Nonlinear Functional Analysis and Applications
Vol. 24, No. 3 (2019), pp. 477-483
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

SOME NEW FIXED POINT THEOREMS FOR GENERALIZED CONTRACTIONS INVOLVING RATIONAL EXPRESSIONS IN COMPLEX VALUED b-METRIC SPACES

Anil Kumar Dubey[1], Urmila Mishra[2] and Won Hee Lim[3]

[1]Department of Mathematics
Bhilai Institute of Technology, Bhilai House
Durg (Chhattisgarh), India
e-mail: anilkumardbye@rediffmail.com

[2]Department of Mathematics
Vishwavidyalaya Engineering College
Lakhanpur-Ambikapur(Chhattisgarh), India
e-mail: mishra.urmila@gmail.com

[3]Department of Mathematics
Kyungnam University, Changwon, Gyeongnam, 51767, Korea
e-mail: worry36@kyungnam.ac.kr

Abstract. In this paper, we present some fixed point theorems for generalized contractions involving rational expressions in the framework of complex valued b-metric spaces. Our results generalize some existing results.

1. Introduction

In 2011, Azam et al. [2] introduced the concept of complex valued metric space and proved some fixed point results for mappings satisfying a rational inequality. This concept is useful in different branches of mathematics, including applied mathematics, number theory, algebraic geometry etc. Large

[0]Received December 3, 2018. Revised March 15, 2019.
[0]2010 Mathematics Subject Classification: 47H10, 54H25.
[0]Keywords: Partially ordered, complex valued b-metric spaces, fixed point, Cauchy sequence.
[0]Corresponding author: U. Mishra(mishra.urmila@gmail.com).

number of papers have been published containing fixed point results for a single and a pair of self-mappings with different rational contraction conditions in complex valued metric space (see [9, 11, 13, 14]). After the establishment of complex valued metric spaces, Rao et al. [10] introduced the complex valued b-metric spaces and then several authors have contributed in this directions (see [1, 3, 4, 5, 6, 7, 8] and [12]).

In this manuscript,we prove some fixed point theorems having rational type contraction conditions in the notion of partially ordered complex valued b-metric space.

2. Basic facts and definitions

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows: $z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2)$. It follows that $\precsim$ exist if one of the following conditions are satisfied:

(C_1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$;
(C_2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$;
(C_3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$;
(C_4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of $(C_2), (C_3)$ and (C_4) is satisfied and we will write $z_1 \prec z_2$ if only (C_4) is satisfied. Note that

(i) $0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|$,
(ii) $z_1 \precsim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$,
(iii) If $a, b \in \mathbb{R}$ and $a \leq b \Rightarrow az \precsim bz$ for all $z \in \mathbb{C}$.

The following definition is recently introduced by Rao et al. [10].

Definition 2.1. ([10]) Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{C}$ is called complex valued b-metric if the following conditions are satisfied:

($M1$) $0 \precsim d(x, y)$ and $d(x, y) = 0 \Leftrightarrow x = y$ for all $x, y \in X$;
($M2$) $d(x, y) = d(y, x)$ for all $x, y \in X$;
($M3$) $d(x, y) \precsim s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then the pair (X, d) is called a complex valued b-metric space.

Example 2.2. ([10]) Let $X = [0, 1]$. Define the mapping $d : X \times X \to \mathbb{C}$ by $d(x, y) = |x - y|^2 + i|x - y|^2$, for all $x, y \in X$. Then (X, d) is a complex valued b-metric space with $s = 2$.

All other definitions like convergent sequence, Cauchy sequence, complete complex valued b-metric space we refer [10].

Lemma 2.3. ([10]) *Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 2.4. ([10]) *Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.*

Definition 2.5. Let (X, d) be complex valued b-metric space, $T : X \to X$ and $x \in X$. Then the function T is continuous at x if for any sequence $\{x_n\}$ in X with $x_n \to x$, $Tx_n \to Tx$.

Definition 2.6. Let $(X, \precsim)$ be a partially ordered set and $T : X \to X$. The mapping T is said to be non-decreasing if for all $x_1, x_2 \in X, x_1 \precsim x_2$ implies $Tx_1 \precsim Tx_2$ and non-increasing if for all $x_1, x_2 \in X, x_1 \precsim x_2$ implies $Tx_1 \succsim Tx_2$.

Now we are ready to state and prove our main result.

