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GENERALIZED OSTROWSKI TYPE INEQUALITIES FOR INTERVAL VALUED FUNCTIONS AND ITS APPLICATION FOR ERROR ESTIMATION TO SOME QUADRATURE RULES

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Abstract. This work is devoted to study some Ostrowski type inequalities for generalized Hukukara differentiable interval valued functions. Mean valued theorem for gH-differentiable interval valued functions is also derived. Established results generalize some pioneer results in the existing state-of-art. Moreover, error estimations to quadrature rule of Riemann type, Simpson type and Trapezoidal type are obtained for interval valued functions by the application of one of the derived inequality. In addition an illustrative example is also furnished which demonstrates the viability of the hypothesis of one of our results.

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1. Introduction

Let $\Im : [\vartheta_1, \vartheta_2] \to R$ be differentiable on $[\vartheta_1, \vartheta_2]$ and $\Im'$ is bounded on $(\vartheta_1, \vartheta_2)$. Then for all $y \in [\vartheta_1, \vartheta_2]$ the Ostrowski inequality [18] is given by

$$\left| \frac{1}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} \Im(t)dt - \Im(y) \right| \leq \left(\frac{1}{4} + \left(\frac{y - \frac{\vartheta_1 + \vartheta_2}{2}}{\vartheta_2 - \vartheta_1} \right)^2 \right) (\vartheta_2 - \vartheta_1) \|\Im'\|_\infty. \tag{1.1}$$

The constant $\frac{1}{4}$ can't be replaced by any smaller constant and hence it is the best possible constant for the inequality (1.1) (see, [19]). The Ostrowski inequality has been investigated for its different versions and applications throughout the course of time by numerous researchers. Some of its versions and applications can be seen in [16], [7], [9] and in the references there in. A detailed discussion about the topic can be seen in Dragomir and Rassias [19] and Anastassiou [6].

On the other hand, set valued analysis is very important tool to study the classical and applied mathematics ([11], [12]). Optimal control theory, mathematical programming, dynamical games were motivating reasons for researchers to present advanced set valued analysis [10]. In particular, Interval analysis plays a very important role in the study of interval valued optimization (e.g., see [3], [4], [21] and the references there in), interval valued differential equations, random set, fuzzy theory etc. The first book was written by Moore [17] to deal the interval analysis. Now deriving the Ostrowski type inequality for interval valued differentiable functions is surely an interesting move. In fact, this will wider the application domain of Ostrowski inequality. In this direction, in particular, utilizing the idea of Hukuhara differentiability of interval valued functions the Ostrowski type inequalities were extended to the fuzzy valued functions by Anastassiou [5]. Since an interval valued function is also a fuzzy valued function, therefore the fuzzy Ostrowski type inequalities presented in Anastassiou [5] are also valid for interval valued functions. However, the concept of H-derivative for interval valued functions is very restrictive ([2], [22]). The most general concept of differentiability of interval valued functions is generalized Hukuhara differentiability (gH-differentiability in shot) ([2], [24], [14]). Utilizing this concept, Chalco-Cano et al. [23] derived some Ostrowski type inequalities for gH-differentiable interval valued functions. Ostrowski type inequalities and its applications have also been discussed in ([25], [1]) for interval valued functions in the settings of gH-differentiability. The present paper aims to discuss Ostrowski type inequalities for interval valued functions which are assumed to be gH-differentiable. Our findings improve and generalize the results in ([23], [25]). Moreover, as a consequence of our study, we investigate an error estimation to quadrature rule of Riemann, Simpson and

Trapezoidal type for interval valued functions. The paper is categorized in the following sections:

Section 2 presents interval arithmetic. Section 3 is set to present calculus for interval valued functions. Section 4 is devoted to derive main results. Section 5 presents the applications of Theorem 4.3 obtained in section 4 in order to derive error estimations to a the quadrature rules of Riemann type, Simpson type and Trapezoidal type for interval valued functions. Section 6 presents a numerical example. Finally, we conclude in section 7.

