Nonlinear Functional Analysis and Applications
Vol. 24, No. 3 (2019), pp. 485-509
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

A GENERAL FUNCTIONAL EQUATION

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Abstract. In this paper, we introduce the following functional equation

$$\sum_{i=0}^{k}(-1)^i \left(\begin{array}{c} k \\ i \end{array} \right) f(x+(j-i)y) = k!f(y).$$

where $k \in \mathbb{N}$ and $j = [\frac{k+1}{2}]$. We achieve the general solution of the above functional equation.

1. Introduction and preliminaries

Aczél [1] and Kannappan [6] have treated systematically the following Cauchy equations. We assume that $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x * y) = f(x) \circ f(y) \tag{1.1}$$

[0]Received December 4, 2018. Revised March 5, 2019.
[0]2010 Mathematics Subject Classification: 39B22.
[0]Keywords: Functional equation, general solution, general functional equation.
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such that $*$ or $\circ$ are addition '+' or multiply '.'. If $*$ and $\circ$ is equal to '+', then the equation (1.1) is said an additive functional equation. The additive functional equation is one of equations that have been extensively studied or explored and was solved by, among numerous authors, Aczél, Banach, Gauss, Hamel, Kuczma, Legendre and others, under various hypotheses of the function, domain, and range. For additional information, we refer interesting reader to Kannappan [8] and Kuczma [9].

Among the normed linear spaces, inner product spaces play an important role. The interesting question when a normed linear space is an inner product space led to several characterizations of inner product spaces starting with Fréchet [3], Jordan and von Neumann [5], etc. Functional equations are instrumental in many characterizations. The basic algebraic (norm) condition that makes the normed linear space an inner product space is the parallelogram identity, also known as the J ordan-von Neumann identity (or the Appolonius law),

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for $x, y \in X$ where X is a normed linear spaces. This translates into a functional equation well known as the quadratic functional equation,

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.2}$$

for $x, y \in X$ where X is a normed linear spaces. Many authors have studied the quadratic functional equation (1.2) under various hypotheses of the function, domain and range (see [7, 8, 10, 11, 15]).

In 2001, Rassias [14] introduced the cubic functional equation

$$f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) = 6f(y) \tag{1.3}$$

and investigated the solution and the Ulam-Hyers stability problem for these cubic mappings. It is easy to show that the function $f(x) = x^3$ satisfies the functional equation (1.3), which is called a cubic functional equation and every solution of the cubic functional equation is said to be a cubic mapping.

The quartic functional equation

$$f(x+2y) + f(x-2y) = 4f(x+y) + 4f(x-y) + 6f(x) + 24f(y) \tag{1.4}$$

was introduced by Rassias [13]. It is easy to show that the function $f(x) = x^4$ is a solution of (1.4). Every solution of the quartic functional equation is said to be a quartic mapping.

Xu *et al.* [16] have proved the general solutions and stability of the quintic functional equation

$$\begin{aligned} &f(x+3y) - 5f(x+2y) + 10f(x+y) - 10f(x) + 5f(x-y) - f(x-2y) \\ &= 120f(y) \end{aligned} \tag{1.5}$$

and the sextic functional equation

$$\begin{aligned} & f(x+3y)-6f(x+2y)+15f(x+y)-20f(x)+15f(x-y) \\ & -6f(x-2y)+f(x+3y)=720f(y). \end{aligned} \tag{1.6}$$

Since $f(x)=x^5$ is a solution of (1.5), we say that it is a quintic functional equation. Similarly, $f(x)=x^6$ is a solution of (1.6), and we say that it is a sextic functional equation. Every solution of the quintic or sextic functional equation is said to be a quintic or sextic mapping, respectively.

We extensively generalize the above results. Assume that $k \in \mathbb{N}$ and $j=[\frac{k+1}{2}]$. We introduce the following functional equation.

$$\sum_{i=0}^{k}(-1)^i \binom{k}{i} f\left(x+(j-i)y\right)=k!f(y). \tag{1.7}$$

If we set $k=1,2,3,4,5$ or 6 in the functional equation (1.7), then we get the additive functional equation, the quadratic functional equation (1.2), the cubic functional equation (1.3), the quartic functional equation (1.4), the quintic functional equation (1.5) or the sextic functional equation (1.6), respectively. See [2, 4, 12, 17] for more information on functional equations.

In this paper, we find the general solution of the functional equation (1.7).

2. Solution of the functional equation (1.7)

In the rest of this paper, unless otherwise explicitly stated, we will assume that $f:\mathbb{R}\to\mathbb{R}$, $k\in\mathbb{N}$ and $j=[\frac{k+1}{2}]$. We start our work with the following lemmas.

Lemma 2.1. *For any $1\le i\le j$, we have*

$$\sum_{d=1}^{2i}(-1)^d \binom{k}{2i+1-d}\binom{k}{d-1}+(-1)^i\binom{k}{i}=\binom{k}{2i} \tag{2.1}$$

and

$$\sum_{d=1}^{2i-1}(-1)^{d-1}\binom{k}{2i-d}\binom{k}{d-1}=\binom{k}{2i-1}. \tag{2.2}$$

Proof. Consider the following identity.

$$(1-z)^k(1+z)^k=(1-z^2)^k. \tag{2.3}$$

It is obvious that the coefficient of z^{2i} of the right side of the functional equation (2.3) is equal to

$$(-1)^i\binom{k}{i} \tag{2.4}$$

and the coefficient of z^{2i} of the left side of the functional equation (2.3) is equal to

$$\sum_{d=0}^{2i}(-1)^d\binom{k}{2i-d}\binom{k}{d}. \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$\sum_{d=0}^{2i}(-1)^d\binom{k}{2i-d}\binom{k}{d} = (-1)^i\binom{k}{i}. \tag{2.6}$$

Then by (2.6), we obtain (2.1).

For second assertion, we consider the functional equation (2.3) again. The coefficient of z^{2i-1} of the left side of (2.3) is equal to

$$\sum_{d=0}^{2i-1}(-1)^d\binom{k}{2i-d-1}\binom{k}{d}.$$

On the other hand, the coefficient of z^{2i-1} of the right side of (2.3) is equal to zero. Thus we have $\sum_{d=0}^{2i-1}(-1)^d\binom{k}{2i-d-1}\binom{k}{d} = 0$. Therefore, we have

$$\binom{k}{2i-1} = \sum_{d=0}^{2i-2}(-1)^d\binom{k}{2i-d-1}\binom{k}{d}.$$

So we conclude that (2.2) holds. □

Lemma 2.2. *For any $1 \le i \le j$, we have*

$$(-1)^i\left\{\binom{k}{i} - \binom{k}{k-i+1}\right\} + \sum_{d=1}^{2i}(-1)^d\binom{k}{2i+1-d} \times\left\{\binom{k}{d-1} + \binom{k}{d-3}\right\} = \binom{k}{2i} + \binom{k}{2i-2} \tag{2.7}$$

and

$$\sum_{d=1}^{2i-1}(-1)^{d-1}\binom{k}{2i-d}\left\{\binom{k}{d-1} + \binom{k}{d-3}\right\} = \binom{k}{2i-1} + \binom{k}{2i-3}. \tag{2.8}$$

