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EXISTENCE AND ITERATION OF POSITIVE SOLUTIONS FOR A NONLINEAR MULTI-POINT BOUNDARY VALUE PROBLEM WITH A p-LAPLACIAN OPERATOR

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Abstract. In this paper, we study the existence and iteration of positive solutions for a nonlinear multi-point boundary value problem with a p-Laplacian operator. By using the monotone iterative technique, we obtain not only the existence of positive solutions for the problem, but also establish iterative schemes for approximating the solutions,which is different from the previous ones.

1. Introduction

In this paper we are interested in the existence of positive solutions for the following nonlinear multi-point boundary value problem with a p-Laplacian operator:

$$(\phi_p(u^{'}(t)))^{'} + q(t)f(t, u(t), u^{'}(t)) = 0. \ \ t \in (0, 1) \tag{1.1}$$

$$\phi_p(u(0)) - \sum_{i=1}^{m-2} \alpha_i \phi_p(u^{'}(\xi_i)) = 0, \ \phi_p(u(1)) + \sum_{i=1}^{m-2} \beta_i \phi_p(u^{'}(\xi_i)) = 0, \tag{1.2}$$

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where $\phi_p(s)$ is a p-Laplacian operator, that is, $\phi_p(s) = |s|^{p-2}s, p > 1; \xi_i \in (0,1)$with $0 < \xi_1 < \xi_2 < \xi_3 < \cdots < \xi_{m-2} < 1$ and α_i, β_i, f satisfy:

(H1) $\alpha_i, \beta_i \in [0,1)$, $0 \le \sum_{i=1}^{m-2} \alpha_i < 1,\ 0 \le \sum_{i=1}^{m-2} \beta_i < 1$;

(H2) Assume that $f \in C([0,1] \times [0,+\infty), [0,+\infty)), q(t)$ is nonnegative continuous function defined on$(0,1)$ and $q(t) \ne 0$ on any subinterval of $(0,1)$. In addition, $\int_0^1 q(t)dt < \infty$.

In recent years, because of the wide mathematical and physical background [1,9], the existence of positive solutions for nonlinear boundary value problems with a p-Laplacian operator has received wide attention, and there exist a very large number of papers devoted to the existence of solutions of the p-Laplacian operator with two, three, and four-point boundary conditions, for example,

$u(0) = 0,\ u(1) = 0,$

$u(0) - B_0(u^{'}(0)) = 0,\ u^{'}(1) = 0,$

$u(0) - B_0(u^{'}(0)) = 0,\ u(1) - B_1(u^{'}(1)) = 0,$

$u^{'}(0) = 0,\ u(1) + B_1(u^{'}(1)) = 0,$

and

$\alpha\phi_p(u(0)) - \beta\phi_p(u^{'}(0)) = 0,\ \gamma\phi_p(u(1)) + \delta\phi_p(u^{'}(1)) = 0,$

$\mu\phi_p(u(0)) - \omega\phi_p(u^{'}(\xi)) = 0,\ \rho\phi_p(u(1)) + \tau\phi_p(u^{'}(\eta)) = 0.$

For further knowledge, see [10,6]. the methods and techniques employed in these papers involve the use of Leray-Shauder degree theory [7], the upper and lower solution method [2], the fixed point theorem in a cone [6],the monotone iterative technique [8,3], and the variational method [11]. However, there are few papers dealing with the existence of positive solutions for multi-point boundary value problem.

Motivated by the method of [8], this paper is concerned with the existence and iteration of positive solutions for the boundary value problem with p-Laplacian (1.1)-(1.2).

2. Some definitions

In this section we provide some background material from the theory cones in Banach space.

If $P \in E$ is a cone, we denote the order induced by P on E by $\le$. That is $x \le y$ if and only if $y - x \in P$.

Definition 2.1. Given a cone P in a real Banach spaces E, a functional $\psi : P \to R$ is said to be increasing on P, if $\psi(x) \le \psi(y)$, for all $x, y \in P$ with $x \le y$.

Definition 2.2. The map α is said to be concave on [0,1], if

$$\alpha(tu + (1-t)v) \ge t\alpha(u) + (1-t)\alpha(v)$$

for all $u, v \in [0,1]$ and $t \in [0,1]$.

