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REGULARIZATION FOR A SYSTEM OF INVERSE-STRONGLY MONOTONE OPERATOR EQUATIONS

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Abstract. In this paper, we introduce a regularization process of finding a common element of a system of operator equations for inverse-strongly monotone operators in real Banach spaces, and then give a convergence theorem. The convergence rates of regularized solutions are estimated by using a regularization parameter-choice that is based upon the generalized discrepancy principle. Further, we consider an iterative regularization method of zero order for solving system of inverse-strongly monotone operator equations in real Hilbert spaces.

1. Introduction

Let X be a real reflexive Banach space having property E-S (i.e. weak and norm convergences of any sequence in X imply its strong convergences). Let X and its dual space be strictly convex. For the sake of simplicity, the norms of X and X^* are denoted by the same symbol $\|.\|$. We write $\langle x^*, x\rangle$ instead of $x^*(x)$ for $x^* \in X^*$ and $x \in X$.

Let $A_j : X \to X^*$ be a family of hemicontinuous monotone operators defined on X, $f_j \in X^*$, $j = 1, ..., N$. Set $S_j = \{\bar{x} \in X : A_j(\bar{x}) = f_j\}$. It is easy to see that S_j is closed convex subset in X (see [10]). Assume that $S = \cap_{j=1}^N S_j \neq \emptyset$. We consider the following problem

$$\text{finding an element } x^0 \in S. \tag{1.1}$$

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Without additional conditions on A_j such as the strongly or uniformly monotone property, each equation $A_j(x) = f_j$ is an ill-posed problem. By this, we mean that the solution set S_j does not depend continuously on the data (A_j, f_j). Therefore, to find a solution of this equation, we have to use stable methods. One of these methods is the Tikhonov regularization in the form (see [1])

$$A_j^h(x) + \alpha U^s(x - x_*) = f_j^\delta, \tag{1.2}$$

where $\alpha > 0$ is a regularization parameter, U^s is the generalized duality mapping of X, A_j^h is a monotone bounded hemicontinuous operator on X, (A_j^h, f_j^δ) are approximation of (A_j, f_j) in the sense that

$$\|A_j^h(x) - A_j(x)\| \leq hg(\|x\|), \quad \|f_j^\delta - f_j\| \leq \delta \tag{1.3}$$

with levels $(h, \delta) \to 0$, $g(t)$ is a non-negative bounded function for $t \geq 0$ and x_* is in X which plays the role of a criterion of selection.

Let $\tau = (h, \delta)$. For each j, equation (1.2) has a unique solution $x_j^{\alpha,\tau}$ and if h/α, δ/α, $\alpha \to 0$ then $x_j^{\alpha,\tau} \to x_j \in S_j$ with x_*-minimal norm (see [1]), i.e.

$$\|x_j - x_*\| = \min_{x \in S_j} \|x - x_*\|, \quad j = 1, ..., N.$$

In this paper, we consider the more general problem, that is to find a common element x_α^τ of the solution sets of equations involving inverse-strongly monotone operators such that $x_\alpha^\tau \to x^0 \in S$ as $h, \delta, \alpha \to 0$ and estimate the value of $\|x_\alpha^\tau - x^0\|$ based on regularization parameter choice by the generalized discrepancy principle. Moreover, we propose an iterative regularization method of zero order that is a parallel algorithm. This algorithm generates a sequence $\{z_n\}$ from an arbitrary initial $z_0 \in H$, where H is a real Hilbert space. The sequence $\{z_n\}$ is shown to converge to $x^0 \in S$.

We now recall some definitions (see [5, 13]).

Definition 1.1. An operator $A : D(A) \equiv X \to X^*$ is called inverse-strongly monotone if

$$\langle A(x) - A(y), x - y \rangle \geq m_A \|A(x) - A(y)\|^2, \ \forall x, y \in X, \ m_A > 0, \tag{1.4}$$

where m_A is a positive constant.

Definition 1.2. An operator $U^s : X \to X^*$ is called the generalized duality mapping of X if

$$U^s(x) = \{x^* \in X^* : \langle x^*, x \rangle = \|x^*\|^{s-1}\|x\| = \|x\|^s\}, \ s \geq 2.$$

Assume that the generalized duality mapping U^s satisfies the following condition

$$\langle U^s(x) - U^s(y), x - y \rangle \geq m_U \|x - y\|^s, \ \forall x, y \in X, \tag{1.5}$$

where m_U is a positive constant. It is well-known that when X is a Hilbert space then $U^s = I$, $s = 2$ and $m_s = 1$, where I denotes the identity operator in the setting space (see [2]).

2. Main result

For approximations to a solution of (1.1), we introduce the following regularized problem of finding an $x_\alpha^\tau \in X$ such that (see Nguyen Buong [9])

$$\begin{gathered}\sum_{j=1}^{N} \alpha^{\lambda_j}(A_j^h(x_\alpha^\tau) - f_j^\delta) + \alpha U^s(x_\alpha^\tau - x_*) = \theta, \\ \lambda_1 = 0 < \lambda_j < \lambda_{j+1} < 1, \quad j = 2, ..., N-1.\end{gathered} \tag{2.1}$$

We have the following result.

Lemma 2.1. *Let X be an E-S space with strictly convex dual space X^*, $A_j^h : X \to X^*$ be a monotone bounded hemicontinuous operator for all $h > 0$, $U^s : X \to X^*$ be a generalized duality mapping and $f_j^\delta \in X^*$ for all $\delta > 0$. Then Problem (2.1) has an unique solution x_α^τ for all $\alpha > 0$.*

Proof. Since A_j^h is a monotone bounded hemicontinuous operator so it is a maximal monotone (see [5]). This implies that $\sum_{j=1}^N \alpha^{\lambda_j} A_j^h + \alpha U^s$ is also maximal monotone and coercitive (see [5, 6]). Then Theorem 1.7.4 in [3] guaranties the solvability of equation (2.1) in the sense of inclusion. Let x_α^τ be a solution of (2.1). It is unique because the operator $\sum_{j=1}^N \alpha^{\lambda_j} A_j^h + \alpha U^s$ is strictly monotone. □

The solution x_α^τ satisfying (2.1) will be view as the regularized solution of problem (1.1).