2. Interval arithmetic

Let $\mathcal{K}_c$ denote the family of all compact convex nonempty interval on R, that is,

$$\mathcal{K}_c = \{A = [\lambda^L, \lambda^U] | \lambda^L, \lambda^U \in R, \lambda^L \leq \lambda^U\}.$$

The Hausdorff metric on $\mathcal{K}_c$ is defined by

$$H(A, B) = \max\{d(A, B), d(B, A)\},$$

where $A, B \in \mathcal{K}_c$, $d(A, B) = \max_{a \in A} d(a, B)$ and $d(a, B) = \min_{b \in B} |a - b|$. Then $(\mathcal{K}_c, H)$ forms a complete metric space (see, Aubin and Cellina [11]). Assume that $A = [\lambda^L, \lambda^U]$, $B = [\beta^L, \beta^U] \in \mathcal{K}_c$ and $K \in R$, then by definition we have

$$A + B = \{\lambda + \beta : \lambda \in A \text{ and } \beta \in B\} = [\lambda^L + \beta^L, \lambda^U + \beta^U], \tag{2.1}$$

$$KA = K[\lambda^L, \lambda^U] = \begin{cases} [K\lambda^L, K\lambda^U], \text{if } \ K \geq 0, \\ \\ [K\lambda^U, K\lambda^L], \text{if } \ K < 0. \end{cases} \tag{2.2}$$

Aubin and Cellena [11] and Assev [20] have shown that the space $\mathcal{K}_c$ is not a linear space with operations (2.1) and (2.2). Since it does not contain inverse element and therefore subtraction is not well defined. However $\mathcal{K}_c$ is a quasilinear space (see, [20]). In view of (2.1), we see that $-A = -[\lambda^L, \lambda^U] = [-\lambda^U, -\lambda^L]$ and $A - B = A + (-B) = [\lambda^L - \beta^U, \lambda^U - \beta^L]$. However, this definition is having a serious drawback, i.e., is $A - A \neq \{0\}$ in general. This situation is partially solved by the definition called Hukuhara difference. It states that the Hukuhara difference $A \ominus_H B = C = [\gamma^L, \gamma^U]$ exists if $\lambda^L - \beta^L \leq \lambda^U - \beta^U$, where $\gamma^L = \lambda^L - \beta^L$ and $\gamma^U = \lambda^U - \beta^U$ [8].

Next, in Stefanini and Bede [14], the concept of the generalization of H-difference of two intervals has been introduced as follows.

Definition 2.1. ([14]) Let A,$B \in \mathcal{K}_c$. The generalized Hukuhara difference (gH-difference) is defined as

$$A \ominus_{gH} B = C \Longleftrightarrow \begin{cases} & (i)\, A = B + C, \\ or & (ii) B = A + (-1)C. \end{cases}$$

Also, for any two intervals $A = [\lambda^L, \lambda^U]$, $B = [\beta^L, \beta^U] \in \mathcal{K}_c$, $A \ominus_{gH} B$ always exists and $A \ominus_{gH} B = [\min\{\lambda^L - \beta^L, \lambda^U - \beta^U\}, \max\{\lambda^L - \beta^L, \lambda^U - \beta^U\}]$.

For more details on the topic one is referred to ([14], [15]).

3. Calculus for interval valued functions

The function $\Im : R^n \to \mathcal{K}_c$ defined on Euclidean space R^n is said to be the interval valued function. That is $\Im(x) = \Im(x_1, ..., x_n)$ is a closed interval in R for each $x \in R^n$. The interval valued function $\Im(x)$ can also be written as $\Im(x) = [\Im^L(x), \Im^U(x)]$, where $\Im^L$ and $\Im^U$ are real valued functions such that $\Im^L(x) \leq \Im^U(x)$ for every $x \in R^n$ and are known as lower and upper end point functions respectively.