Proof. Replacing $2i$ by $2i-2$ in (2.1) and adding the outcome to (2.1), we obtain (2.7). Similarly, replacing $2i-1$ by $2i-3$ in (2.2) and adding the outcome to (2.2), we get (2.8). □

Lemma 2.3. *Let $k = 2j-1$. Then for all $1 \le t \le i-2$, we have the following results:*

(1) *If $j = 2i$, then we have*

$$\begin{aligned}
&\sum_{d=1}^{j}(-1)^{d-1}\left\{\binom{k}{k-2t+1-d}-\binom{k}{k+2t+3-d}\right\}\\
&\quad\times\left\{\binom{k}{d-1}+\binom{k}{d-3}\right\}\\
&=\left\{\binom{k}{j}+\binom{k}{j-2}\right\}\left\{\binom{k}{j-2t-2}-\binom{k}{j-2t-1}\right\}
\end{aligned}\tag{2.9}$$

and

$$\begin{aligned}
&(-1)^{j-t-1}\left\{\binom{k}{j-t-1}-\binom{k}{j+t+1}\right\}\\
&\quad+\sum_{d=1}^{j}(-1)^{d}\left\{\binom{k}{k-2t-d}-\binom{k}{k+2t+4-d}\right\}\\
&\quad\times\left\{\binom{k}{d-1}+\binom{k}{d-3}\right\}\\
&=\left\{\binom{k}{j}+\binom{k}{j-2}\right\}\left\{\binom{k}{j-2t-2}-\binom{k}{j-2t-3}\right\}.
\end{aligned}\tag{2.10}$$

(2) *If $j = 2i-1$, then we have*

$$\begin{aligned}
&(-1)^{j-t-1}\left\{\binom{k}{j-t-1}-\binom{k}{j+t+1}\right\}\\
&\quad+\sum_{d=1}^{j}(-1)^{d}\left\{\binom{k}{k-2t-d}-\binom{k}{k+2t+4-d}\right\}\\
&\quad\times\left\{\binom{k}{d-1}+\binom{k}{d-3}\right\}\\
&=-\left\{\binom{k}{j}+\binom{k}{j-2}\right\}\left\{\binom{k}{j-2t-2}-\binom{k}{j-2t-3}\right\}
\end{aligned}\tag{2.11}$$

and

$$\begin{aligned}&\sum_{d=1}^{j}(-1)^{d-1}\Big\{\binom{k}{k-2t+1-d}-\binom{k}{k+2t+3-d}\Big\}\\&\quad\times\Big\{\binom{k}{d-1}+\binom{k}{d-3}\Big\}\\&=-\Big\{\binom{k}{j}+\binom{k}{j-2}\Big\}\Big\{\binom{k}{j-2t-2}-\binom{k}{j-2t-1}\Big\}.\end{aligned}\tag{2.12}$$

Proof. By (2.9), for all $1 \le t \le i-2$, we have

$$\begin{aligned}&\sum_{d=1}^{j}(-1)^{d-1}\Big\{\binom{k}{k-2t+1-d}-\binom{k}{k+2t+3-d}\Big\}\\&\quad\times\Big\{\binom{k}{d-1}+\binom{k}{d-3}\Big\}\\&=\sum_{d=1}^{j}(-1)^{d-1}\binom{k}{2j-2t-d}\Big\{\binom{k}{d-1}+\binom{k}{d-3}\Big\}\\&\quad+\sum_{d=2t+3}^{j}(-1)^{d}\binom{k}{d-2t-3}\Big\{\binom{k}{d-1}+\binom{k}{d-3}\Big\}.\end{aligned}\tag{2.13}$$

Note that $\binom{k}{d}=\binom{k}{k-d}$ for all $0 \le d \le k$ and $\binom{k}{d}=0$ for all $d>k$.

The second term of the right hand of (2.13) can be written as follows.

$$\begin{aligned}&\sum_{d=2t+3}^{j}(-1)^{d}\binom{k}{d-2t-3}\Big\{\binom{k}{d-1}+\binom{k}{d-3}\Big\}\\&=\binom{k}{0}\Big\{\binom{k}{2t+2}+\binom{k}{2t}\Big\}\\&\quad+\sum_{d=2t+4}^{j}(-1)^{d}\binom{k}{d-2t-3}\Big\{\binom{k}{d-1}+\binom{k}{d-3}\Big\}\\&=\binom{k}{0}\Big\{\binom{k}{2t+2}+\binom{k}{2t}\Big\}\\&\quad+\sum_{d=j+3}^{k-2t}(-1)^{d}\binom{k}{2j-2t-d}\Big\{\binom{k}{d-1}+\binom{k}{d-3}\Big\}.\end{aligned}\tag{2.14}$$

Using (2.8), (2.13) and (2.14), we get

$$
\begin{aligned}
&\sum_{d=1}^{j}(-1)^{d-1}\left\{\binom{k}{k-2t+1-d}-\binom{k}{k+2t+3-d}\right\}\\
&\quad\times\left\{\binom{k}{d-1}+\binom{k}{d-3}\right\}\\
&=\sum_{d=1}^{j}(-1)^{d-1}\binom{k}{2j-2t-d}\left\{\binom{k}{d-1}+\binom{k}{d-3}\right\}\\
&\quad+\binom{k}{0}\left\{\binom{k}{2t+2}+\binom{k}{2t}\right\}\\
&\quad+\sum_{d=j+3}^{k-2t}(-1)^{d}\binom{k}{2j-2t-d}\left\{\binom{k}{d-1}+\binom{k}{d-3}\right\}\\
&=\sum_{d=1}^{k-2t}(-1)^{d}\binom{k}{2j-2t-d}\left\{\binom{k}{d-1}+\binom{k}{d-3}\right\}\\
&\quad-\binom{k}{j-2t-1}\left\{\binom{k}{j}+\binom{k}{j-2}\right\}\\
&\quad+\binom{k}{j-2t-2}\left\{\binom{k}{j+1}+\binom{k}{j-1}\right\}\\
&\quad-\binom{k}{0}\left\{\binom{k}{2t+2}+\binom{k}{2t}\right\}\\
&=\binom{k}{k-2t}+\binom{k}{k-2t-2}-\binom{k}{2t}-\binom{k}{2t+2}\\
&\quad+\left\{\binom{k}{j-2t-2}-\binom{k}{j-2t-1}\right\}\left\{\binom{k}{j}+\binom{k}{j-2}\right\}\\
&=\left\{\binom{k}{j-2t-2}-\binom{k}{j-2t-1}\right\}\left\{\binom{k}{j}+\binom{k}{j-2}\right\}.
\end{aligned}
$$

Therefore, the proof of (2.9) is complete.

