1)\|\|y_0\| \\
&+ \int_0^{t_1} \|\Phi(t_2 - s) - \Phi(t_1 - s)\| h_q(s)\mathrm{d}s \\
&+ \int_{t_1}^{t_2} \|\Phi(t_2 - s)\| h_q(s)\mathrm{d}s \qquad (4.2)
\end{aligned}$$

and

$$\|(Fu)'(t)\| \leq \|A\Phi(t)\|\|u_0\| + \|\Phi_0(t)\|\|v_0\| + \int_0^t \|\Phi_2(t-s)\|h_q(s)\mathrm{d}s.$$
$$\leq n_4\|u_0\| + n_1\|v_0\| + n_5\int_0^t h_q(s)\mathrm{d}s$$
$$\leq n_6 + n_5\int_0^t h_q(s)\mathrm{d}s.$$

Hence it follows that

$$\begin{aligned}
&\|{}^C\!D^\beta(Fu)(t_2) \ - \ {}^C\!D^\beta(Fu)(t_1)\| \\
&= \Big\|\frac{1}{\Gamma(1-\beta)}\int_0^{t_2}(t_2-s)^{-\beta}(Fu)'(s)\mathrm{d}s \\
&\quad - \frac{1}{\Gamma(1-\beta)}\int_0^{t_1}(t_1-s)^{-\beta}(Fu)'(s)\mathrm{d}s\Big\| \\
&\leq \frac{1}{\Gamma(1-\beta)}\Big\|\int_{t_1}^{t_2}(t_2-s)^{-\beta}(Fu)'(s)\mathrm{d}s\Big\| \\
&\quad + \frac{1}{\Gamma(1-\beta)}\Big\|\int_0^{t_1}\Big((t_2-s)^{-\beta}-(t_1-s)^{-\beta}\Big)(Fu)'(s)\mathrm{d}s\Big\| \\
&\leq \frac{1}{\Gamma(1-\beta)}\int_{t_1}^{t_2}(t_2-s)^{-\beta}\|(Fu)'(s)\|\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(1-\beta)}\int_0^{t_1}\Big((t_2-s)^{-\beta}-(t_1-s)^{-\beta}\Big)\|(Fu)'(s)\|\mathrm{d}s \\
&\leq \frac{n_6}{\Gamma(1-\beta)}\int_{t_1}^{t_2}(t_2-s)^{-\beta}\mathrm{d}s + \frac{n_5}{\Gamma(1-\beta)}\int_{t_1}^{t_2}(t_2-s)^{-\beta}\left(\int_0^s h_q(\tau)\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad + \frac{n_6}{\Gamma(1-\beta)}\int_0^{t_1}\Big((t_2-s)^{-\beta}-(t_1-s)^{-\beta}\Big)\mathrm{d}s \\
&\quad + \frac{n_5}{\Gamma(1-\beta)}\int_0^{t_1}\Big((t_2-s)^{-\beta}-(t_1-s)^{-\beta}\Big)\left(\int_0^s h_q(\tau)\mathrm{d}\tau\right)\mathrm{d}s \\
&\leq \frac{n_6}{\Gamma(2-\beta)}(t_2^{1-\beta}-t_1^{1-\beta}) + \frac{1}{\Gamma(1-\beta)}\int_{t_1}^{t_2}(t_2-s)^{-\beta}\left(\int_0^s h_q(\tau)\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(2-\beta)}\int_0^{t_1}\Big((t_2-\tau)^{1-\beta}-(t_2-t_1)^{1-\beta}-(t_1-\tau)^{1-\beta}\Big)h_q(\tau)\mathrm{d}\tau. \quad (4.3)
\end{aligned}$$

The right-hand sides of (4.2) and (4.3) tend to zero as $t_2 \to t_1$. Thus F maps B_q into an equicontinuous family of functions. It is easy to see that the family FB_q is uniformly bounded.

Next we show that F is a compact operator. It suffices to show that the closure of FB_q is compact.

Let $0 \le t \le T$ be fixed and ϵ be a real number satisfying $0 < \epsilon < t$. For $u \in B_q$, we define

$$(F_\epsilon u)(t) = \Phi_0(t)u_0 + \Phi_1(t)v_0 + \int_0^{t-\epsilon} \Phi(t-s)f(s, u(s), {}^C D^\beta u(s))\mathrm{d}s.$$

Note that using the same methods as in the procedure above, we obtain the boundedness and equicontinuous property of F_ϵ which implies that the set $S_\epsilon(t) = \{(F_\epsilon u)(t) : \ u \in B_q\}$ is relatively compact in X for every $0 < \epsilon < t$.