Let $\Im : R^n \to \mathcal{K}_c$ be an interval valued function. Then, $\Im$ is said to be continuous at x_0 if $\lim_{x \to x_0} \Im(x) = \Im(x_0)$, where the limits are taken in the metric space $(\mathcal{K}_c, H)$. Consequently, $\Im$ is continuous at x_0 if and only if $\Im^L$ and $\Im^U$ are continuous functions at x_0.

Now, we denote by $C([\vartheta_1, \vartheta_2], \mathcal{K}_c)$ the family of continuous interval valued functions defined on interval $[\vartheta_1, \vartheta_2]$. Then, $C([\vartheta_1, \vartheta_2], \mathcal{K}_c)$ is a quasilinear spaces [20]. For this space, we define a quasinorm $\|.\|_\infty$ as follows

$$\|\Im\|_\infty = \sup_{t \in [\vartheta_1, \vartheta_2]} H(\Im(t), \{0\}).$$

For more details on the topic one is referred to [20].

Definition 3.1. ([11]) The integral (Aumann integral) of an interval valued function $\Im : [\vartheta_1, \vartheta_2] \to \mathcal{K}_c$ is defined as

$$\int_{x_1}^{x_2} \Im(x) dx = \left\{ \int_{x_1}^{x_2} G(x) dx | G \in S(\Im) \right\},$$

where $S(\Im)$ is the set of all integrable selectors of $\Im$, that is $S(\Im) = \{G : [\vartheta_1, \vartheta_2] \to R | G$ is integrable and $G(x) \in \Im(x)$ $a.e.\}$. If $S(\Im) \neq \phi$, then the integral exist and $\Im$ is said to be Aumann integrable.

We remark that if $\Im$ is measurable, then it has a measurable selector (see, [11], [12]) which is integrable. Therefore $S(\Im) \neq \phi$. However, we have the following theorem.

Theorem 3.2. ([11]) *Assume that* $\Im : [\vartheta_1, \vartheta_2] \to \mathcal{K}_c$ *is a measurable and integrally bounded interval valued function. Then it is integrable and* $\int_{\vartheta_1}^{\vartheta_2} \Im(x)dx \in \mathcal{K}_c$.

Corollary 3.3. ([11]) *A continuous interval valued function* $\Im : [\vartheta_1, \vartheta_2] \to \mathcal{K}_c$ *is integrable.*

The integral (Aumann integral) of interval valued functions satisfies the following properties.

Proposition 3.4. ([11]) *Assume that* $\Im_1, \Im_2 : [\vartheta_1, \vartheta_2] \to \mathcal{K}_c$ *are two measurable and integrally bounded interval valued functions. Then*

(1) $\int_{x_1}^{x_2} (\Im_1(x) + \Im_2(x))dx = \int_{x_1}^{x_2} \Im_1(x)dx + \int_{x_1}^{x_2} \Im_2(x)dx$,

(2) $\int_{x_1}^{x_2} \Im_1(x)dx = \int_{x_1}^{\mu} \Im_1(x)dx + \int_{\mu}^{x_2} \Im_1(x)dx, \ x_1 < \mu < x_2$.

Theorem 3.5. ([2]) *Assume that* $\Im : [\vartheta_1, \vartheta_2] \to \mathcal{K}_c$ *is a measurable and integrally bounded interval valued function such that* $\Im(x) = [\Im^L(x), \Im^U(x)]$. *Then* $\Im^L$ *and* $\Im^U$ *are integrable functions and*

$$\int_{x_1}^{x_2} \Im(x)dx = \left[\int_{x_1}^{x_2} \Im^L(x)dx, \int_{x_1}^{x_2} \Im^U(x)dx \right].$$

Next, based on H-difference, the H-derivative (derivative in the sense Hukuhara) of interval valued functions is having limitations, details can be seen in ([2], [23], [22], [24]). To avoid such difficulty, [14] introduced generalized Hukuhara differentiability of interval valued function and studied its properties.