Theorem 2.1. *Let X and X^* be strictly convex spaces, A_j be a inverse-strongly monotone, A_j^h be a monotone bounded hemicontinuous operator, and $U^s : X \to X^*$ a generalized duality mapping. Assume that (1.3) and (1.5) hold. If*

$$\frac{h+\delta}{\alpha} \to 0 \text{ as } \alpha \to 0, \tag{2.2}$$

then the sequence $\{x_\alpha^\tau\}$ of solutions of the equation (2.1) converges strongly in X to $x^0 \in S$ with x_*-minimal norm.

Proof. For $x \in S$, it follows from (1.1) and (2.1) that

$$\sum_{j=1}^{N} \alpha^{\lambda_j} \langle A_j^h(x_\alpha^\tau) - f_j^\delta - A_j(x) + f_j, x_\alpha^\tau - x\rangle \\ + \alpha\langle U^s(x_\alpha^\tau - x_*) - U^s(x - x_*), x_\alpha^\tau - x\rangle \\ = \alpha\langle U^s(x - x_*), x - x_\alpha^\tau\rangle.$$

Using (1.5) we obtain

$$\alpha m_U \|x_\alpha^\tau - x\|^s \leq \sum_{j=1}^{N} \alpha^{\lambda_j} \langle A_j^h(x_\alpha^\tau) - A_j^h(x) + A_j^h(x) - A_j(x) + f_j - f_j^\delta, x - x_\alpha^\tau\rangle \\ + \alpha\langle U^s(x - x_*), x - x_\alpha^\tau\rangle.$$

It follows from (1.3) and the monotonicity of A_j^h that

$$m_U \|x_\alpha^\tau - x\|^s \leq \frac{1}{\alpha} N(hg(\|x\|) + \delta)\|x - x_\alpha^\tau\| + \langle U^s(x - x_*), x - x_\alpha^\tau\rangle. \quad (2.3)$$

Now from (2.2) and (2.3) we conclude that the sequence $\{x_\alpha^\tau\}$ is bounded. So there exists a subsequence $\{x_\beta^\nu\}$, where $\beta \subseteq \alpha$ and $\nu = (h', \delta') \subseteq \tau$, which weakly converges to some element $\hat{x} \in X$. We also have

$$\frac{h' + \delta'}{\beta} \to 0 \text{ as } \alpha \to 0.$$

First, we prove that $\hat{x} \in S_1$. Indeed, for an arbitrary $x \in X$, by virtue of the monotonicity of A_j^h and the property of U^s and (2.1) we have

$$\langle A_1^{h'}(x) - f_1^{\delta'}, x - x_\beta^\nu\rangle \geq \langle A_1^{h'}(x_\beta^\nu) - f_1^{\delta'}, x - x_\beta^\nu\rangle \\ = \sum_{j=2}^{N} \beta^{\lambda_j} \langle A_j^{h'}(x_\beta^\nu) - f_j^{\delta'}, x_\beta^\nu - x\rangle \\ + \beta\langle U^s(x_\beta^\nu - x_*), x_\beta^\nu - x\rangle \\ \geq \sum_{j=2}^{N} \beta^{\lambda_j} \langle A_j^{h'}(x) - f_j^{\delta'}, x_\beta^\nu - x\rangle + \beta\langle U^s(x - x_*), x_\beta^\nu - x\rangle.$$

Letting $\alpha \to 0$, and so $\beta \to 0$ and $\nu \to 0$, we obtain from the last inequality and (1.3) the limit inequality

$$\langle A_1(x) - f_1, x - \hat{x}\rangle \geq 0, \quad \forall x \in X.$$

Consequently, by Minty's lemma $\hat{x} \in S_1$ (see [14]). Now, we shall prove that $\hat{x} \in S_j$, $j = 2..., N$. Indeed, by (2.1) and making use of the monotonicity of

$A_j^{h'}$, it follows that

$$\begin{aligned}\langle A_2^{h'}(x_\beta^\nu) - f_2^{\delta'}, x_\beta^\nu - x\rangle &+ \sum_{j=3}^N \beta^{\lambda_j-\lambda_2}\langle A_j^{h'}(x_\beta^\nu) - f_j^{\delta'}, x_\beta^\nu - x\rangle \\ &+ \beta^{1-\lambda_2}\langle U^s(x_\beta^\nu - x_*), x_\beta^\nu - x\rangle \\ &= \frac{1}{\beta^{\lambda_2}}\langle A_1^{h'}(x_\beta^\nu) - A_1^{h'}(x) + A_1^{h'}(x) - A_1(x) + f_1 - f_1^{\delta'}, x - x_\beta^\nu\rangle \\ &\le \frac{\beta^{1-\lambda_2}}{\beta}(h'g(\|x\|) + \delta')\|x - x_\beta^\nu\|, \quad \forall x \in S_1.\end{aligned}$$

After letting $\alpha \to 0$ we obtain

$$\langle A_2(\hat{x}) - f_2, \hat{x} - x\rangle \le 0, \quad \forall x \in S_1. \tag{2.4}$$

Let $\tilde{x}$ be an element in $S_1 \cap S_2$. It follows from (2.4) that

$$0 = \langle A_2(\tilde{x}) - f_2, \tilde{x} - \hat{x}\rangle \ge \langle A_2(\hat{x}) - f_2, \tilde{x} - \hat{x}\rangle \ge 0.$$

Hence,

$$\langle A_2(\hat{x}) - f_2, \tilde{x} - \hat{x}\rangle = 0 = \langle A_2(\tilde{x}) - f_2, \tilde{x} - \hat{x}\rangle.$$

Consequently $\langle A_2(\tilde{x}) - A_2(\hat{x}), \tilde{x} - \hat{x}\rangle = 0$. Using the inverse-strongly monotonicity of A_2 we have

$$0 = \langle A_2(\tilde{x}) - A_2(\hat{x}), \tilde{x} - \hat{x}\rangle \ge m_{A_2}\|A_2(\tilde{x}) - A_2(\hat{x})\|^2 \ge 0.$$

Therefore,

$$A_2(\hat{x}) - f_2 = A_2(\tilde{x}) - f_2 = 0.$$

So, $\hat{x} \in S_2$.