3. Main results

Theorem 3.1. *Let $(X, \precsim)$ be a partially ordered set and suppose that there exist a complex valued b-metric d on X such that (X, d) is a complete complex valued b-metric space. Let the mapping $T : X \to X$ be a continuous and non-decreasing mapping. Suppose that there exist non-negative real numbers $\alpha, \beta, \gamma, \delta, \lambda$ with $s(\alpha + \beta + \gamma) + \delta + s(s + 1)\lambda < 1$ such that for all $x, y \in X$ with $x \precsim y$,*

$$\begin{aligned} d(Tx, Ty) \quad &\precsim \quad \alpha d(x, y) + \beta\Big[\frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{d(x, Ty) + d(y, Tx)}\Big] \\ &\quad + \gamma d(x, Tx) + \delta d(y, Ty) + \lambda\Big[d(x, Ty) + d(y, Tx)\Big] \qquad (3.1) \end{aligned}$$

if there exist $x_0 \in X$ with $x_0 \precsim Tx_0$, then T has a fixed point.

Proof. Let $x_0 = Tx_0$. Then it is obvious that x_0 is a fixed point of T. Suppose that $x_0 \prec Tx_0$. Then we construct the sequence $\{x_n\}$ in X such that

$$x_{n+1} = Tx_n \quad \text{for} \quad \text{every} \quad n \geq 0. \qquad (3.2)$$

Since T is a non-decreasing mapping, by induction we get,

$$x_0 \prec Tx_0 = x_1 \precsim Tx_1 = x_2 \precsim Tx_2 = x_3 \precsim ... \precsim Tx_{n-1} = x_n \precsim Tx_n = x_{n+1}. \quad (3.3)$$

If there exist some $n \geq 1$ such that $x_{n+1} = x_n$, then from (3.2), $x_{n+1} = Tx_n = x_n$, that is x_n is a fixed point of T and the proof is finished.

Now, we assume that $x_{n+1} \neq x_n$ for all $n \geq 1$. Since $x_n \prec x_{n+1}$, all $n \geq 1$, by (3.1) we have,

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \\ &\precsim \alpha d(x_n, x_{n+1}) \\ &\quad + \beta \Big[\frac{d(x_n, Tx_n) d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_{n+1}) d(x_{n+1}, Tx_n)}{d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)} \Big] \\ &\quad + \gamma d(x_n, Tx_n) + \delta d(x_{n+1}, Tx_{n+1}) \\ &\quad + \lambda [d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)] \\ &= \alpha d(x_n, x_{n+1}) \\ &\quad + \beta \Big[\frac{d(x_n, x_{n+1}) d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+2}) d(x_{n+1}, x_{n+1})}{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})} \Big] \\ &\quad + \gamma d(x_n, x_{n+1}) + \delta d(x_{n+1}, x_{n+2}) \\ &\quad + \lambda [d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})] \\ &\precsim \alpha d(x_n, x_{n+1}) + \beta d(x_n, x_{n+1}) + \gamma d(x_n, x_{n+1}) + \delta d(x_{n+1}, x_{n+2}) \\ &\quad + s\lambda [d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \end{aligned}$$

which implies that,

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\precsim \Big(\frac{\alpha + \beta + \gamma + s\lambda}{1 - \delta - s\lambda} \Big) d(x_n, x_{n+1}) \\ &= \mu d(x_n, x_{n+1}) \end{aligned} \tag{3.4}$$

where $\mu = \left(\frac{\alpha+\beta+\gamma+s\lambda}{1-\delta-s\lambda} \right)$, since $s(\alpha + \beta + \gamma) + \delta + s(s+1)\lambda < 1$, it follows that $0 < \mu < \frac{1}{s}$. By induction, we have

$$d(x_{n+1}, x_{n+2}) \precsim \mu d(x_n, x_{n+1}) \precsim \mu^2 d(x_{n-1}, x_n) \precsim ... \precsim \mu^{n+1} d(x_0, x_1)$$

for $m > n$

$$\begin{aligned} d(x_n, x_m) &\precsim s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\ &\precsim sd(x_n, x_{n+1}) + s^2 [d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ &\precsim sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) \\ &\quad + ... + s^{m-n} d(x_{m-1}, x_m) \\ &\precsim (s\mu^n + s^2 \mu^{n+1} + ... + s^{m-n} \mu^{m-1}) d(x_0, x_1) \\ &\precsim s\mu^n [1 + (s\mu) + (s\mu)^2 + ... + (s\mu)^{m-n-1}] d(x_0, x_1) \\ &\precsim \frac{s\mu^n}{1 - s\mu} d(x_0, x_1). \end{aligned}$$