Similarly, we can prove (2.10), (2.11) and (2.12). □

Lemma 2.4. *Let $k = 2j$. Then for all $1 \le t \le i-2$, we have the following results.*

(1) *If $j = 2i$, then we have*

$$\begin{aligned}&\sum_{d=1}^{j}(-1)^{d-1}\binom{k}{d-1}\left\{\binom{k}{k-2t-d}+\binom{k}{k+2t+2-d}\right\}\\&=-\binom{k}{j}\binom{k}{j-2t}\end{aligned}\tag{2.15}$$

and

$$\begin{aligned}&(-1)^{j-t-1}\binom{k}{j-t-1}\\&\qquad=\sum_{d=1}^{j}(-1)^{d}\binom{k}{d-1}\left\{\binom{k}{k-2t-1-d}\right.\\&\qquad\quad\left.+\binom{k}{k+2t+3-d}\right\}\\&\qquad=\binom{k}{j}\binom{k}{j-2t-1}.\end{aligned}\tag{2.16}$$

(2) *If $j = 2i-1$, then we have*

$$\begin{aligned}&(-1)^{j-t-1}\binom{k}{j-t-1}\\&+\textstyle\sum_{d=1}^{j}(-1)^{d}\binom{k}{d-1}\left\{\binom{k}{k-2t-1-d}+\binom{k}{k+2t+3-d}\right\}\\&=-\binom{k}{j}\binom{k}{j-2t-2}\end{aligned}\tag{2.17}$$

and

$$\begin{aligned}&\sum_{d=1}^{j}(-1)^{d-1}\binom{k}{d-1}\left\{\binom{k}{k-2t-d}+\binom{k}{k+2t+2-d}\right\}\\&=\binom{k}{j}\binom{k}{j-2t-1}.\end{aligned}\tag{2.18}$$

Proof. Since $k = 2j$ and $j = 2i$, by (2.15), for all $1 \leq t \leq i-2$, we have

$$\begin{aligned}
&\sum_{d=1}^{j}(-1)^{d-1}\binom{k}{d-1}\left\{\binom{k}{k-2t-d}-\binom{k}{k+2t+2-d}\right\}\\
&=\sum_{d=1}^{j}(-1)^{d-1}\binom{k}{d-1}\binom{k}{k-2t-d} \qquad (2.19)\\
&+\sum_{d=2t+2}^{j}(-1)^{d-1}\binom{k}{d-1}\binom{k}{d-2t-2}.
\end{aligned}$$

The second term of the right hand of (2.19) can be written as follows.

$$\begin{aligned}
&\sum_{d=2t+2}^{j}(-1)^{d-1}\binom{k}{d-1}\binom{k}{d-2t-2}\\
&=-\binom{k}{2t+1}\binom{k}{0}+\sum_{d=2t+3}^{j}(-1)^{d-1}\binom{k}{d-1}\binom{k}{d-2t-2} \qquad (2.20)\\
&=-\binom{k}{2t+1}\binom{k}{0}+\sum_{d=j+2}^{2j-2t-1}(-1)^{d-1}\binom{k}{d-1}\binom{k}{k-2t-2}.
\end{aligned}$$

Using (2.2), (2.20) and (2.19), we get

$$\begin{aligned}
&\sum_{d=1}^{j}(-1)^{d-1}\binom{k}{d-1}\left\{\binom{k}{k-2t-d}-\binom{k}{k+2t+2-d}\right\}\\
&=-\binom{k}{2t+1}\binom{k}{0}+\sum_{d=1}^{j}(-1)^{d-1}\binom{k}{d-1}\binom{k}{k-2t-d}\\
&\quad+\sum_{d=j+2}^{2j-2t-1}(-1)^{d-1}\binom{k}{d-1}\binom{k}{k-2t-2}\\
&=-\binom{k}{2t+1}\binom{k}{0}-\binom{k}{j}\binom{k}{j-2t-1}\\
&\quad+\sum_{d=1}^{2j-2t-1}(-1)^{d-1}\binom{k}{d-1}\binom{k}{k-2t-d}\\
&=-\binom{k}{2t+1}\binom{k}{0}-\binom{k}{j}\binom{k}{j-2t-1}+\binom{k}{2t+1}\\
&=-\binom{k}{j}\binom{k}{j-2t-1}.
\end{aligned}$$

Therefore, the proof of (2.15) is complete.

Similarly, we can prove (2.16), (2.17) and (2.18). □

Lemma 2.5. *Assume f satisfies the functional equation* (1.7). *Then we have the following statements*

(1) $f(0) = 0$.
(2) *If $k = 2j$, then we have $f(-x) = f(x)$.*
(3) *If $k = 2j - 1$, then we have $f(-x) = -f(x)$.*

Proof. Letting $x = y = 0$ in (1.7), we get $\sum_{i=0}^{k}(-1)^i \left(\begin{array}{c} k \\ i \end{array} \right) f(0) = k!f(0)$. Then we conclude $0 = (1-1)^k f(0) = k!f(0)$. Thus we obtain $f(0) = 0$. Suppose $k = 2j$. Replacing $x = 0$ and $y = x$ in (1.7), we find

$$\sum_{i=0}^{k}(-1)^i \left(\begin{array}{c} k \\ i \end{array} \right) f\left((j-i)x\right) = k!f(x). \tag{2.21}$$

Also, if we let $x = 0$ and $y = -x$ in (1.7), then we get

$$\sum_{i=0}^{k}(-1)^i \left(\begin{array}{c} k \\ i \end{array} \right) f\left(-(j-i)x\right) = k!f(-x). \tag{2.22}$$

By (2.21) and (2.22), we get

$$\begin{aligned} k!f(-x) &= \sum_{i=0}^{k}(-1)^i \left(\begin{array}{c} k \\ i \end{array} \right) f\left(-(j-i)x\right) \\ &= \sum_{i=0}^{k}(-1)^{k-i} \left(\begin{array}{c} k \\ k-i \end{array} \right) f\left(-(j-k+i)x\right) \\ &= \sum_{i=0}^{k}(-1)^{k} \left(\begin{array}{c} k \\ i \end{array} \right) f\left((j-i)x\right) = k!f(x). \end{aligned}$$

Therefore, we obtain $f(-x) = f(x)$.