Moreover, for every $u \in B_q$,

$$\begin{aligned}\|(Fu)(t) - (F_\epsilon u)(t)\| &\le \left\| \int_{t-\epsilon}^t \Phi(t-s)f(s, u(s), {}^C D^\beta u(s))\mathrm{d}s \right\| \\ &\le \int_{t-\epsilon}^t \|\Phi(t-s)\| h_q(s)\mathrm{d}s.\end{aligned}$$

Also

$$\begin{aligned}\|(Fu)'(t) - (F_\epsilon u)'(t)\| &\le \left\| \int_{t-\epsilon}^t \Phi_2(t-s)f(s, u(s), {}^C D^\beta u(s))\mathrm{d}s \right\| \\ &\le \int_{t-\epsilon}^t \|\Phi_2(t-s)\| h_q(s)\mathrm{d}s.\end{aligned}$$

Since $\|(Fu)(t) - (F_\epsilon u)(t)\| \to 0$ and $\|(Fu)'(t) - (F_\epsilon u)'(t)\| \to 0$ as $\epsilon \to 0$, this implies that

$$\begin{aligned}&\|{}^C D^\beta (Fu)(t) - {}^C D^\beta (F_\epsilon u)(t)\| \\ &\le \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \|(Fu)'(t) - (F_\epsilon u)'(t))\|\mathrm{d}s \quad \to 0 \text{ as } \epsilon \to 0.\end{aligned}$$

So relatively compact sets $S_\epsilon(t) = \{(F_\epsilon u)(t) : \ u \in B_q\}$ are arbitrarily close to the set $\{(Fu)(t) : \ u \in B_q\}$. Hence $\{(Fu)(t) : \ u \in B_q\}$ is compact in Z by the Arzela-Ascoli theorem.

Next it remains to show that F is continuous. Let $\{u_n\}$ be a sequence in Z such that $\|u_n - u\| \to 0$ as $n \to \infty$. Then there is an integer k such that $\|u_n\| \le k$, $\|{}^C D^\beta u_n\| \le k$ for all n and $t \in J$. So $\|u(t)\| \le k$, $\|{}^C D^\beta u(t)\| \le k$ and u, ${}^C D^\beta u \in Z$. By $(H1)$,

$$f(t, u_n(t), {}^C D^\beta u_n(t)) \to f(t, u(t), {}^C D^\beta u(t)),$$

for each $t \in J$. Since

$$\|f(t, u_n(t), {}^C D^\beta u_n(t)) - f(t, u(t), {}^C D^\beta u(t))\| \le 2h_k(t),$$

we have, by the dominated convergence theorem,

$$\begin{aligned}
&\|(Fu_n)(t)–(Fu)(t)\| \\
&= \sup_{t\in J}\left\| \int_0^t \Phi(t-s)\left[f(s,u_n(s),{}^C\!D^\beta u_n(s)) - f(s,u(s),{}^C\!D^\beta u(s))\right] \mathrm{d}s\right\| \\
&\leq \int_0^T \left\|\Phi(t-s)\left[f(s,u_n(s),{}^C\!D^\beta u_n(s)) - f(s,u(s),{}^C\!D^\beta u(s))\right]\right\| \mathrm{d}s.
\end{aligned}$$

Also

$$\begin{aligned}
&\|(Fu_n)'(t)–(Fu)'(t)\| \\
&= \sup_{t\in J}\left\| \int_0^t \Phi_2(t-s)\left[f(s,u_n(s),{}^C\!D^\beta u_n(s)) - f(s,u(s),{}^C\!D^\beta u(s))\right] \mathrm{d}s\right\| \\
&\leq \int_0^T \left\|\Phi_2(t-s)\left[f(s,u_n(s),{}^C\!D^\beta u_n(s)) - f(s,u(s),{}^C\!D^\beta u(s))\right]\right\| \mathrm{d}s.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\|{}^C\!D^\beta(Fu_n)(t) - {}^C\!D^\beta(Fu)(t)\| \\
&\leq \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\|(Fu_n)'(t)-(Fu)'(t)\|\mathrm{d}s \quad \to 0 \text{ as } n\to\infty.
\end{aligned}$$