Definition 3.6. ([14]) The gH-derivative of an interval valued function $\Im : [\vartheta_1, \vartheta_2] \to \mathcal{K}_c$ at $x_0 \in [\vartheta_1, \vartheta_2]$ is defined as

$$\Im'(x_0) = \lim_{h \to 0} \frac{\Im(x_0 + h) \ominus_{gH} \Im(x_0)}{h}. \tag{3.1}$$

If $\Im'(x_0)$ exists in $\mathcal{K}_c$, then we say that $\Im$ is generalized Hukuhara differentiable (gH-differentiable) at x_0.

Further we have the following result.

Theorem 3.7. ([22]) *Assume that* $\Im : [\vartheta_1, \vartheta_2] \to \mathcal{K}_c$ *is an interval valued function such that* $\Im(x) = [\Im^L(x), \Im^U(x)]$. *Then,* $\Im$ *is* gH*-differentiable at* $x_0 \in [\vartheta_1, \vartheta_2]$ *if and only if one of the following cases holds.*

(1) $\Im^L$ *and* $\Im^U$ *are differentiable at* x_0 *and*

$$\Im'(x_0) = [\min\{\Im^{L'}(x_0), \Im^{U'}(x_0)\}, \max\{\Im^{L'}(x_0), \Im^{U'}(x_0)\}];$$

(2) *The derivatives* ${\Im^{L}}'_{-}(x_0), {\Im^{L}}'_{+}(x_0), {\Im^{U}}'_{-}(x_0)$ *and* ${\Im^{U}}'_{+}(x_0)$ *exist and satisfy* ${\Im^{L}}'_{-}(x_0) = {\Im^{U}}'_{+}(x_0)$ *and* ${\Im^{L}}'_{+}(x_0) = {\Im^{U}}'_{-}(x_0)$.

$$\Im'(x_0) = [\min\{{\Im^{L}}'_{-}(x_0), {\Im^{U}}'_{-}(x_0)\}, \max\{{\Im^{L}}'_{-}(x_0), {\Im^{U}}'_{-}(x_0)\}]$$

$$= [\min\{{\Im^{L}}'_{+}(x_0), {\Im^{U}}'_{+}(x_0)\}, \max\{{\Im^{L}}'_{+}(x_0), {\Im^{U}}'_{+}(x_0)\}].$$

Definition 3.8. ([14]) We say that

(1) $\Im(x) = [\Im^L(x), \Im^U(x)]$ is differentiable at $x_0 \in [\vartheta_1, \vartheta_2]$ in the first form if $\Im^L$ and $\Im^U$ are differentiable at x_0 and

$$\Im'(x_0) = [{\Im^{L}}'(x_0), {\Im^{U}}'(x_0)].$$

(2) $\Im(x) = [\Im^L(x), \Im^U(x)]$ is differentiable at $x_0 \in [\vartheta_1, \vartheta_2]$ in the second form if $\Im^L$ and $\Im^U$ are differentiable at x_0 and

$$\Im'(x_0) = [{\Im^{U}}'(x_0), {\Im^{L}}'(x_0)].$$

Furthermore, a point $x_0 \in [\vartheta_1, \vartheta_2]$ is said to be a switching point for the differentiability of $\Im$, if in any neighborhood N_0 of x_0 there exist points $x_1 < x_0 < x_2$ such that

type I : $\Im$ is differentiable at x_1 in the first form while it is not differentiable in the second form, and $\Im$ is differentiable at x_2 in the second form while it is not differentiable in the first form, or

type II : $\Im$ is differentiable at x_1 in the second form while it is not differentiable in the first form, and $\Im$ is differentiable at x_2 in the first form while it is not differentiable in the second form.

Next theorem is the interval version of the second fundamental theorem.