In the same manner, we can prove $f(-x) = -f(x)$ if $k = 2j - 1$. □

Lemma 2.6. *Assume f satisfies* (1.7), *and that $k = 2j - 1$. Then we have the following results.*

(1) *For even steps and for $i \geq 1$ and $j \geq 2i$, we have*

$$
\begin{aligned}
&\Big\{(-1)^i\left(\binom{k}{i}-\binom{k}{k-i+1}\right)\\
&+\sum_{d=1}^{2i}(-1)^d\binom{k}{2i-d+1}\left(\binom{k}{d-1}+\binom{k}{d-3}\right)\Big\}\\
&\times f\left((k-2i+1)x\right)\\
&+\sum_{s=i}^{j-max\{2,i\}}\sum_{d=1}^{2i}(-1)^{d-1}\binom{k}{2s+2-d}\left(\binom{k}{d-1}+\binom{k}{d-3}\right)\\
&\times f\left((k-2s)x\right)\\
&+\sum_{r=i+1}^{j-max\{2,i\}}\Big\{\sum_{d=1}^{2i}(-1)^d\binom{k}{2r+1-d}\left(\binom{k}{d-1}+\binom{k}{d-3}\right)\\
&+(-1)^r\left(\binom{k}{r}-\binom{k}{r-1}\right)\Big\}f\left((k-2r+1)x\right)\\
&+\sum_{t=1}^{2i-4}\Big\{\sum_{d=1}^{2i}(-1)^{d+t}\left(\binom{k}{t+d}-\binom{k}{d-t-4}\right)\\
&\times\left(\binom{k}{d-1}+\binom{k}{d-3}\right) \qquad (2.23)\\
&+(-1)^{j-1}\cos(\frac{t\pi}{2})\left(\binom{k}{j-[\frac{t}{2}]-1}-\binom{k}{j+[\frac{t}{2}]+1}\right)\Big\}f\left((t+2)x\right)\\
&-\Big\{k!+(-1)^j\left(\binom{k}{j-1}-\binom{k}{j+1}\right)\\
&+\sum_{d=1}^{2i}(-1)^{d-1}\left(\binom{k}{d}-\binom{k}{d-4}\right)\left(\binom{k}{d-1}+\binom{k}{d-3}\right)\Big\}\\
&\times f\left(2x\right)\\
&+\Big\{k!\sum_{d=1}^{2i}\left(\binom{k}{d-1}+\binom{k}{d-3}\right)\\
&+\sum_{d=1}^{2i}(-1)^{d-1}\left(\binom{k}{d-1}-\binom{k}{d-3}\right)\\
&\times\left(\binom{k}{d-1}+\binom{k}{d-3}\right)\Big\}f\left(x\right)\\
&=0.
\end{aligned}
$$

(2) *For odd steps and for $i \geq 0$ and $j \geq 2i+1$, we have*

$$
\begin{aligned}
&\sum_{d=1}^{2i+1}(-1)^{d}\binom{k}{2i-d+2}\left(\binom{k}{d-1}+\binom{k}{d-3}\right)f\left((k-2i)x\right)\\
&+\sum_{s=i+1}^{j-max\{2,i+1\}}\sum_{d=1}^{2i+1}(-1)^{d-1}\binom{k}{2s+2-d}\\
&\times\left(\binom{k}{d-1}+\binom{k}{d-3}\right)f\left((k-2s)x\right)\\
&+\sum_{r=i+1}^{j-max\{2,i\}}\Bigg\{\sum_{d=1}^{2i+1}(-1)^{d}\binom{k}{2r+1-d}\left(\binom{k}{d-1}+\binom{k}{d-3}\right)\\
&+(-1)^{r}\left(\binom{k}{r}+\binom{k}{r-1}\right)\Bigg\}f\left((k-2r+1)x\right)\\
&+\sum_{t=1}^{2i-3}\Bigg\{\sum_{d=1}^{2i+1}(-1)^{d+t}\left(\binom{k}{t+d}-\binom{k}{d-t-4}\right)\\
&\times\left(\binom{k}{d-1}+\binom{k}{d-3}\right)\\
&+(-1)^{j-1}\cos(\frac{t\pi}{2})\left(\binom{k}{j-[\frac{t}{2}]-1}+\binom{k}{j+[\frac{t}{2}]+1}\right)\Bigg\}f\left((t+2)x\right)\\
&-\Bigg\{k!+(-1)^{j}\left(\binom{k}{j-1}-\binom{k}{j+1}\right)\\
&+\sum_{d=1}^{2i+1}(-1)^{d-1}\left(\binom{k}{d}-\binom{k}{d-4}\right)\left(\binom{k}{d-1}+\binom{k}{d-3}\right)\Bigg\}\\
&\times f\left(2x\right)\\
&+\Bigg\{k!\sum_{d=1}^{2i+1}\left(\binom{k}{d-1}+\binom{k}{d-3}\right)\\
&+\sum_{d=1}^{2i+1}(-1)^{d-1}\left(\binom{k}{d-1}-\binom{k}{d-3}\right)\\
&\times\left(\binom{k}{d-1}+\binom{k}{d-3}\right)\Bigg\}f\left(x\right)\\
&=0.
\end{aligned}
\tag{2.24}
$$

Proof. We prove the assertion by induction. Assume that f satisfies (1.7).

Replacing (x, y) by $(0, 2x)$ in (1.7) and by $f(-x) = -f(x)$, we have

$$\begin{pmatrix} k \\ 0 \end{pmatrix} f(2jx) + \sum_{t=1}^{k} (-1)^t \begin{pmatrix} k \\ t \end{pmatrix} f\left(2(j-t)x\right) - k! f(2x) \\ = 0. \tag{2.25}$$

By $f(-x) = -f(x)$, we can rewrite the right of (2.25) as follows.

$$\begin{pmatrix} k \\ 0 \end{pmatrix} f(2jx) - \left\{ k! + (-1)^j \left(\begin{pmatrix} k \\ j-1 \end{pmatrix} - \begin{pmatrix} k \\ j+1 \end{pmatrix} \right) \right\} f\left(2x\right) \\ + \sum_{t=1}^{j-2} (-1)^t \left(\begin{pmatrix} k \\ t \end{pmatrix} - \begin{pmatrix} k \\ t-1 \end{pmatrix} \right) f\left(2(j-t)x\right) = 0. \tag{2.26}$$

Replacing (x, y) by (jx, x) in (1.7), and multiplying the result by $\begin{pmatrix} k \\ 0 \end{pmatrix}$, we get

$$\begin{pmatrix} k \\ 0 \end{pmatrix} \left\{ f(2jx) + \sum_{t=1}^{k-1} (-1)^t \begin{pmatrix} k \\ t \end{pmatrix} f\left((k-t+1)x\right) - (k!+1) f(x) \right\} = 0. \tag{2.27}$$

It follows from (2.26) and (2.27) that

$$\begin{pmatrix} k \\ 0 \end{pmatrix} (k!+1) f(x) - \begin{pmatrix} k \\ 0 \end{pmatrix} \sum_{t=1}^{k-1} (-1)^t \begin{pmatrix} k \\ t \end{pmatrix} f\left((k-t+1)x\right) \\ + \sum_{t=1}^{j-2} (-1)^t \left(\begin{pmatrix} k \\ t \end{pmatrix} - \begin{pmatrix} k \\ t-1 \end{pmatrix} \right) f\left(2(j-t)x\right) = 0. \tag{2.28}$$

The first summation of (2.28) can be written as follows.