Since $0 \leq \mu < \frac{1}{s}$, we conclude that $\left(\frac{s\mu^n}{1-s\mu} \right) \to 0$ as $n \to \infty$, which implies that $\{x_n\}$ is a Cauchy sequence. From the completeness of X, there exists a point $z \in X$ such that

$$x_n \to z \quad \text{as} \quad n \to \infty. \tag{3.5}$$

The continuity of T implies that $Tz = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = z$ that is z is a fixed point of T. □

If continuity is dropped for underlying mapping then we get the following result:

Theorem 3.2. *Let $(X, \precsim)$ be a partially ordered set and suppose that there exist a complex valued b-metric d on X such that (X, d) is a complete complex valued b-metric space. Let the mapping $T : X \to X$ be a non-decreasing mapping. Assume that if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then $x_n \precsim x$, for all $n \in \mathbb{N}$. Suppose that (3.1) hold for all $x, y \in X$ with $x \precsim y$. If there exist $x_0 \in X$ with $x_0 \precsim Tx_0$, then T has a fixed point.*

Proof. We take the similar approach as Theorem 3.1 and prove that $\{x_n\}$ is non-decreasing sequence such that $x_n \to z \in X$. Then $x_n \precsim z$, for all $n \in \mathbb{N}$. From inequality (3.1), we have

$$\begin{aligned} d(x_{n+1}, Tz) &= d(Tx_n, Tz) \\ &\precsim \alpha d(x_n, z) + \beta\Big[\frac{d(x_n, Tx_n)d(x_n, Tz) + d(z, Tz)d(z, Tx_n)}{d(x_n, Tz) + d(z, Tx_n)}\Big] \\ &\quad +\gamma d(x_n, Tx_n) + \delta d(z, Tz) + \lambda[d(x_n, Tz) + d(z, Tx_n)] \\ &= \alpha d(x_n, z) + \beta\Big[\frac{d(x_n, x_{n+1})d(x_n, Tz) + d(z, Tz)d(z, x_{n+1})}{d(x_n, Tz) + d(z, x_{n+1})}\Big] \\ &\quad +\gamma d(x_n, x_{n+1}) + \delta d(z, Tz) + \lambda[d(x_n, Tz) + d(z, x_{n+1})]. \end{aligned}$$

Taking the limit as $n \to \infty$ and using (3.5), we have

$$\begin{aligned} d(z, Tz) &\precsim \alpha d(z, z) + \beta\Big[\frac{d(z, z)d(z, Tz) + d(z, Tz)d(z, z)}{d(z, Tz) + d(z, z)}\Big] \\ &\quad +\gamma d(z, z) + \delta d(z, Tz) + \lambda[d(z, Tz) + d(z, z)] \\ &\precsim \delta d(z, Tz) + \lambda d(z, Tz) = (\delta + \lambda)d(z, Tz). \end{aligned}$$

Since $(\delta + \lambda) < 1$, it is contradiction unless $d(z, Tz) = 0$. Therefore $Tz = z$ and hence z is fixed point of T. □

If we set $\beta = \gamma = \delta = 0$ in inequality (3.1) of Theorem 3.1 then we get following Corollary.

Corollary 3.3. *Let $(X, \precsim)$ be a partially ordered set and suppost that there exist a complex valued b-metric d on X such that (X, d) is a complete complex valued b-metric space. Let the mapping $T : X \to X$ be a continuous and non-decreasing mapping. Suppose there exist non-negative real numbers α and λ with $\alpha + 2s\lambda < \frac{1}{s}$ such that, for all $x, y \in X$ with $x \precsim y$,*

$$d(Tx, Ty) \precsim \alpha d(x, y) + \lambda[d(x, Ty) + d(y, Tx)].$$

If there exist $x_0 \in X$ with $x_0 \precsim Tx_0$, then T has a fixed point.

Example 3.4. Let $X = \{0, \frac{1}{2}, 2\}$ and partial order $\precsim$ is defined as $x \precsim y$ if and only if $x \geq y$. Let the complex valued b-metric d be given by $d(x,y) = |x-y|^2(1+i)$ for all $x, y \in X$. Let $s = 2$ and $T : X \to X$ be defined as below:

$$T(0) = 0 \quad \text{and} \quad T\Big(\frac{1}{2}\Big) = 0.$$

Take $x = \frac{1}{2}, y = 0, T(0) = 0$ and $T\Big(\frac{1}{2}\Big) = 0$ in Corollary 3.3, then we have

$$\begin{aligned} d(Tx,Ty) = 0 &\precsim \alpha\Big|\frac{1}{2} - 0\Big|^2(1+i) + \lambda\Big[\Big|\frac{1}{2} - 0\Big|^2 + |0-0|^2\Big](1+i) \\ &= (\alpha+\lambda)\frac{1+i}{4}. \end{aligned}$$

This implies that $\alpha + \lambda \geq 0$. If we take $\alpha = \lambda = \frac{1}{16}$, then all the conditions of Corollary 3.3 are satisfied and of course 0 is the fixed point of T.