Set $\tilde{S}_i = \cap_{k=1}^i S_k$. Then, $\tilde{S}_i$ is also closed convex, and $\tilde{S}_i \neq \emptyset$. Now, suppose that $\hat{x} \in \tilde{S}_i$, and we need to show that $\hat{x}$ belongs to S_{i+1}. Again, by virtue of (2.1) for $x \in \tilde{S}_i$, we can write

$$\begin{aligned}\langle A_{i+1}^{h'}(x_\beta^\nu) - f_{i+1}^{\delta'}, x_\beta^\nu - x\rangle &+ \sum_{j=i+2}^N \beta^{\lambda_j-\lambda_{i+1}}\langle A_j^{h'}(x_\beta^\nu) - f_j^{\delta'}, x_\beta^\nu - x\rangle \\ &+ \beta^{1-\lambda_{i+1}}\langle U^s(x_\beta^\nu - x_*), x_\beta^\nu - x\rangle \\ &= \sum_{k=1}^i \beta^{\lambda_k-\lambda_{i+1}}\langle A_k^{h'}(x_\beta^\nu) - f_k^{\delta'}, x - x_\beta^\nu\rangle \\ &\le \frac{1}{\beta}\sum_{k=1}^i \beta^{\lambda_k+1-\lambda_{i+1}}\langle A_k^{h'}(x) - A_k(x) + f_k - f_k^{\delta'}, x - x_\beta^\nu\rangle \\ &\le \frac{1}{\beta}N\big(h'g(\|x\|) + \delta'\big)\|x - x_\beta^\nu\|.\end{aligned}$$

Therefore, by letting $\alpha \to 0$ we have

$$\langle A_{i+1}(\hat{x}) - f_{i+1}, \hat{x} - x\rangle \leq 0, \quad \forall x \in \tilde{S}_i.$$

By an argument analogous to the previous one, we get $\hat{x} \in S_{i+1}$, which means that $\hat{x} \in S$.

On the other hand, it follows from (2.3) that

$$\langle U^s(x - x_*), x - \hat{x}\rangle \geq 0, \quad \forall x \in S.$$

S_j is closed convex, so is S. Replacing x by $t\hat{x} + (1-t)x$, $t \in (0,1)$ in the last inequality, dividing by $(1-t)$ and letting t to 1, we obtain

$$\langle U^s(\hat{x} - x_*), x - \hat{x}\rangle \geq 0, \quad \forall x \in S.$$

Hence $\|\hat{x} - x_*\| \leq \|x - x_*\|$, $\forall x \in S$. Because of the convexity and the closedness of S, and the strictly convexity of X we deduce that $\hat{x} = x^0$. So, all sequence $\{x^\tau_\alpha\}$ converges weakly to x^0. It follows from (2.3) that the sequence $\{x^\tau_\alpha\}$ converges strongly to x^0. This completes the proof. □

Now, we consider the problem of choosing $\tilde{\alpha} = \alpha(h,\delta)$ such that

$$\lim_{h,\delta\to 0} \alpha(h,\delta) = 0 \text{ and } \lim_{h,\delta\to 0} \frac{h+\delta}{\alpha(h,\delta)} = 0.$$

To solve this problem, we use the function for selecting $\tilde{\alpha} = \alpha(h,\delta)$ by generalized discrepancy principle, i.e. the relation $\tilde{\alpha} = \alpha(h,\delta)$ is constructed on the basis of the following equation

$$\rho(\tilde{\alpha}) = (h+\delta)^p \tilde{\alpha}^{-q}, \quad p, q > 0, \tag{2.5}$$

with $\rho(\tilde{\alpha}) = \tilde{\alpha}(c + \|x^\tau_{\tilde{\alpha}} - x_*\|^{s-1})$, where $x^\tau_{\tilde{\alpha}}$ is the solution of (2.1) with $\alpha = \tilde{\alpha}$, c is some positive constant. Note that the generalized discrepancy principle was presented in [11] for linear ill-posed problems and then it was developed for nonlinear ones in [7]. We have the following results.

Lemma 2.2. *Let X be an E-S space with a strictly convex dual space X^*, $A^h_j : X \to X^*$ be a monotone bounded hemicontinuous operator and $U^s : X \to X^*$ with condition (1.5) holds. Then the function $\rho(\alpha) = \alpha(c + \|x^\tau_\alpha - x_*\|)$ is single-valued and continuous for $\alpha \geq \alpha_0 > 0$, where x^τ_α is the solution of (2.1).*