$$\sum_{t=1}^{k-1} (-1)^t \begin{pmatrix} k \\ t \end{pmatrix} f\left((k-t+1)x\right) \\ = \sum_{r=1}^{j-2} \begin{pmatrix} k \\ 2r \end{pmatrix} f\left((k-2r+1)x\right) - \sum_{s=0}^{j-2} (-1)^t \begin{pmatrix} k \\ 2s+1 \end{pmatrix} f\left((k-2s)x\right). \tag{2.29}$$

By (2.28) and (2.29), we conclude

$$\begin{aligned}
&(k!+1)f(x)+\binom{k}{1}f(kx)+\sum_{s=1}^{j-2}\binom{k}{2s+1}f\left((k-2s)x\right)\\
&-\sum_{r=1}^{j-2}\left\{\binom{k}{2r}+(-1)^{r-1}\left(\binom{k}{r}-\binom{k}{r-1}\right)\right\}\\
&\times f\left((k-2r+1)x\right)\\
&-\left\{k!+(-1)^{j}\left(\binom{k}{j-1}-\binom{k}{j+1}+\binom{k}{1}\right)\right\}f\left(2x\right)\\
&=0,
\end{aligned} \tag{2.30}$$

which proves (2.24) for $i=1$.

Note that if $a_0>a_1$, then $\sum_{i=a_0}^{a_1}F(i)=0$. Replacing (x,y) by $((j-1)x,x)$ in (1.7), and multiplying the result by $\binom{k}{1}$, we get

$$\begin{aligned}
&\binom{k}{1}\left\{f(kx)+\sum_{t=1}^{k-2}(-1)^{t}\binom{k}{t}f\left((k-t)x\right)-(k!-\binom{k}{1})f(x)\right\}\\
&=0.
\end{aligned} \tag{2.31}$$

The summation of (2.31) can be written as follows.

$$\begin{aligned}
\sum_{t=1}^{k-2}(-1)^{t}\binom{k}{t}f\left((k-t)x\right)=&\sum_{s=1}^{j-2}(-1)^{t}\binom{k}{2s}f\left((k-2s)x\right)\\
&-\sum_{r=1}^{j-2}(-1)^{t}\binom{k}{2r-1}f\left((k-2r+1)x\right).
\end{aligned} \tag{2.32}$$

It follows from (2.30), (2.31) and (2.32) that

$$\begin{aligned}
&\left\{(\binom{k}{1}+1)k!-\binom{k}{1}\binom{k}{k-1}+1\right\}f(x)\\
&+\left(\binom{k}{1}\binom{k}{1}-\binom{k}{2}-\binom{k}{1}+\binom{k}{k}\right)f((k-1)x)\\
&+\sum_{s=1}^{j-2}\left(\binom{k}{2s+1}-\binom{k}{1}\binom{k}{2s}\right)f\left((k-2s)x\right)
\end{aligned}$$

$$+\sum_{r=2}^{j-2}\left\{\binom{k}{1}\binom{k}{2r-1}-\binom{k}{2r}+(-1)^r\left(\binom{k}{r}-\binom{k}{r-1}\right)\right\}$$
$$\times f\left((k-2r+1)x\right)$$
$$-\left\{k!+\binom{k}{1}+(-1)^j\left(\binom{k}{j-1}-\binom{k}{j+1}\right)-\binom{k}{1}\binom{k}{2}\right\}$$
$$\times f\left(2x\right)$$
$$=0,$$

which proves (2.23) for $i=1$. Here, assume that the induction hypothesis is true for the even step $2i$ (i.e., (2.23)) and prove the odd step $2i+1$ (i.e., (2.24)). Replacing (x,y) by $((j-2i)x,x)$ in (1.7), and multiplying the result by $\left(\binom{k}{2i}+\binom{k}{2i-2}\right)$, we find

$$\left(\binom{k}{2i}+\binom{k}{2i-2}\right)\left\{f((k-2i+1)x)+\sum_{t=1}^{k-4i+1}(-1)^t\binom{k}{t}\right.$$
$$\times f\left((k-t-2i+1)x\right)+\sum_{d=1}^{2i-2}(-1)^d\left(\binom{k}{4i-d-1}-\binom{k}{d-1}\right) \tag{2.33}$$
$$\left.\times f((2i-d)x)-\left(k!+\binom{k}{2i}-\binom{k}{2i-2}\right)f(x)\right\}$$
$$=0.$$

The first summation of (2.33) can be written as follows.

$$\begin{aligned}&\sum_{t=1}^{k-4i+1}(-1)^t\binom{k}{t}f\left((k-t-2i+1)x\right)\\&=\sum_{r=i+1}^{j-max\{2,i\}}\binom{k}{2r-2i}f((k-2r+1)x)\\&\quad-\sum_{s=i}^{j-i-1}\binom{k}{2s-2i+1}f((k-2s)x).\end{aligned} \tag{2.34}$$

By (2.33), (2.23) and (2.34), we obtain (2.24).

On the other hand, assume that the induction hypothesis is true for the odd step $2i+1$ (i.e., (2.24)) and prove the even step $2i$ (i.e., (2.23)).

Replacing (x,y) by $((j-2i-1)x,x)$ in (1.7), and multiplying the result by $\left(\binom{k}{2i+1}+\binom{k}{2i-1}\right)$, we get

$$\begin{aligned}&\left(\binom{k}{2i+1}+\binom{k}{2i-1}\right)\Bigg\{f((k-2i)x)\\&\quad+\sum_{t=1}^{k-4i-1}(-1)^t\binom{k}{t}f\left((k-t-2i)x\right)\\&\quad+\sum_{d=1}^{2i-1}(-1)^d\left(\binom{k}{2i+d+1}-\binom{k}{2i-d-1}\right)f((d+1)x)\\&\quad-\left(k!-\binom{k}{2i+1}-\binom{k}{2i-1}\right)f(x)\Bigg\}\\&=0.\end{aligned}\tag{2.35}$$

The first summation of (2.35) can be written as follows.

$$\begin{aligned}&\sum_{t=1}^{k-4i-1}(-1)^t\binom{k}{t}f\left((k-t-2i)x\right)\\&=\sum_{s=i+1}^{j-max\{2,i+1\}}\binom{k}{2s-2i}f((k-2s)x)\\&\quad-\sum_{r=i+1}^{j-max\{2,i+1\}}\binom{k}{2r-2i-1}f((k-2r+1)x).\end{aligned}\tag{2.36}$$