Theorem 3.9. ([24]) *Assume that* $\Im : [\vartheta_1, \vartheta_2] \to \mathcal{K}_c$ *is an interval valued function. If* $\Im$ *is* gH*-differentiable in the first form (or second form) in* $[\vartheta_1, \vartheta_2]$ *then*

$$\int_{\vartheta_1}^{\vartheta_2} \Im'(x)dx = \Im(\vartheta_2) \ominus_{gH} \Im(\vartheta_1).$$

Next, we present a version of mean valued theorem for gH-differentiable interval valued functions. For this we consider consider a function $\Omega : X \times X \to X$, where X is an open subset in R.

In the rest of this paper whenever we say $\Im : [\vartheta_1, \vartheta_1 + \Omega(\vartheta_1, \vartheta_2)] \to \mathcal{K}_C$, it means that $\vartheta_1, \vartheta_2 \in X$ with $\vartheta_1 < \vartheta_1 + \Omega(\vartheta_1, \vartheta_2)$ where X is an open subset of R and $\Omega : X \times X \to X$.

Theorem 3.10. *Assume that* $\Im : [\vartheta_1, \vartheta_1+\Omega(\vartheta_1,\vartheta_2)] \to \mathcal{K}_C$ *is a* gH*-differentiability interval value function defined on* $[\vartheta_1, \vartheta_1+\Omega(\vartheta_1,\vartheta_2)]$ *with a finite number of switching points at* $\vartheta_1 = \rho_0 < \rho_1 < \dots < \rho_r = \vartheta_1+\Omega(\vartheta_1,\vartheta_2)$*. Assume that* $\Im'$ *is continuous. Then*

$$H(\Im(\vartheta_1+\Omega(\vartheta_1,\vartheta_2)), \Im(\vartheta_1)) \le \|\Im'\|_\infty \Omega(\vartheta_1,\vartheta_2). \tag{3.2}$$

Proof. We shall prove this result by induction and we will induct on r. Assume that $r = 0$, that is there is no switching point in $[\vartheta_1, \vartheta_1+\Omega(\vartheta_1,\vartheta_2)]$. Then by Theorem 3.9 we have

$$\begin{aligned}
H(\Im(\vartheta_1+\Omega(\vartheta_1,\vartheta_2)), \Im(\vartheta_1)) &= H(\Im(\vartheta_1+\Omega(\vartheta_1,\vartheta_2)) \ominus_{gH} \Im(\vartheta_1), \{0\}) \\
&= H\left(\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)} \Im'(x)dx, \{0\}\right) \\
&= H\left(\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)} \Im'(x)dx, \int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)} \{0\}dx\right) \\
&\le \int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)} H(\Im'(x), \{0\})dx \\
&\le \|\Im'\|_\infty \Omega(\vartheta_1,\vartheta_2).
\end{aligned}$$

Which establishes the result for $r = 0$. Now assume that $r = 1$, then there exist one switching point in $[\vartheta_1, \vartheta_1+\Omega(\vartheta_1,\vartheta_2)]$. Let ρ_1 be the switching point in $[\vartheta_1, \vartheta_1+\Omega(\vartheta_1,\vartheta_2)]$. Now assume that $\Im$ is differentiable on $[\vartheta_1, \rho_1]$ in the first form and $\Im$ is differentiable on $[\rho_1, \vartheta_1+\Omega(\vartheta_1,\vartheta_2)]$ in the second form. Then

$$\begin{aligned}
&H(\Im(\vartheta_1+\Omega(\vartheta_1,\vartheta_2)), \Im(\vartheta_1)) \\
&\le H(\Im(\vartheta_1+\Omega(\vartheta_1,\vartheta_2)), \Im(\rho_1)) + H(\Im(\rho_1), \Im(\vartheta_1)) \\
&\le (\vartheta_1+\Omega(\vartheta_1,\vartheta_2) - \rho_1) \sup_{x\in[\rho_1, \vartheta_1+\Omega(\vartheta_1,\vartheta_2)]} H(\Im'(x), \{0\}) \\
&\quad + (\rho_1 - \vartheta_1) \sup_{x\in[\vartheta_1,\rho_1]} H(\Im'(x), \{0\}) \\
&\le (\vartheta_1+\Omega(\vartheta_1,\vartheta_2) - \vartheta_1) \sup_{x\in[\vartheta_1, \vartheta_1+\Omega(\vartheta_1,\vartheta_2)]} H(\Im'(x), \{0\}) \\
&\le \|\Im'\|_\infty \Omega(\vartheta_1,\vartheta_2).
\end{aligned}$$