Proof. Single-valued solvability of the equation (2.1) implies the continuity property of the function $\rho(\alpha)$. Let $\alpha_1, \alpha_2 \geq \alpha_0$ be arbitrary ($\alpha_0 > 0$). It follows from (2.1) that

$$\sum_{j=1}^{N}\alpha_1^{\lambda_j}\langle A_j^h(x_{\alpha_1}^\tau)-f_j^\delta, x_{\alpha_1}^\tau - x_{\alpha_2}^\tau\rangle + \alpha_1\langle U^s(x_{\alpha_1}^\tau - x_*), x_{\alpha_1}^\tau - x_{\alpha_2}^\tau\rangle$$
$$+\sum_{j=1}^{N}\alpha_2^{\lambda_j}\langle A_j^h(x_{\alpha_2}^\tau)-f_j^\delta, x_{\alpha_2}^\tau - x_{\alpha_1}^\tau\rangle + \alpha_2\langle U^s(x_{\alpha_2}^\tau - x_*), x_{\alpha_2}^\tau - x_{\alpha_1}^\tau\rangle$$
$$+\sum_{j=1}^{N}\alpha_2^{\lambda_j}\langle A_j^h(x_{\alpha_1}^\tau)-f_j^\delta, x_{\alpha_2}^\tau - x_{\alpha_1}^\tau\rangle + \sum_{j=1}^{N}\alpha_2^{\lambda_j}\langle A_j^h(x_{\alpha_1}^\tau)-f_j^\delta, x_{\alpha_1}^\tau - x_{\alpha_2}^\tau\rangle = 0,$$

where $x_{\alpha_1}^\tau$ and $x_{\alpha_2}^\tau$ are solutions of (2.1) with $\alpha = \alpha_1$ and $\alpha = \alpha_2$. Using the monotonicity of A_j^h we have

$$\begin{aligned}\alpha_1\langle U^s(x_{\alpha_1}^\tau - x_*) - U^s(x_{\alpha_2}^\tau - x_*), x_{\alpha_1}^\tau - x_{\alpha_2}^\tau\rangle \\ \le (\alpha_2-\alpha_1)\langle U^s(x_{\alpha_2}^\tau - x_*), x_{\alpha_1}^\tau - x_{\alpha_2}^\tau\rangle \\ + \sum_{j=1}^{N}(\alpha_2^{\lambda_j}-\alpha_1^{\lambda_j})\langle A_j^h(x_{\alpha_1}^\tau)-f_j^\delta, x_{\alpha_1}^\tau - x_{\alpha_2}^\tau\rangle.\end{aligned}$$

It follows from (1.5) and the last inequality that

$$\begin{aligned}m_U\|x_{\alpha_1}^\tau - x_{\alpha_2}^\tau\|^{s-1} \le \frac{|\alpha_2-\alpha_1|}{\alpha_0}\|x_{\alpha_2}^\tau - x_*\|^{s-1} \\ + \sum_{j=1}^{N}\frac{|\alpha_2^{\lambda_j}-\alpha_1^{\lambda_j}|}{\alpha_0}\|A_j^h(x_{\alpha_1}^\tau)-f_j^\delta\|.\end{aligned}$$

Obviously, $x_{\alpha_1}^\tau \to x_{\alpha_2}^\tau$ as $\alpha_1 \to \alpha_2$. It means that the function $\|x_\alpha^\tau - x_*\|$ is continuous on $[\alpha_0; +\infty)$. Therefore, $\rho(\alpha)$ is also continuous on $[\alpha_0; +\infty)$. □

Theorem 2.2. *Let X and X^* be strictly convex spaces, A_j^h be a monotone bounded hemicontinuous operator, $U^s : X \to X^*$ be a duality mapping. Assume that (1.3) and (1.5) hold. Then*

(i) There exists at least a solution $\tilde\alpha$ of the equation (2.5);

(ii) Let $\tau \to 0$. Then

(1) $\tilde\alpha \to 0$;

(2) If $0 < p < q$ then $\frac{h+\delta}{\tilde\alpha} \to 0$, $x_{\tilde\alpha}^\tau \to x^0 \in S$ with x_*-minimal norm and there exits constants $C_1, C_2 > 0$ such that for sufficiently small $h, \delta > 0$ the relation

$$C_1 \le (h+\delta)^p\alpha^{-1-q}(h,\delta) \le C_2$$

holds.

Proof. (i) For $0 < \alpha < 1$, it follows from (2.1) that

$$\sum_{j=1}^{N} \alpha^{\lambda_j} \langle A_j^h(x_\alpha^\tau) - f_j^\delta, x_\alpha^\tau - x_* \rangle + \alpha \langle U^s(x_\alpha^\tau - x_*), x_\alpha^\tau - x_* \rangle = 0.$$

Hence,

$$\begin{aligned} &\alpha \langle U^s(x_\alpha^\tau - x_*), x_\alpha^\tau - x_* \rangle \\ &\leq \sum_{j=1}^{N} \alpha^{\lambda_j} \langle A_j^h(x_*) - f_j^\delta, x_* - x_\alpha^\tau \rangle \\ &= \sum_{j=1}^{N} \alpha^{\lambda_j} \langle A_j^h(x_*) - A_j(x_*) + A_j(x_*) - f_j + f_j - f_j^\delta, x_* - x_\alpha^\tau \rangle. \end{aligned}$$

We invoke (1.3), (1.5) and the last inequality to deduce that

$$\alpha \| x_\alpha^\tau - x_* \|^{s-1} \leq N\big(hg(\|x_*\|) + \|A_j(x_*) - f_j\| + \delta\big). \tag{2.6}$$

It follows from (2.6) and the form of $\rho(\alpha)$ that

$$\begin{aligned} \alpha^q \rho(\alpha) &= \alpha^{1+q}(c + \|x_\alpha^\tau - x_*\|^{s-1}) \\ &= c\alpha^{1+q} + \alpha^q \alpha \|x_\alpha^\tau - x_*\|^{s-1} \\ &\leq c\alpha^{1+q} + \alpha^q N\big(hg(\|x_*\|) + \|A_j(x_*) - f_j\| + \delta\big). \end{aligned}$$

Therefore, $\lim_{\alpha \to +0} \alpha^q \rho(\alpha) = 0$.

On the other hand,

$$\lim_{\alpha \to +\infty} \alpha^q \rho(\alpha) \geq c \lim_{\alpha \to +\infty} \alpha^{1+q} = +\infty.$$

Since $\rho(\alpha)$ is continuous, there exists at least one $\tilde{\alpha}$ which satisfies (2.5).

ii) It follows from (2.5) and the form of $\rho(\tilde{\alpha})$ that

$$\tilde{\alpha} \leq c^{-1/(1+q)}(h + \delta)^{p/(1+q)}.$$

Therefore, $\tilde{\alpha} \to 0$ as $\tau \to 0$.