In order to verify the Theorem 3.1, take the same values as above in Theorem 3.1, we have

$$d(Tx,Ty) = 0 \precsim \alpha\Big(\frac{1+i}{4}\Big) + \beta\Big(\frac{1+i}{4}\Big) + \gamma\Big(\frac{1+i}{4}\Big) + \delta(0) + \lambda\Big(\frac{1+i}{4}\Big).$$

The above inequality is satisfied for $\alpha = \beta = \delta = \frac{1}{6}$ and $\gamma = \lambda = 0$ with $s(\alpha+\beta+\gamma) + \delta + s(s+1)\lambda < 1$, then all the conditions of Theorem 3.1 are satisfied and of course 0 is the fixed point of T.

4. Conclusion

In this paper, we establish some fixed point theorems for generalized contractions involving rational expressions in the setting of complex valued b-metric spaces. We give one example in support of our results. Our results extend and generalize some well known existing results.

Acknowledgments: The authors are thankful to the learned referee for his/her deep observations and their suggestions, which greatly helped us to improve the paper significantly.

References

[1] A. Aiman Mukheimer, *Some common fixed point theorems in complex valued b-metric spaces,* The Sci. World Jour., **2014** (2014), Article ID 587825, 6 pages.

[2] A. Azam, B. Fisher and M. Khan, *Common fixed point theorems in Complex valued Metric Spaces,* Numer. Funct. Anal. Opti., **32**(3) (2011), 243–253.

[3] A.K. Dubey, *Complex valued b-metric spaces and common fixed point theorems under rational contractions,* J. Complex Anal., **2016** (2016), Article ID 9786063, 7 pages.

[4] A.K. Dubey, *Common fixed point results for contractive mappings in complex valued b-metric spaces,* Nonlinear Funct. Anal. Appl., **20**(2) (2015), 257–268.

[5] A.K. Dubey and M. Tripathi, *Common fixed point theorem in complex valued b-metric space for rational contractions,* J. Inf. Math. Sci., **7**(3) (2015), 149–161.

[6] A.K. Dubey, Manjula Tripathi and M.D. Pandey, *Common fixed point results for rational type contraction in complex valued b-metric spaces,* Int. J. Pure Appl. Math., **116**(2) (2017), 447–456.

[7] A.K. Dubey, U. Mishra and M. Tripathi, *Common fixed point of mappings satisfying rational inequality in complex valued b-metric spaces,* Comm. Math. Appl., **8**(3) (2017), 289–300.

[8] A.K. Dubey, M. Tripathi and R.P. Dubey, *Various fixed point theorems in complex valued b-metric spaces,* Int. Jour. Engg. Math., **2016** (2016), Article ID7072606, 7 pages.

[9] H.K. Nashine, M. Imdad and M. Hasan, *Common fixed point theorems under rational contractions in complex valued metric spaces,* J. Nonlinear Sci. Appl., **7** (2014), 42–50.

[10] K.P.R. Rao, P.R. Swamy and J.R. Prasad, *A common fixed point theorem in complex valued b-metric spaces,* Bull. Math. Stat. Research, **1**(1) (2013), 1–8.

[11] F. Rouzkard and M. Imdad, *Some common fixed point theorems on complex valued metric spaces,* Comput. Math.with Appl., **64** (2012), 1866–1874.

[12] D. Singh, O.P. Chauhan, N. Singh and V. Joshi, *Common fixed point theorems in complex valued b-metric spaces,* J. Math. Comput. Sci., **5**(3) (2015), 412–429.

[13] W. Sintunavarat and P. Kumam, *Generalized common fixed point theorems in complex valued metric spaces and applications,* J. Inequalities and Appl., **2012** (2012), doi: 10.1186/1029-242X-2012-84.

[14] W. Sintunavarat, Y.J. Cho and P. Kumam, *Ursyhon integral equations approach by common fixed point in complex valued metric spaces,* Adv. Differ. Equ., **2013:49** (2013), 1–14.