By (2.35), (2.24) and (2.36), we obtain (2.23). □

Lemma 2.7. *Assume f satisfies (1.7) and that $k=2j$. Then*

(1) *for odd steps and for $i\geq 1$ and $j\geq 2i-1$, we have*

$$\begin{aligned}&\left\{\sum_{d=1}^{2i-1}(-1)^{d-1}\binom{k}{2i-d}\binom{k}{d-1}\right\}f\left((k-2i+1)x\right)\\&+\sum_{s=i+1}^{j-max\{1,i-1\}}\sum_{d=1}^{2i-1}(-1)^{d-1}\binom{k}{2s-d}\binom{k}{d-1}f\left((k-2s+1)x\right)\\&+\sum_{r=i}^{j-max\{2,i\}}\left\{(-1)^r\binom{k}{r}+\sum_{d=1}^{2i-1}(-1)^{d-1}\binom{k}{2r+1-d}\binom{k}{d-1}\right\}\\&\times f\left((k-2r)x\right)\end{aligned}$$

$$+\sum_{t=1}^{2i-4}\Big\{\sum_{d=1}^{2i-1}(-1)^{d+t}\left(\binom{k}{d-t-3}-\binom{k}{t+d+1}\right)\binom{k}{d-1}$$
$$+(-1)^{j-1}\cos(\frac{t\pi}{2})\binom{k}{j-[\frac{t}{2}]-1}\Big\}f\left((t+2)x\right)$$
$$-\Big\{\frac{k!}{2}+(-1)^{j}\binom{k}{j-1}+\sum_{d=1}^{2i-1}(-1)^{d-1}\binom{k}{d-1} \tag{2.37}$$
$$\times\left(\binom{k}{d+1}+\binom{k}{d-3}\right)\Big\}f\left(2x\right)$$
$$+\Big\{k!\sum_{d=1}^{2i-1}\binom{k}{d-1}+\binom{k}{2i-1}\binom{k}{2i-2}\Big\}f\left(x\right)$$
$$=0.$$

(2) *for even steps and for $i \geq 1$ and $j \geq 2i$, we have*

$$\Big\{\sum_{d=1}^{2i}(-1)^{d}\binom{k}{2i-d+1}\binom{k}{d-1}+(-1)^{i}\binom{k}{i}\Big\}f\left((k-2i)x\right)$$
$$+\sum_{s=i+1}^{j-1}\sum_{d=1}^{2i}(-1)^{d-1}\binom{k}{2s-d}\binom{k}{d-1}f\left((k-2s+1)x\right)$$
$$+\sum_{r=i+1}^{j-max\{2,i\}}\Big\{(-1)^{r}\binom{k}{r}+\sum_{d=1}^{2i}(-1)^{d}\binom{k}{2r+1-d}\binom{k}{d-1}\Big\}$$
$$\times f\left((k-2r)x\right)$$
$$+\sum_{t=1}^{2i-3}\Big\{\sum_{d=1}^{2i}(-1)^{d+t}\binom{k}{d-1}\left(\binom{k}{t+d+1}-\binom{k}{d-t-3}\right) \tag{2.38}$$
$$+(-1)^{j-1}\cos(\frac{t\pi}{2})\binom{k}{j-[\frac{t}{2}]-1}\Big\}f\left((t+2)x\right)$$
$$-\Big\{\frac{k!}{2}+(-1)^{j}\binom{k}{j-1}$$
$$+\sum_{d=1}^{2i}(-1)^{d-1}\binom{k}{d-1}\left(\binom{k}{d+1}+\binom{k}{d-3}\right)\Big\}f\left(2x\right)$$
$$+\Big\{k!\sum_{d=1}^{2i}\binom{k}{d-1}-\binom{k}{2i-1}\binom{k}{2i}\Big\}f\left(x\right)=0.$$

Proof. We prove the assertion by induction. Assume that f satisfies (1.7).

Replacing (x, y) by $(0, 2x)$ in (1.7), we have

$$2\begin{pmatrix} k \\ 0 \end{pmatrix} f(kx) + \sum_{t=1}^{k-1}(-1)^t \begin{pmatrix} k \\ t \end{pmatrix} f\left(2(j-t)x\right) - k!f(2x) = 0. \qquad (2.39)$$

By $f(-x) = f(x)$, the summation of (2.39) can be written as follows.

$$\begin{aligned}\sum_{t=1}^{k-1}(-1)^t \begin{pmatrix} k \\ t \end{pmatrix} f\left(2(j-t)x\right) &= \sum_{t=1}^{j-1}(-1)^t \begin{pmatrix} k \\ t \end{pmatrix} f\left(2(j-t)x\right) \\ &\quad + \sum_{t=j+1}^{k-1}(-1)^t \begin{pmatrix} k \\ t \end{pmatrix} f\left(2(j-t)x\right) \\ &= \sum_{t=1}^{j-1}(-1)^t \begin{pmatrix} k \\ t \end{pmatrix} f\left(2(j-t)x\right) + \sum_{t=1}^{j-1}(-1)^t \begin{pmatrix} k \\ t \end{pmatrix} f\left(2(j-t)x\right) \\ &= 2\sum_{t=1}^{j-1}(-1)^t \begin{pmatrix} k \\ t \end{pmatrix} f\left(2(j-t)x\right).\end{aligned} \qquad (2.40)$$

By (2.39) and (2.40), we conclude

$$\begin{aligned}&\begin{pmatrix} k \\ 0 \end{pmatrix} f(kx) - \left\{\frac{k!}{2} + (-1)^j \begin{pmatrix} k \\ j-1 \end{pmatrix}\right\} f\left(2x\right) \\ &+ \sum_{t=1}^{j-2}(-1)^t \begin{pmatrix} k \\ t \end{pmatrix} f\left(2(j-t)x\right) = 0.\end{aligned} \qquad (2.41)$$

Replacing (x, y) by (jx, x) in (1.7), and multiplying the result by $\begin{pmatrix} k \\ 0 \end{pmatrix}$, we find

$$\begin{aligned}&\begin{pmatrix} k \\ 0 \end{pmatrix} \left\{ f(kx) + \sum_{t=1}^{k-2}(-1)^t \begin{pmatrix} k \\ t \end{pmatrix} f\left((k-t)x\right) - \left(k! + \begin{pmatrix} k \\ k-1 \end{pmatrix}\right) f(x) \right\} \\ &= 0.\end{aligned} \qquad (2.42)$$

The summation of (2.42) can be written as follows.

$$\begin{aligned}\textstyle\sum_{t=1}^{k-2}(-1)^t \begin{pmatrix} k \\ t \end{pmatrix} f\left((k-t)x\right) &= \textstyle\sum_{r=1}^{j-1} \begin{pmatrix} k \\ 2r \end{pmatrix} f\left((k-2r)x\right) \\ &\quad - \textstyle\sum_{s=1}^{j-1} \begin{pmatrix} k \\ 2s-1 \end{pmatrix} f\left((k-2s+1)x\right).\end{aligned} \qquad (2.43)$$

It follows from (2.41), (2.42) and (2.43) that

$$\begin{aligned}
&\left(k!+\binom{k}{k-1}\right)f(x)-\left(\frac{k!}{2}+(-1)^j\binom{k}{j-1}+\binom{k}{k-2}\right)f(2x)\\
&\quad+\binom{k}{1}f((k-1)x)+\sum_{s=2}^{j-1}\binom{k}{2s-1}f\left((k-2s+1)x\right)\\
&\quad+\sum_{r=1}^{j-2}\left((-1)^r\binom{k}{r}-\binom{k}{2r}\right)f\left((k-2r)x\right)\\
&=0,
\end{aligned} \tag{2.44}$$

which proves (2.37) for $i=1$.