If $0 < p < q$, it follows from (2.5) and (2.6) that

$$\begin{aligned} \left[\frac{h+\delta}{\tilde{\alpha}}\right]^p &= \big[(h+\delta)^p \tilde{\alpha}^{-q}\big]\tilde{\alpha}^{q-p} \\ &= [\tilde{\alpha}c + \tilde{\alpha}\|x_{\tilde{\alpha}}^\tau - x_*\|^{s-1}]\tilde{\alpha}^{q-p} \\ &\leq c\tilde{\alpha}^{1+q-p} + \tilde{\alpha}^{q-p} N\big(hg(\|x_*\|) + \|A_j(x_*) - f_j\| + \delta\big). \end{aligned}$$

So

$$\lim_{h,\delta \to 0} \left[\frac{h+\delta}{\tilde{\alpha}}\right]^p = 0.$$

By Theorem 2.1 the sequence $x_{\tilde{\alpha}}^{\tau}$ converges to $x^0 \in S$ with x_*-minimal norm as $h, \delta \to 0$.
Clearly,

$$(h+\delta)^p \tilde{\alpha}^{-1-q} = \tilde{\alpha}^{-1}\rho(\tilde{\alpha}) = (c + \|x_{\tilde{\alpha}}^{\tau} - x_*\|^{s-1}),$$

therefore there exists a positive constant C_2 in the theorem. On the other hand, because $c > 0$ there exists a positive constant C_1 in the theorem. This completes the proof. □

To estimate the convergence rates for the sequence $\{x_{\tilde{\alpha}}^{\tau}\}$, we assume that there exists a positive constant $\tilde{\tau}$ such that

$$\|A_1(y) - A_1(x) - A_1'(x)(y-x)\| \le \tilde{\tau}\|A_1(y) - A_1(x)\|, \quad \forall x \in S, \tag{2.7}$$

and y belongs to some neighbourhood of S.

Note that, Hanke, Neubauer and Scherzer [12] gave a first convergence analysis of the Landweber iteration method for a class of nonlinear operators with (2.7) when $\tilde{\tau} < 1/2$. The use of this assumption to estimate the convergence rates of the regularized solutions of ill-posed inverse-strongly monotone variational inequalities in Banach space was considered in [8].

Theorem 2.3. *Let X and X^* be strictly convex spaces, A_j^h be a monotone bounded hemicontinuous operator, $U^s : X \to X^*$ be a duality mapping. Assume that (1.3) and (1.5) hold and,*

(i) A_1 is Fréchet continuously differentiable with (2.7) for $x = x^0$;

(ii) there exists $z \in X$ such that $A_1'(x^0)^* z = U^s(x^0 - x_*)$;

(iii) the parameter $\tilde{\alpha} = \alpha(h,\delta)$ is chosen by (2.5) with $0 < p < q$.

Then,

$$\|x_{\tilde{\alpha}}^{\tau} - x^0\| = O\big((h+\delta)^{\mu_1}\big), \quad \mu_1 = \min\left\{\frac{1+q-p}{s(1+q)}, \frac{\lambda_2 p}{s(1+q)}\right\}.$$

Proof. Replacing x by x^0 in (2.3) we obtain

$$m_U\|x_{\tilde{\alpha}}^{\tau} - x^0\|^s \le \frac{1}{\tilde{\alpha}} N\big(hg(\|x^0\|) + \delta\big)\|x^0 - x_{\tilde{\alpha}}^{\tau}\| + \langle U^s(x^0 - x_*), x^0 - x_{\tilde{\alpha}}^{\tau}\rangle. \tag{2.8}$$

Using conditions $(i), (ii)$ we can write

$$\begin{aligned}
\langle U^s(x^0 - x_*), x^0 - x_{\tilde{\alpha}}^{\tau}\rangle &= \langle z, A_1'(x^0)(x^0 - x_{\tilde{\alpha}}^{\tau})\rangle \\
&\le \|z\|(\tilde{\tau}+1)\|A_1(x_{\tilde{\alpha}}^{\tau}) - A_1(x^0)\| \\
&\le \|z\|(\tilde{\tau}+1)\big(hg(\|x_{\tilde{\alpha}}^{\tau}\|) + \|A_1^h(x_{\tilde{\alpha}}^{\tau}) - f_1^{\delta}\| + \delta\big) \\
&\le \|z\|(\tilde{\tau}+1)\Big[\sum_{j=2}^{N} \tilde{\alpha}^{\lambda_j}\|A_j^h(x_{\tilde{\alpha}}^{\tau}) - f_j^{\delta}\| \\
&\quad + \tilde{\alpha}\|x_{\tilde{\alpha}}^{\tau} - x_*\|^{s-1} + hg(\|x_{\tilde{\alpha}}^{\tau}\|) + \delta\Big].
\end{aligned} \tag{2.9}$$

Combining with (2.8), the inequality (2.9) becomes

$$\begin{aligned} m_U\|x^\tau_{\tilde\alpha}-x^0\|^s \le{}& \frac{1}{\tilde\alpha}N\big(hg(\|x^0\|)+\delta\big)\|x^0-x^\tau_{\tilde\alpha}\| \\ &+\|z\|(\tilde\tau+1)\Bigg[\sum_{j=2}^N \tilde\alpha^{\lambda_j}\|A^h_j(x^\tau_{\tilde\alpha})-f^\delta_j\| \\ &+\tilde\alpha\|x^\tau_{\tilde\alpha}-x_*\|^{s-1}+hg(\|x^\tau_{\tilde\alpha}\|)+\delta\Bigg]. \end{aligned} \tag{2.10}$$