Thus F is continuous. Finally the set $\zeta(F) = \{u \in Z: \ u = \lambda F u, \ \lambda \in (0,1)\}$ is bounded as shown in the first step. By Schaefer's fixed point theorem, the operator F has a fixed point in Z. This fixed point is then the solution of (4.1). This completes the proof. □

5. Examples

Example 5.1. Consider the fractional differential equations

$$\begin{cases} {}^C\!D^{3/2}u_1(t) = 4u_2(t) + \dfrac{u_1}{u_1^2+\sin t}\ , \\[2ex] {}^C\!D^{3/2}u_2(t) = 3u_1(t) + \dfrac{u_1}{u_1^2+t}\ , \end{cases} \tag{5.1}$$

with initial conditions $\begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} u_1'(0) \\ u_2'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for $t \in [0,1]$.

It has the following form

$$\begin{cases} {}^C\!D^{3/2}u(t) = Au(t) + f(t,u),\ t\in[0,1], \\[2ex] \qquad u(0) = u_0,\ u'(0) = v_0, \end{cases} \tag{5.2}$$

where $A=\begin{bmatrix} 0 & 4 \\ 3 & 0 \end{bmatrix}$, $f(t,u)=\begin{bmatrix} \dfrac{u_1}{u_1^2+\sin t} \\ \\ \dfrac{u_2}{u_2^2+t} \end{bmatrix}$, $u(t)=\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$.

Using Mittag-Leffler matrix function for a given matrix A, we get

$$\Phi(1-s)=\begin{bmatrix} L_1(s) & L_2(s) \\ L_3(s) & L_4(s) \end{bmatrix},$$

where

$$\begin{aligned} L_1(s) &= (t-s)^{\frac{1}{2}} E_{3,\frac{3}{2}}(12(t-s)^3), \\ L_2(s) &= 4e^{12(t-s)}, \\ L_3(s) &= 3e^{12(t-s)}, \\ L_4(s) &= (t-s)^{\frac{1}{2}} E_{3,\frac{3}{2}}(12(t-s)^3). \end{aligned}$$

Further the nonlinear function f is bounded, continuous and satisfies conditions of Theorem 4.5. Hence there exist a solution to the nonlinear equation (5.1).

Example 5.2. Consider the system of fractional differential equation of the form

$$\begin{cases} {}^C\!D^{5/4}u_1(t) = u_1(t) - u_2(t) + \dfrac{\exp(-2t)\left(|u_1| + |{}^C\!D^{1/2}u_1(t)|\right)}{1+|u_2(t)|}, \\ \\ {}^C\!D^{5/4}u_2(t) = u_2(t) + \dfrac{\exp(-2t)\left(|u_2| + |{}^C\!D^{1/2}u_2(t)|\right)}{1+|u_1(t)|}, \end{cases} \tag{5.3}$$

with initial conditions $\begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} u_1'(0) \\ u_2'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for $t \in [0,5]$.

It has the following form

$$\begin{cases} {}^C\!D^{5/4}u(t) = Au(t) + f(t,u(t),{}^CD^{1/2}u(t)),\ t\in[0,5], \\ \\ \qquad u(0)=u_0,\ u'(0)=v_0 \end{cases} \tag{5.4}$$

where $A=\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $u(t)=\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$, and

$$f(t,u(t),{}^CD^{1/2}u(t)) = \begin{bmatrix} \dfrac{\exp(-2t)\left(|u_1| + |{}^C\!D^{1/2}u_1(t)|\right)}{1+|u_2(t)|} \\ \dfrac{\exp(-2t)\left(|u_2| + |{}^C\!D^{1/2}u_2(t)|\right)}{1+|u_1(t)|} \end{bmatrix}.$$

Using Mittag-Leffler matrix function for a given matrix A, we get

$$\Phi(5-t) = \begin{bmatrix} N(t) & 0 \\ 0 & N(t) \end{bmatrix},$$

where $N(t) = (5-t)^{1/4} E_{5/4,5/4}((5-t)^{5/4})$. Further the nonlinear function f is continuous and satisfies the hypotheses of Theorem 4.5. Hence the equation (5.3) has a solution on $[0, 3]$.

References

[1] K. Balachandran and S.M. Anthoni,*Existence of solutions of second order neutral functional differential equations,* Tamkang J. of Math., **30** (1999), 299-309.

[2] K. Balachandran and S. Ilamaran, *Existence and uniqueness of mild and strong solutions of a semilonear evolution equation with nonlocal conditions,* Indian J. Pure and Appl. Math., **25** (1994), 411-418.