3. Existence and iteration of positive solutions to (1,1)-(1.2)

Let $E = C^{'}[0,1]$. Then E is a Banach space with the norm

$$\|u\| := \max\left\{\max_{0\le t\le 1}|u(t)|,\ \max_{0\le t\le 1}|u^{'}(t)|\right\}.$$

We note that, from the nonnegativity of q and f, a solution of (1,1) and (1.2) is nonnegative and concave on [0,1]. And define

$$P = \{u \in E; u(t) \ge 0, u(t) \ \text{ is } \ \text{concave} \ \ \text{function}, t \in [0,1]\},$$

then P is a cone.

For notational convenience, we denote

$$A = \phi_p^{-1}\left(\int_0^1 q(r)dr\right)\max\left\{\left(\phi_p^{-1}(\sum_{i=1}^{m-2}\alpha_i)+1\right),\left(\phi_p^{-1}(\sum_{i=1}^{m-2}\beta_i)+1\right)\right\}.$$

We will prove the following existence result.

Theorem 3.1 *Assume that (H_1), (H_2) hold, and there exists $\alpha > 0$, such that*

(S1) $f(t,x_1,y_1) \le f(t,x_2,y_2)$ *for any* $0 \le t \le 1, 0 \le x_1 \le x_2 \le a, 0 \le |y_1| \le |y_2| \le a$;
(S2) $\max_{0\le t\le 1} f(t,a,a) \le \phi_p(\frac{a}{A})$;
(S3) $f(t,0,0) \ne 0$ *for* $0 \le t \le 1$.

Then the boundary value problem (1.1)-(1.2) has two positive, concave solutions w^* and v^*, such that

$$0 < w^* \le a, \quad 0 < |(w^*)^{'}| \le a$$

and

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} T^n w_0 = w^*, \lim_{n\to\infty}(w_n)^{'} = \lim_{n\to\infty}(T^n w_0)^{'} = (w^*)^{'},$$

where

$$w_0(t) = a\frac{\min\left\{\left(\phi_p^{-1}(\sum_{i=1}^{m-2}\alpha_i)+t\right),\left(\phi_p^{-1}(\sum_{i=1}^{m-2}\beta_i)+1-t\right)\right\}}{\max\left\{\left(\phi_p^{-1}(\sum_{i=1}^{m-2}\alpha_i)+1\right),\left(\phi_p^{-1}(\sum_{i=1}^{m-2}\beta_i)+1\right)\right\}}, \quad 0 \le t \le 1,$$

and

$$0 < v^* \le a, \quad 0 < |(v^*)^{'}| \le a$$

and

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} T^n v_0 = v^*, \lim_{n\to\infty}(v_n)^{'} = \lim_{n\to\infty}(T^n v_0)^{'} = (v^*)^{'},$$

where $v_0 = 0 \quad 0 \leq t \leq 1$,*and*

$$(Tu)(t) = \begin{cases} \phi_p^{-1}\left(\sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^{\sigma} q(r)f(r,u(r),u^{'}(r))dr\right) \\ \qquad + \int_0^t \phi_p^{-1}\left(\int_s^{\sigma} q(r)f(r,u(r),u^{'}(r))dr\right) ds, t \in [0,\sigma], \\ \phi_p^{-1}\left(\sum_{i=1}^{m-2} \beta_i \int_{\sigma}^{\xi_i} q(r)f(r,u(r),u^{'}(r))dr\right) \\ \qquad + \int_t^1 \phi_p^{-1}\left(\int_{\sigma}^{s} q(r)f(r,u(r),u^{'}(r))dr\right) ds, t \in [\sigma,1], \end{cases} \tag{3.1}$$

where $\sigma \in [0,1]$ *is the unique solution of the equation*

$$\begin{aligned} &\phi_p^{-1}\left(\sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^{x} q(r)f(r,u(r),u^{'}(r))dr\right) \\ &\qquad + \int_0^t \phi_p^{-1}\left(\int_s^{x} q(r)f(r,u(r),u^{'}(r))dr)ds\right) \\ &= \phi_p^{-1}\left(\sum_{i=1}^{m-2} \beta_i \int_{x}^{\xi_i} q(r)f(r,u(r),u^{'}(r))dr\right) \\ &\qquad + \int_t^1 \phi_p^{-1}\left(\int_x^{s} q(r)f(r,u(r),u^{'}(r))dr)ds\right). \end{aligned}$$

The iterative schemes in above Theorem 3.1 are $w_1 = Tw_0, w_2 = Tw_1 = T^2w_0$, $w_{n+1} = Tw_n = T^n w_0, n = 1,2,\cdots$, and $v_1 = Tv_0, v_2 = Tv_1 = T^2 v_0, v_{n+1} = Tv_n = T^n v_0, n = 1,2,\cdots$. They start off with a known piecewise linear function and zero function, respectively.