Now, it follows from Theorem 2.2 that

$$\tilde\alpha \le C_1^{-1/(1+q)}(h+\delta)^{p/(1+q)},$$

and

$$\begin{aligned} \frac{h+\delta}{\tilde\alpha} &\le C_2(h+\delta)^{1-p}\tilde\alpha^q \\ &\le C_2C_1^{-q/(1+q)}(h+\delta)^{1-p/(1+q)}. \end{aligned}$$

Therefore,

$$m_U\|x^\tau_{\tilde\alpha}-x^0\|^s \le \overline{C_1}(h+\delta)^{1-p/(1+q)}\|x^0-x^\tau_{\tilde\alpha}\|+\overline{C_2}(h+\delta)^{\lambda_2 p/(1+q)},$$

where $\overline{C_i}, i=1,2$ are the positive constants. Using the implication

$$a,b,c\ge 0,\ \ s>t,\ \ a^s\le ba^t+c \Longrightarrow a^s=O(b^{s/(s-t)}+c),$$

we obtain

$$\|x^\tau_{\tilde\alpha}-x^0\| = O\big((h+\delta)^{\mu_1}\big).$$

□

Remark 2.1. If α is chosen a priory such that $\alpha\sim(h+\delta)^\eta$, $0<\eta<1$, it follows from (2.10) that

$$\|x^\tau_{\alpha}-x^0\| = O\big((h+\delta)^{\mu_2}\big),\ \ \mu_2=\min\Big\{\frac{1-\eta}{s-1},\frac{\lambda_2\eta}{s}\Big\}.$$

And now, we consider the following iterative regularization method of zero order, where z_{n+1} is defined by

$$z_{n+1}=z_n-\beta_n\Bigg[\sum_{j=1}^N \alpha_n^{\lambda_j}(A_j(z_n)-f_j)+\alpha_n(z_n-x_*)\Bigg],\ \ z_0\in H, \tag{2.11}$$

where H is a real Hilbert space, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers.

We consider the operator equation

$$\sum_{j=1}^{N} \alpha_n^{\lambda_j}(A_j(x) - f_j) + \alpha_n(x - x_*) = \theta. \tag{2.12}$$

Theorem 2.4. *Let X and X^* be strictly convex spaces, $A_j : X \to X^*$ be a monotone bounded hemicontinuous operator and inverse-strongly monotone. Assume that (1.3) holds. Then*

(i) For each $\alpha_n > 0$, Problem (2.12) has a unique solution x_n;

(ii) If $0 < \alpha_n \leq 1$, $\alpha_n \to 0$ as $n \to +\infty$, then $\lim_{n\to+\infty} x_n = x^0 \in S$ with x_* -minimal norm and

$$\|x_{n+1} - x_n\| = O\left(\frac{|\alpha_{n+1} - \alpha_n|}{\alpha_n}\right),$$

where x_{n+1} is a solution of (2.12) when α_n is replaced by α_{n+1}.

Proof. (i) By an argument analogous to that used in the proof of the equation (2.1), we deduce that the equation (2.12) has a unique solution denoted by x_n.
(ii) The proof of the first part is anologus to Theorem 2.1.

Let x_{n+1} be a solution of (2.12) when α_n is replaced by α_{n+1}. It follows from (2.12) that

$$\begin{aligned}&\sum_{j=1}^{N} \alpha_n^{\lambda_j}\langle A_j(x_n) - f_j, x_n - x_{n+1}\rangle + \alpha_n\langle x_n - x_*, x_n - x_{n+1}\rangle\\ &+\sum_{j=1}^{N} \alpha_{n+1}^{\lambda_j}\langle A_j(x_{n+1}) - f_j, x_{n+1} - x_n\rangle + \alpha_{n+1}\langle x_{n+1} - x_*, x_{n+1} - x_n\rangle\\ &+\sum_{j=1}^{N} \alpha_{n+1}^{\lambda_j}\langle A_j(x_n) - f_j, x_{n+1} - x_n\rangle + \sum_{j=1}^{N} \alpha_{n+1}^{\lambda_j}\langle A_j(x_n) - f_j, x_n - x_{n+1}\rangle = 0.\end{aligned}$$

Because of the monotonicity of A_j and the last inequality, we obtain

$$\begin{aligned}&\sum_{j=1}^{N}(\alpha_n^{\lambda_j} - \alpha_{n+1}^{\lambda_j})\langle A_j(x_n) - f_j, x_n - x_{n+1}\rangle + \alpha_n\langle x_n - x_*, x_n - x_{n+1}\rangle\\ &\qquad + \alpha_{n+1}\langle x_{n+1} - x_*, x_{n+1} - x_n\rangle \leq 0.\end{aligned}$$

Hence,

$$\alpha_n\langle x_n - x_{n+1}, x_n - x_{n+1}\rangle \le (\alpha_n - \alpha_{n+1})\langle x_{n+1} - x_*, x_{n+1} - x_n\rangle + \sum_{j=2}^{N}(\alpha_n^{\lambda_j} - \alpha_{n+1}^{\lambda_j})\langle A_j(x_n) - f_j, x_{n+1} - x_n\rangle.$$

So

$$\|x_n - x_{n+1}\| \le \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n}\|x_{n+1} - x_*\| + \frac{K}{\alpha_n}\sum_{j=2}^{N}|\alpha_n^{\lambda_j} - \alpha_{n+1}^{\lambda_j}|, \qquad (2.13)$$

where K is a positive constant such that $K = \max_{2\le j\le N}\|A_j(x_n) - f_j\|$.