Replacing (x,y) by $((j-1)x,x)$ in (1.7) and multiplying the result by $\binom{k}{1}$, we get

$$\begin{aligned}
\binom{k}{1}\Big\{f((k-1)x)+\sum_{t=1}^{k-3}(-1)^t\binom{k}{t}f\left((k-t-1)x\right)\\
-(k!-\binom{k}{k}-\binom{k}{k-2})f(x)\Big\}=0.
\end{aligned} \tag{2.45}$$

The summation of (2.45) can be written as follows.

$$\begin{aligned}
\textstyle\sum_{t=1}^{k-3}(-1)^t\binom{k}{t}f\left((k-t-1)x\right)=\sum_{s=2}^{j-1}\binom{k}{2s-2}f\left((k-2s+1)x\right)\\
-\textstyle\sum_{r=1}^{j-1}\binom{k}{2r-1}f\left((k-2r)x\right).
\end{aligned} \tag{2.46}$$

It follows from (2.44), (2.45) and by (2.46) that

$$\begin{aligned}
&\left\{(\binom{k}{1}+1)k!-\binom{k}{1}\binom{k}{2}\right\}f(x)\\
&\quad+\left\{\binom{k}{1}\binom{k}{1}-\binom{k}{2}+\binom{k}{1}\right\}f((k-1)x)\\
&\quad+\sum_{s=2}^{j-1}\left\{\binom{k}{2s-1}-\binom{k}{1}\binom{k}{2s-2}\right\}f\left((k-2s+1)x\right)\\
&\quad+\sum_{r=2}^{j-2}\left\{\binom{k}{1}\binom{k}{2r-1}-\binom{k}{2r}+(-1)^r\binom{k}{r}\right\}f\left((k-2r+1)x\right)\\
&\quad-\left\{\frac{k!}{2}+\binom{k}{2}+(-1)^j\binom{k}{j-1}-\binom{k}{1}\binom{k}{3}\right\}f\left(2x\right)=0,
\end{aligned}$$

which proves (2.38) for $i = 1$. Here assume that the induction hypothesis is true for the even step $2i$ (i.e., (2.38)) and prove the odd step $2i+1$ (i.e., (2.37)).

Replacing (x, y) by $((j-2i)x, x)$ in (1.7), and multiplying the result by $\binom{k}{2i}$, we find

$$\begin{aligned}&\binom{k}{2i}\Big\{f((k-2i)x)+\sum_{t=1}^{k-4i-1}(-1)^t\binom{k}{t}f\left((k-t-2i)x\right)\\&\quad+\sum_{d=1}^{2i-1}(-1)^{d-1}\left(\binom{k}{4i-d+1}+\binom{k}{d-1}\right)f((2i+1-d)x)\qquad(2.47)\\&\quad-\left(k!+\binom{k}{2i+1}+\binom{k}{2i-2}\right)f(x)\Big\}\\&=0.\end{aligned}$$

The first summation of (2.47) can be written as follows.

$$\begin{aligned}&\sum_{t=1}^{k-4i-1}(-1)^t\binom{k}{t}f\left((k-t-2i)x\right)=\sum_{r=i+1}^{j-i-1}\binom{k}{2r-2i}f((k-2r)x)\\&-\sum_{s=i+1}^{j-i}\binom{k}{2s-2i-1}f((k-2s+1)x).\qquad(2.48)\end{aligned}$$

By (2.47), (2.38) and (2.48), we obtain (2.37).

On the other hand, assume that the induction hypothesis is true for the odd step $2i+1$ (i.e., (2.37)) and prove the even step $2i$ (i.e., (2.38)). Replacing (x, y) by $((j-2i+1)x, x)$ in (1.7), and multiplying the result by $\binom{k}{2i-1}$, we get

$$\begin{aligned}&\binom{k}{2i-1}\Big\{f((k-2i+1)x)+\sum_{t=1}^{k-4i+1}(-1)^t\binom{k}{t}f\left((k-t-2i+1)x\right)\\&\quad+\sum_{d=1}^{2i-2}(-1)^{d}\left(\binom{k}{4i-d-1}+\binom{k}{d-1}\right)f((2i-d)x)\qquad(2.49)\\&\quad-\left(k!-\binom{k}{2i}-\binom{k}{2i-2}\right)f(x)\Big\}\\&=0.\end{aligned}$$

The first summation of (2.49) can be written as follows.

$$\begin{aligned}&\sum_{t=1}^{k-4i+1}(-1)^t\binom{k}{t}f\left((k-t-2i+1)x\right) \\ &=\sum_{s=i+1}^{j-i}\binom{k}{2s-2i}f((k-2s+1)x)-\sum_{r=i}^{j-i}\binom{k}{2r-2i+1}f((k-2r)x).\end{aligned}\tag{2.50}$$

By (2.49), (2.37) and (2.50), we obtain (2.38). □

In the following theorem, we investigate the general solution of the functional equation (1.7).

Theorem 2.8. *Assume that f satisfies the functional equation* (1.7)*. Then we have $f(2x)=2^k f(x)$.*

Proof. First, we assume that $k=2j-1$ and $j=2i$ with $i\geq 2$. Using (2.23), we have

$$\begin{aligned}&\Big\{(-1)^i\left(\binom{k}{i}-\binom{k}{k-i+1}\right) \\ &+\sum_{d=1}^{j}(-1)^d\binom{k}{j-d+1}\left(\binom{k}{d-1}+\binom{k}{d-3}\right)\Big\}f\left(jx\right) \\ &+\sum_{s=i}^{i}\sum_{d=1}^{j}(-1)^{d-1}\binom{k}{2s+2-d} \\ &\times\left(\binom{k}{d-1}+\binom{k}{d-3}\right)f\left((k-2s)x\right) \\ &+\sum_{t=1}^{j-4}\Big\{\sum_{d=1}^{j}(-1)^{d+t}\left(\binom{k}{d+t}-\binom{k}{d-t-4}\right) \\ &\times\left(\binom{k}{d-1}+\binom{k}{d-3}\right) \\ &+(-1)^{j-1}\cos(\frac{t\pi}{2})\left(\binom{k}{j-[\frac{t}{2}]-1}-\binom{k}{j+[\frac{t}{2}]+1}\right)\Big\}f\left((t+2)x\right) \\ &-\Big\{k!+(-1)^j\left(\binom{k}{j-1}-\binom{k}{j+1}\right) \\ &+\sum_{d=1}^{j}(-1)^{d-1}\left(\binom{k}{d}-\binom{k}{d-4}\right)\left(\binom{k}{d-1}+\binom{k}{d-3}\right)\Big\}f\left(2x\right)\end{aligned}\tag{2.51}$$

$$
\begin{aligned}
&+\left\{k!\sum_{d=1}^{j}\left(\binom{k}{d-1}+\binom{k}{d-3}\right)\right.\\
&\left.+\sum_{d=1}^{j}(-1)^{d-1}\left(\binom{k}{d-1}-\binom{k}{d-3}\right)\left(\binom{k}{d-1}+\binom{k}{d-3}\right)\right\}\\
&=0.
\end{aligned}
$$