Proof. We define an operator $T : P \to P$ by (3.1), then from the definition of T, we deduce that for each $u \in P$, there is $Tu \in C^{'}[0,1]$ which is nonnegative and satisfies (1.2). Because

$$(Tu)^{'}(t) = \begin{cases} \phi_p^{-1}\left(\int_t^{\sigma} q(r)f(r,u(r),u^{'}(r))dr\right), & t \in [0,\sigma], \\ -\phi_p^{-1}\left(\int_{\sigma}^{t} q(r)f(r,u(r),u^{'}(r))dr\right), & t \in [\sigma,1], \end{cases} \tag{3.2}$$

we can see that $(Tu)^{'}(t)$ is continuous and nonincreasing on $[0,1]$,that is, $(Tu)(t)$ is concave on $[0,1]$, so we have $Tu \in P$. We also obtain that $(Tu)(\sigma)$ is the maximum value of Tu on $[0,1]$, since $(Tu)^{'}(\sigma) = 0$.

In what follows, we will prove that $T : P \to P$ is a completely continuous operator. The continuity of T is obvious because of the continuity of f and q. Now, we prove T is compact. Let $\Omega \in p$ be a bounded set. Then, there exists R, such that $\Omega \in \{x \in P | \|x\| \leq R\}$. For any $x \in \Omega$, we have

$$\begin{aligned} 0 &\leq \int_0^1 q(s)f(s,u(s),u^{'}(s))ds \\ &\leq \max_{s\in[0,1],u\in[0,R],v\in[R,R]} f(s,u,v)q(s) \\ &=: K. \end{aligned}$$

Then, we have

$$|Tu| \leq \phi_p^{-1}(k) \max\left\{ \phi_p^{-1} \sum_{i=1}^{m-2} \alpha_i + 1, \phi_p^{-1} \sum_{i=1}^{m-2} \beta i + 1 \right\},$$

$$|(Tu)^{'}| \leq \phi_p^{-1}(k),$$

and

$$|(\phi_p(Tu)^{'})^{'}| \leq k.$$

The Arzelà-Ascoli theorem guarantees that $T\Omega$ is relatively compact, which means T is compact. Then $T : P \to P$ is a completely continuous, and each fixed point of T in P is a solution of (1.1) (1.2).

We denote

$$P_a = \{u \in P | \|u\| < a\},$$

and

$$\overline{p}_a = \{u \in P | \|u\| \leq a\}.$$

Then, in what follows, we first prove that $T : \overline{p}_a \to \overline{p}_a$. If $u \in \overline{p}_a$, then $\|u\| \leq a$, we have

$$0 \leq u(t) \leq \max_{0\leq t\leq 1} |u(t)| \leq \|u\| \leq a,$$

and

$$|u^{'}(t)| \leq \max_{0\leq t\leq 1} |u^{'}(t)| \leq \|u\| \leq a,$$

So by $(S_1), (S_2)$ we have

$$0 \leq f(t,u(t),u^{'}(t)) \leq f(t,a,a) \leq \max_{0\leq t\leq 1} f(t,a,a) \leq \phi_p\left(\frac{a}{A}\right),\ for\ 0 \leq t \leq 1.$$

In fact,

$$\|Tu\| = \max\left\{\max_{0} |(Tu)(t)|, \max_{0\le t\le 1} |u^{'}(t)|\right\}$$
$$= \max\left\{(Tu)(\sigma), (Tu)^{'}(0), -(Tu)^{'}(1)\right\}.$$