On the other hand, it follows from (2.12) that

$$\sum_{j=1}^{N}\alpha_{n+1}^{\lambda_j}\langle A_j(x_{n+1}) - f_j, x_{n+1} - x\rangle + \alpha_{n+1}\langle x_{n+1} - x_*, x_{n+1} - x\rangle$$
$$= \sum_{j=1}^{N}\alpha_{n+1}^{\lambda_j}\langle A_j(x_{n+1}) - A_j(x) + f_j - f_j, x_{n+1} - x\rangle + \alpha_{n+1}\langle x_{n+1} - x_*, x_{n+1} - x\rangle = 0,\ \forall x \in S.$$

Using the monotone property of A_j the last equality have the form

$$\alpha_{n+1}\langle x_{n+1} - x_*, x_{n+1} - x\rangle = \sum_{j=1}^{N}\alpha_{n+1}^{\lambda_j}\langle A_j(x_{n+1}) - A_j(x), x - x_{n+1}\rangle \le 0,\ \forall x \in S.$$

Therefore,

$$\|x_{n+1} - x_*\| \le \|x - x_*\|,\ \forall x \in S. \qquad (2.14)$$

Combining (2.13), (2.14) and the Lagrange's mean-value theorem for the differentiable function $\varphi(\nu) = \nu^\gamma$, $0 < \gamma < 1$, $\nu \in [1; +\infty)$ on $[\alpha_n; \alpha_{n+1}]$ we get

$$\|x_{n+1} - x_n\| \le M\frac{|\alpha_{n+1} - \alpha_n|}{\alpha_n},$$

where

$$M = \|x^0 - x_*\| + K(N-1).$$

The proof is complete. □

We need the following result (see [4]).

Lemma 2.3. *Let $\{u_k\}, \{a_k\}, \{b_k\}$ be the sequences of positive numbers satisfying the following conditions:*

(i) $u_{k+1} \le (1 - a_k)u_k + b_k, \quad 0 \le a_k \le 1,$

(ii) $\sum\limits_{k=1}^{\infty} a_k = +\infty, \quad \lim\limits_{k\to+\infty} \dfrac{b_k}{a_k} = 0.$

Then, $\lim\limits_{k\to+\infty} u_k = 0.$

Theorem 2.5. *Assume that* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *in the problem (2.11) satisfy the following conditions:*

(i) $1 \geq \alpha_n \searrow 0,\ \beta_n \to 0$ *as* $n \to +\infty$ *;*

(ii) $\lim\limits_{n\to+\infty} \dfrac{|\alpha_{n+1} - \alpha_n|}{\beta_n \alpha_n^2} = 0,\ \lim\limits_{n\to+\infty} \dfrac{\beta_n}{\alpha_n} = 0;$

(iii) $\sum\limits_{n=1}^{\infty} \alpha_n \beta_n = +\infty.$

Then $\{z_n\}$ *generated from (2.11) converges in* H *to* $x^0 \in S$ *as* $n \to +\infty$.

Proof. First, we have $\|z_n - x^0\| \leq \|z_n - x_n\| + \|x_n - x^0\|$. The second term in right-hand side of this estimate tends to zero as $n \to \infty$, by Theorem 2.3. So we only have to proof that z_n approximates x_n as $n \to \infty$.
Let $\Delta_n = \|z_n - x_n\|$. Obviuously,

$$\begin{aligned}\Delta_{n+1} &= \|z_{n+1} - x_{n+1}\| \\ &= \|z_n - x_n - \beta_n\Big[\sum_{j=1}^{N} \alpha_n^{\lambda_j}(A_j(z_n) - f_j) + \alpha_n(z_n - x_*)\Big] \\ &\quad - (x_{n+1} - x_n)\|, \qquad (2.15)\\ &\leq \Big\|z_n - x_n - \beta_n\Big[\sum_{j=1}^{N} \alpha_n^{\lambda_j}(A_j(z_n) - f_j) + \alpha_n(z_n - x_*)\Big]\Big\| \\ &\quad + \|x_{n+1} - x_n\|,\end{aligned}$$

where

$$\begin{aligned}&\Big\|z_n - x_n - \beta_n\Big[\sum_{j=1}^{N} \alpha_n^{\lambda_j}(A_j(z_n) - f_j) + \alpha_n(z_n - x_*)\Big]\Big\|^2 \\ &= \|z_n - x_n\|^2 + \beta_n^2\Big\|\sum_{j=1}^{N} \alpha_n^{\lambda_j}(A_j(z_n) - f_j) + \alpha_n(z_n - x_*)\Big\|^2 \\ &\quad - 2\beta_n\Big\langle z_n - x_n, \sum_{j=1}^{N} \alpha_n^{\lambda_j}(A_j(z_n) - f_j) + \alpha_n(z_n - x_*) \\ &\quad - \Big[\sum_{j=1}^{N} \alpha_n^{\lambda_j}(A_j(x_n) - f_j) + \alpha_n(x_n - x_*)\Big]\Big\rangle\end{aligned}$$

$$\leq (1-2\beta_n\alpha_n)\|z_n - x_n\|^2 + \beta_n^2 \Big\| \sum_{j=1}^{N} \alpha_n^{\lambda_j}(A_j(z_n) - f_j) + \alpha_n(z_n - x_*)\Big\|^2. \tag{2.16}$$

Since A_j is inverse-strongly monotone, A_j is Lipschitz continuous, and

$$\Big\| \sum_{j=1}^{N} \alpha_n^{\lambda_j}(A_j(z_n) - f_j) + \alpha_n(z_n - x_*)\Big\|^2$$
$$= \Big\| \sum_{j=1}^{N} \alpha_n^{\lambda_j}(A_j(z_n) - f_j) + \alpha_n(z_n - x_*) - \sum_{j=1}^{N} \alpha_n^{\lambda_j}(A_j(x_n) - f_j) - \alpha_n(x_n - x_*)\Big\|^2$$
$$\leq \Big(\sum_{j=1}^{N} \alpha_n^{\lambda_j} \frac{1}{m_{A_j}} \|z_n - x_n\| \Big)^2 + \alpha_n^2 \|z_n - x_n\|^2 + 2\alpha_n \sum_{j=1}^{N} \alpha_n^{\lambda_j} \frac{1}{m_{A_j}} \|z_n - x_n\|^2$$
$$\leq c\|z_n - x_n\|^2,$$

where c is positive constant. Combining (2.15), (2.16), the last inequality and the Theorem 2.3 yields that

$$\Delta_{n+1} \leq \Big(\Delta_n^2(1 - 2\beta_n\alpha_n + c\beta_n^2)\Big)^{1/2} + M\frac{|\alpha_{n+1} - \alpha_n|}{\alpha_n}.$$