[3] K. Balachandran and J.H. Kim, *Nonlocal Cauchy problem for second order integrodifferential evolution equations in Banach spaces,* Taiwanese J. of Math., **11** (2007), 1343-1357.

[4] K. Balachandran and J.Y. Park, *Existence of solutions of second order nonlinear differential equations with nonlocal conditions in Banach Spaces,* Indian J. Pure and Appl. Math., **32** (2001), 1883-1891.

[5] K. Balachandran, M.M. Matar and J.J.Trujillo, *Note on controllability of linear fractional dynamical systems,* J. of Control and Decis., **3** (2016), 267-279.

[6] J. Bochenek, *An abstract nonlinear second order differential equation,* Ann. Polonici Math., **54** (1991), 155-166.

[7] J. Bochenek, *Second order semilinear Volterra integrodifferential equation in Banach space,* Ann. Polonici Math., **57** (1992), 231-241.

[8] J. Bochenek, *Cosine operator functions and condition* (F)*,* Selected Problems of Mathematics, Cracow University of Technology Monographs, Anniversary Issue, **6** (1995), 13-24.

[9] J. Bochenek and T. Winiarska, *Second order evolution equations with parameter,* Ann. Polonici Math., **59** (1994), 41-52.

[10] L. Byszewski, *On some application of the Bochenek theorem,* Universitatis Iagellonicae Acta Mathematica, **45** (2007), 147-153.

[11] L. Byszewski, *An abstract nonlocal functional-differential second order evolution problem,* Technical Trans. Math., **1** (2017), 147-154.

[12] L. Byszewski and T. Winiarska, *An abstract nonlocal second order evolution problem,* Opuscula Mathematica, **32** (2012), 75-82.

[13] H.O. Fattorini, *Ordinary differential equation in linear topological space,* J. of Differ. Equ., **5** (1968), 72-105.

[14] H.O. Fattorini, *Second Order Linear Differential Equations in Banach Spaces,* North-Holland, Amsterdam, 1985.

[15] E. Hernandez, *A second order impulsive Cauchy problem,* Cadernos de Matematica, **2** (2001), 229-238.

[16] E. Hernandez, *Existence results for a partial second order functional differential equation with impulses,* Dyna. of Continuous, Discrete and Impu. Syst., **14** (2007), 229-250.

[17] E. Hernandez, *Existence of solutions for a second order abstract functional differential equation with state-dependent delay,* Electronic J. of Differ. Equ., **21** (2007), 1-10.

[18] E. Hernandez, *Existence of solutions for an abstract second order differential equation with nonlocal conditions,* Electronic J. of Differ. Equ., **96** (2009), 1-10.

[19] E. Hernandez, K. Balachandran and N. Annapoorani, *Existence results for a damped second order abstract functional differential equation with impulses,* Math. and Comput. Model., **50** (2009), 1583-1594.

[20] E. Hernandez and H.R. Henriquez, *Existence results for second-order differential equations with nonlocal conditions in Banach spaces*, Funkcialaj Ekvacioj, **52** (2009), 113-137.

[21] E. Hernandez, H.R. Henriquez and M.A. McKibben, *Existence results for abstract impulsive second order neutral functional differential equations,* Nonlinear Anal. TMA., **70** (2009), 2736-2751.

[22] H.R. Henriquez, *Existence of solutions of non-autonomous second order functional differential equations with infinite delay,* Nonlinear Anal. TMA., **74** (2011), 3333-3352.

[23] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam 2006.

[24] J. Kisynski, *On cosine operator functions and one parameter group of operators,* Studia Math., **49** (1972), 93-105.

[25] E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley & Sons Inc., New York, 1978.

[26] B. Nagy, *Cosine operator functions and the abstract Cauchy problem,* Periodica Math. Hung., **7** (1976), 213-217.

[27] S.K. Ntouyas and P.C. Tsamatos, *Global existence for second order semilinear ordinary and delay integrodifferential equations with nonlocal conditions,* Appl. Anal., **67** (1997), 245-257.

[28] J.T. Sandefur, *Existence and uniqueness of solutions of second order nonlinear differential equations,* SIAM J. Math. Anal., **14** (1983), 477-487.

[29] C.C. Travis and G.F. Webb, *Compactness, regularity and uniform continuity properties of strongly continuous cosine families,* Houston J. of Math., **3** (1977), 555-567.

[30] C.C. Travis and G.F. Webb, *Cosine family and abstract nonlinear second order differential equations,* Acta Math. Hung., **32** (1978), 75-96.