Then, by(3.1) and (3.2) we have

$$\begin{aligned}
&(Tu)(\sigma)\\
=&\phi_p^{-1}\left(\sum_{i=1}^{m-2}\alpha_i \int_{\xi_i}^{\sigma} q(r)f(r,u(r),u^{'}(r))dr\right)\\
&+\int_0^{\sigma}\phi_p^{-1}\left(\int_s^{\sigma} q(r)f(r,u(r),u^{'}(r))dr\right)ds\\
=&\phi_p^{-1}\left(\sum_{i=1}^{m-2}\beta_i \int_{\sigma}^{\xi_i} q(r)f(r,u(r),u^{'}(r))dr\right)\\
&+\int_{\sigma}^{1}\phi_p^{-1}\left(\int_{\sigma}^{s} q(r)f(r,u(r),u^{'}(r))dr\right)ds\\
\le& \max\left\{\frac{a}{A}\phi_p^{-1}\left(\sum_{i=1}^{m-2}\alpha_i\int_0^1 q(r)dr\right)+\frac{a}{A}\phi_p^{-1}\left(\int_0^1 q(r)dr\right),\right.\\
&\left.\frac{a}{A}\phi_p^{-1}\left(\sum_{i=1}^{m-2}\beta_i\int_0^1 q(r)dr\right)+\frac{a}{A}\phi_p^{-1}\left(\int_0^1 q(r)dr\right)\right\}\\
\le& \frac{a}{A}\phi_p^{-1}\left(\int_0^1 q(r)dr\right)\max\left\{\left(\phi_p^{-1}\sum_{i=1}^{m-2}\alpha_i+1\right),\left(\phi_p^{-1}\sum_{i=1}^{m-2}\beta_i+1\right)\right\}\\
=& a,
\end{aligned}$$

and

$$(Tu)^{'}(0) = \phi_p^{-1}\left(\int_0^{\sigma} q(r)f(r,u(r),u^{'}(r))dr\right) \le \frac{a}{A}\phi_p^{-1}\left(\int_0^1 q(r)dr\right) \le a,$$
$$(Tu)^{'}(1) = \phi_p^{-1}\left(\int_{\sigma}^{1} q(r)f(r,u(r),u^{'}(r))dr\right) \le \frac{a}{A}\phi_p^{-1}\left(\int_0^1 q(r)dr\right) \le a,$$

Thus,we obtain that

$$\|Tu\| \le a.$$

We have shown that $T : \overline{p}_a \to \overline{p}_a$.

Let

$$w_0(t) = a\frac{\min\{(\phi_p^{-1}(\sum_{i=1}^{m-2}\alpha_i)+t),(\phi_p^{-1}(\sum_{i=1}^{m-2}\beta_i)+1-t)\}}{\max\{(\phi_p^{-1}(\sum_{i=1}^{m-2}\alpha_i)+1),(\phi_p^{-1}(\sum_{i=1}^{m-2}\beta_i)+1)\}}, \quad 0\le t\le 1.$$

Let $w_1 = Tw_0, w_2 = Tw_1 = T^2w_0$. Then denote $w_{n+1} = Tw_n = T^n w_0, n = 1, 2 \ldots$. Since $T : \overline{p}_a \to \overline{p}_a$,we have $w_n \in T\overline{p}_a \subseteq \overline{p}_a, n = 1, 2 \ldots$. Since T is completely continuous, $\{w_n\}_{n=1}^{\infty}$ is a sequentially compact set. Since

$$\begin{aligned}
w_1(t) &= Tw_0(t)\\
&= \begin{cases}
\phi_p^{-1}\left(\displaystyle\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{\sigma} q(r)f(r,w_0(r),w_0^{'}(r))dr\right)\\
\qquad +\displaystyle\int_0^t \phi_p^{-1}\left(\int_s^{\sigma} q(r)f(r,w_0(r),w_0^{'}(r))dr\right)ds, \quad t\in[0,\sigma],\\
\phi_p^{-1}\left(\displaystyle\sum_{i=1}^{m-2}\beta_i\int_{\sigma}^{\xi_i} q(r)f(r,w_0(r),w_0^{'}(r))dr\right)\\
\qquad +\displaystyle\int_t^1 \phi_p^{-1}\left(\int_{\sigma}^{s} q(r)f(r,w_0(r),w_0^{'}(r))dr\right)ds, \quad t\in[\sigma,1],
\end{cases}\\
&\le \min\left\{\frac{a}{A}\phi_p^{-1}\left(\sum_{i=1}^{m-2}\alpha_i\int_0^1 q(r)dr\right)+t\frac{a}{A}\phi_p^{-1}\left(\int_0^1 q(r)dr\right),\right.\\
&\qquad \left.\frac{a}{A}\phi_p^{-1}\left(\sum_{i=1}^{m-2}\beta_i\int_0^1 q(r)dr\right)+(1-t)\frac{a}{A}\phi_p^{-1}\left(\int_0^1 q(r)dr\right)\right\}\\
&\le \frac{a}{A}\phi_p^{-1}\left(\int_0^1 q(r)dr\right)\min\left\{\left(\phi_p^{-1}\sum_{i=1}^{m-2}\alpha_i+t\right),\left(\phi_p^{-1}\sum_{i=1}^{m-2}\beta_i+1-t\right)\right\}\\
&= a\frac{\min\{(\phi_p^{-1}\sum_{i=1}^{m-2}\alpha_i+t),(\phi_p^{-1}\sum_{i=1}^{m-2}\beta_i+1-t)\}}{\max\{(\phi_p^{-1}\sum_{i=1}^{m-2}\alpha_i+1),(\phi_p^{-1}\sum_{i=1}^{m-2}\beta_i+1)\}}\\
&= w_0(t), \quad 0\le t\le 1,
\end{aligned} .$$