By taking the squares of the both sides of the last inequality and then applying the elementary estimate (see [4])

$$(a+b)^2 \leq (1 + \alpha_n\beta_n)a^2 + (1 + \frac{1}{\alpha_n\beta_n})b^2$$

we obtain that

$$\Delta_{n+1}^2 \leq \Delta_n^2(1 - \beta_n\alpha_n + c\beta_n^2 - 2\alpha_n^2\beta_n^2 + c\alpha_n\beta_n^3) + \big(1 + \frac{1}{\beta_n\alpha_n}\big)M^2\frac{|\alpha_{n+1} - \alpha_n|^2}{\alpha_n^2}. \tag{2.17}$$

The conditions of Lemma 2.3 for the numerical sequence $\{\Delta_n\}$ are true because of (2.17) and conditions $(i) - (iii)$ with

$$a_n = \alpha_n\beta_n - c\beta_n^2 + 2\alpha_n^2\beta_n^2 - c\alpha_n\beta_n^3$$
$$b_n = \big(1 + \frac{1}{\beta_n\alpha_n}\big)M^2\frac{|\alpha_{n+1} - \alpha_n|^2}{\alpha_n^2}.$$

The proof is complete. □

Remark 2.2. The sequences $\beta_n = (1+n)^{-1/2}$ and $\alpha_n = (1+n)^{-p}$, $0 < 2p < 1/N$ satisfy all conditions in Theorem 2.5.

3. Numerical example

We now apply the obtained results of the previous sections to solve the convex optimization problem: find an element $x^0 \in H$ such that

$$\varphi_j(x^0) = \min_{x \in H} \varphi_j(x) \quad j = 1, ..., N, \tag{3.1}$$

where φ_j is weakly lower semi-continuous proper convex function on a real Hilbert space H.

We consider the case, when the function $\varphi_j : L^2[0,1] \to \mathbb{R} \cup \{+\infty\}$ is defined by $\varphi_j(x) = f\big(\frac{1}{2}\langle B_j x, x\rangle\big)$, $j = 1, 2$, where $f : \mathbb{R} \to \mathbb{R}$ is chosen as follows

$$f(t) = \begin{cases} 0 & , \quad t \le b_0, \\ \dfrac{(t-b_0)^2}{2\nu} & , \quad b_0 < t \le b_0 + \nu, \\ t - b_0 - \dfrac{\nu}{2} & , \quad t > b_0 + \nu, \end{cases}$$

with $\nu > 0$ is sufficiently small, and $B_j : \ L^2[0,1] \to L^2[0,1]$ are difined by $B_j x(t) = \int\limits_0^1 k_j(t,s)x(s)ds$,

$$k_1(t,s) = \begin{cases} t(1-s) & , \quad \text{if} \quad t \le s, \\ s(1-t) & , \quad \text{if} \quad s < t, \end{cases}$$

and

$$k_2(t,s) = \begin{cases} \dfrac{(1-s)^2 st^2}{2} - \dfrac{(1-s)^2 t^3 (1+2s)}{6} + \dfrac{(t-s)^3}{6}, & \text{if} \quad t \ge s, \\ \dfrac{s^2(1-s)(1-t)^2}{2} + \dfrac{s^2(1-t)^3(2s-3)}{6} + \dfrac{(s-t)^3}{6}, & \text{if} \quad t < s. \end{cases}$$

Then x^0 is a solution to the problem (3.1) if and only if $x^0 \in S$ with $A_j(x) = f'\big(\frac{1}{2}\langle B_j x, x\rangle\big) B_j(x)$.

We apply the iterative regularization method (2.11) as follow

$$z_{m+1} = z_m - \beta_m \big[\tilde{A}_1 z_m + \alpha_m \tilde{A}_2 z_m + \alpha_m^2 z_m\big], \quad z_0 \in \mathbb{R}^M, \tag{3.2}$$

where $\tilde{A}_j(x) = f'\big(\frac{1}{2}\langle \tilde{B}_j\tilde{x}, \tilde{x}\rangle\big)\tilde{B}_j(\tilde{x})$ with

$$\tilde{B}_j = (\ell k_j(t_k, t_l))_{k,l=1}^M$$
$$\tilde{x} = (\tilde{x}_1, ..., \tilde{x}_M)^T$$
$$\tilde{x}_k \sim x(t_k),\ k = 1, ..., M,\ \ell = \frac{1}{M}.$$

By choosing $\alpha_m = (1+m)^{-p},\ 0 < p < \frac{1}{4},\ \beta_m = (1+m)^{-1/2},\ z_0 = (5,\ 5,\ ...,\ 5)^T \in \mathbb{R}^M$ and $b_0 = \frac{10^{-3}}{3}$, $\nu = 10^{-2}$ we obtain the results.

m	err	$\|x^0 - z_m\|$
32	0.00067782	0.0077909
64	5.3403×10^{-5}	0.0010353
128	2.7676×10^{-6}	8.9792×10^{-5}
256	8.6222×10^{-8}	4.6575×10^{-6}

Table 2.1: $M = 50$, $p = \frac{1}{9}$

m	err	$\|x^0 - z_m\|$
32	0.00013731	0.001026
64	3.7047×10^{-6}	4.379×10^{-5}
128	3.9267×10^{-8}	7.2603×10^{-7}
256	1.2199×10^{-10}	3.501×10^{-9}

Table 2.2: $M = 50$, $p = \frac{1}{18}$

In these tables, $err = \max_{1\le k\le M} |z_k^{(m-1)} - z_k^{(m)}|$ is error.

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