By (2.7) and $j=2i$, the coefficient of $f(jx)$ in (2.51) can be written as follows,

$$
\text{Coefficient of } f(jx)=\binom{k}{j}+\binom{k}{j-2}. \tag{2.52}
$$

By (2.8) and $k=2j-1$, the coefficient of $f\left((k-2s)x\right)$ in (2.51) can be written as follows.

$$
\begin{aligned}
&\sum_{s=i}^{i}\sum_{d=1}^{j}(-1)^{d-1}\binom{k}{2s+2-d}\left(\binom{k}{d-1}+\binom{k}{d-3}\right)f((k-2s)x)\\
&=\sum_{d=1}^{j}(-1)^{d-1}\binom{k}{j+2-d}\left(\binom{k}{d-1}+\binom{k}{d-3}\right)f((j-1)x)\\
&=\left\{\sum_{d=1}^{j+1}(-1)^{d-1}\binom{k}{j+2-d}\left(\binom{k}{d-1}+\binom{k}{d-3}\right)\right.\\
&\quad\left.-\binom{k}{1}\left(\binom{k}{j}+\binom{k}{j-2}\right)\right\}f\left((j-1)x\right)\\
&=\left\{\binom{k}{j+1}+\binom{k}{j-1}-\binom{k}{1}\left(\binom{k}{j}+\binom{k}{j-2}\right)\right\}\\
&\quad\times f\left((j-1)x\right)\\
&=\left\{-\left(\binom{k}{1}-1\right)\left(\binom{k}{j}+\binom{k}{j-2}\right)\right\}f\left((j-1)x\right).
\end{aligned} \tag{2.53}
$$

For $k=2j-1$, the coefficient of $f\left((t+2)x\right)$ in (2.51) can be written by two parts. By (2.9) and $t=2s-1$ with $1\le s\le i-2$, we have

$$\begin{aligned}&\sum_{d=1}^{j}(-1)^{d+2s-1}\left(\binom{k}{d+2s-1}-\binom{k}{d-2s-3}\right)\\&\quad\times\left(\binom{k}{d-1}+\binom{k}{d-3}\right)\\&=\left(\binom{k}{j}+\binom{k}{j-2}\right)\left(\binom{k}{j-2s-2}-\binom{k}{j-2s-1}\right)\end{aligned}\tag{2.54}$$

and by (2.10) and $t = 2r$ with $1 \leq r \leq i - 2$, we have

$$\begin{aligned}&\sum_{d=1}^{j}(-1)^{d+2r}\left(\binom{k}{d+2r}-\binom{k}{d-2r-4}\right)\\&\quad\times\left(\binom{k}{d-1}+\binom{k}{d-3}\right)\\&\quad+(-1)^{j-r-1}\left(\binom{k}{j-r-1}+\binom{k}{j+r+1}\right)\\&=\left(\binom{k}{j}+\binom{k}{j-2}\right)\left(\binom{k}{j-2r-2}-\binom{k}{j-2r-3}\right).\end{aligned}\tag{2.55}$$

By (2.10), the coefficient of $f(2x)$ in (2.51) can be written as follows,

$$\begin{aligned}&\text{Coeff. of } f(2x)\\&=-k!+\left(\binom{k}{j}+\binom{k}{j-2}\right)\left(\binom{k}{j-2}-\binom{k}{j-3}\right).\end{aligned}\tag{2.56}$$

By (2.9), the coefficient of $f(x)$ in (2.51) can be written as follows,

$$\begin{aligned}\text{Coeff. of } f(x) &= \left(\sum_{d=1}^{j}\binom{k}{d-1}+\sum_{d=3}^{j}\binom{k}{d-3}\right)k!\\&\quad+\left(\binom{k}{j}+\binom{k}{j-2}\right)\left(\binom{k}{j-2}-\binom{k}{j-1}\right)\\&=\left(\sum_{d=0}^{k}\binom{k}{d}-\binom{k}{j}-\binom{k}{j-2}\right)k!\\&\quad+\left(\binom{k}{j}+\binom{k}{j-2}\right)\left(\binom{k}{j-2}-\binom{k}{j-1}\right)\\&=\left(2^k-\binom{k}{j}-\binom{k}{j-2}\right)k!\\&\quad+\left(\binom{k}{j}+\binom{k}{j-2}\right)\left(\binom{k}{j-2}-\binom{k}{j-1}\right).\end{aligned}\tag{2.57}$$

Similarly, we determine the coefficients of (2.24). For $k = 2j - 1$ and $j = 2i$ we conclude

$$\begin{aligned}
&\text{Coeff. of } f(jx) = \binom{k}{j-2} + \binom{k}{j},\\
&\text{Coeff. of } f((j-1)x) = -\left(\binom{k}{j-2} + \binom{k}{j}\right)\left(\binom{k}{1} - \binom{k}{k}\right),\\
&\text{Coeff. of } f((2s+1)x)\\
&= -\left(\binom{k}{j-2} + \binom{k}{j}\right)\left(\binom{k}{j-2s-1} - \binom{k}{j+2s+1}\right),\\
&\text{for } 1 \le s \le i-2,\\
&\text{Coeff. of } f((2r+2)x)\\
&= \left(\binom{k}{j-2} + \binom{k}{j}\right)\left(\binom{k}{j-2r-2} - \binom{k}{j+2r+2}\right),\\
&\text{for } 1 \le r \le i-2,\\
&\text{Coeff. of } f(2x) = \left(\binom{k}{j-2} + \binom{k}{j}\right)\left(\binom{k}{j-2} - \binom{k}{j+2}\right),\\
&\text{Coeff. of } f(x)\\
&= -\left(k! + \binom{k}{j-1} - \binom{k}{j+1}\right)\left(\binom{k}{j} + \binom{k}{j-2}\right).
\end{aligned} \tag{2.58}$$

Finally, by (2.51)–(2.58) and subtracting (2.24) from (2.23), we obtain

$$k!f(2x) = 2^k k!f(x).$$

Therefore, we have $f(2x) = 2^k f(x)$. This is the end of proof in this case.

The proofs of the other situations, $k = 2j - 1$ with $j = 2i + 1$, $k = 2j$ with $j = 2i$ and $k = 2j$ with $j = 2i - 1$, can be obtained in the same way. □

Acknowledgments The third author was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF- 201807042748).

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