and

$$\begin{aligned}
|w_1^{'}| &= |(Tw_0)^{'}(t)|\\
&= \begin{cases}
\left|\phi_p^{-}\left(\displaystyle\int_t^{\sigma} q(r)f(r,w_0(r),w_0^{'}(r))dr\right)\right|, & t\in[0,\sigma],\\
\left|-\phi_p^{-}\left(\displaystyle\int_{\sigma}^{t} q(r)f(r,w_0(r),w_0^{'}(r))dr\right)\right|, & t\in[\sigma,1],
\end{cases}\\
&\le \frac{a}{A}\phi_p^{-1}\left(\int_0^1 q(r)dr\right)
\end{aligned}$$

$$= \frac{a}{\max\{(\phi_p^{-1}(\sum_{i=1}^{m-2}\alpha_i)+1),(\phi_p^{-1}(\sum_{i=1}^{m-2}\beta_i)+1)\}}$$
$$= |w_0^{'}(t)|, \quad 0 \le t \le 1.$$

Then we obtain

$$w_1(t) \le w_0(t), \quad |w_1^{'}(t)| \le |w_0^{'}(t)|, \ 0 \le t \le 1.$$

So, we have

$$w_2(t) = Tw_1(t) \le Tw_0(t) = w_1(t), \ 0 \le t \le 1,$$

and

$$|w_2^{'}(t)| = |(Tw_1)^{'}(t)| \le |(Tw_0)^{'}(t)| = |w_1^{'}(t)|, \ 0 \le t \le 1.$$

Hence by induction, we have

$$w_{n+1}(t) \le w_n(t), \ |w_{n+1}^{'}(t)| \le |w_n^{'}(t)|, \ 0 \le t \le 1, \ n = 1, 2\dots$$

Thus, there exists $w^* \in \overline{p}_a$ such that $w_n \to w^*$. Applying the continuity of T and $w_{n+1} = Tw_n$, we get $Tw^* = w^*$.

Let $v_0(t) = 0, \ 0 \le t \le 1$. Then $v_0(t) \in \overline{p}_a$.Let $v_1 = Tv_0, v_2 = Tv_1 = T^2v_0$. Then we denote $v_{n+1} = Tv_n = T^nv_0, \ n = 1, 2\dots$. Since $T : \overline{p}_a \to \overline{p}_a$,we have $v_n \in T\overline{p}_a \subseteq \overline{p}_a, \ n = 1, 2\dots$. Since T is completely continuous, we see that $\{v_n\}_{n=1}^{\infty}$ is a sequentially compact set.

Since $v_1 = Tv_0 = T0 \in \overline{p}_a$, we have

$$v_1(t) = Tv_0(t) = (T0)(t) \ge 0, \ 0 \le t \le 1,$$

and

$$|v_1^{'}(t)| = |(Tv_0)^{'}(t)| = |(T0)^{'}(t)| \ge 0, \ 0 \le t \le 1.$$

So

$$v_2(t) = Tv_1(t) \ge Tv_0(t) = v_1(t), \ 0 \le t \le 1,$$

and

$$|v_2^{'}(t)| = |(Tv_1)^{'}(t)| \ge |(Tv_0)^{'}(t)| = |v_1^{'}(t)|, \ 0 \le t \le 1.$$

By an induction argument similar to the above we obtain

$v_{n+1}(t) \ge v_n(t), \ |v_{n+1}^{'}(t)| \ge |v_n^{'}(t)|, \ 0 \le t \le 1, \ n = 1, 2\dots$.

Thus, there exists $v^* \in \overline{p}_a$ such that $v_n \to v^*$. Applying the continuity of T and $v_{n+1} = Tv_n$, we get $Tv^* = v^*$.

If $f(t, 0, 0) \ne 0, 0 \le t \le 1$, then the zero function is not the solution of (1.1)(1.2). Hence we have shown that w^* and v^* are two positive, concave solutions of the problem (1.1)(1.2). The proof is completed. □

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