Which establishes the result for $r = 1$. Now assume that the result follows for $r = k$. That is, for k switching points $\vartheta_1 = \rho_0 < \rho_1 < \dots < \rho_k < \rho_{k+1} = \vartheta_1+\Omega(\vartheta_1,\vartheta_2)$, we have

$$H(\Im(\vartheta_1+\Omega(\vartheta_1,\vartheta_2)), \Im(\vartheta_1)) \le (\vartheta_1 - \rho_{k+1}) \sup_{x\in[\vartheta_1,\rho_{k+1}]} H(\Im'(x), \{0\})$$

$$\leq \|\Im'\|_\infty \Omega(\vartheta_1, \vartheta_2). \tag{3.3}$$

Assume that $r = k+1$. That is, there are $k+1$ switching points in $[\vartheta_1, \vartheta_1 + \Omega(\vartheta_1, \vartheta_2)]$. Let $\vartheta_1 = \rho_0 < \rho_1 < ... < \rho_k < \rho_{k+1} < \rho_{k+2} = \vartheta_1 + \Omega(\vartheta_1, \vartheta_2)$ be the switching points such that

(1) if $\Im$ is differentiable on $[\rho_k, \rho_{k+1}]$ in the first form and $\Im$ is differentiable on $[\rho_{k+1}, \vartheta_1 + \Omega(\vartheta_1, \vartheta_2)]$ in the second form, or

(2) if $\Im$ is differentiable on $[\rho_k, \rho_{k+1}]$ in the second form and $\Im$ is differentiable on $[\rho_{k+1}, \vartheta_1 + \Omega(\vartheta_1, \vartheta_2)]$ in the first form.

Then utilizing the properties of Hausdorff metric and (3.3) we have

$$\begin{aligned}
&H(\Im(\vartheta_1+\Omega(\vartheta_1, \vartheta_2)), \Im(\vartheta_1)) \\
&\quad\leq H(\Im(\vartheta_1 + \Omega(\vartheta_1, \vartheta_2)), \Im(\rho_{k+1})) + H(\Im(\rho_{k+1}), \Im(\vartheta_1)) \\
&\quad\leq (\vartheta_1 + \Omega(\vartheta_1, \vartheta_2) - \rho_{k+1}) \sup_{x\in[\rho_{k+1}, \vartheta_1+\Omega(\vartheta_1,\vartheta_2)]} H(\Im'(x), \{0\}) \\
&\qquad + (\rho_{k+1} - \vartheta_1) \sup_{x\in[\vartheta_1, \rho_{k+1}]} H(\Im'(x), \{0\}) \\
&\quad\leq (\vartheta_1 + \Omega(\vartheta_1, \vartheta_2) - \rho_{k+1} + \rho_{k+1} - \vartheta_1) \\
&\qquad \times \sup_{x\in[\vartheta_1, \vartheta_1+\Omega(\vartheta_1,\vartheta_2)]} H(\Im'(x), \{0\}) \\
&\quad\leq \|\Im'\|_\infty \Omega(\vartheta_1, \vartheta_2).
\end{aligned}$$

Which establishes the theorem. □

4. Ostrowski type inequalities

Theorem 4.1. *Assume that the interval valued function* $\Im : [\vartheta_1, \vartheta_1+\Omega(\vartheta_1, \vartheta_2)] \to \mathcal{K}_C$ *is continuously* gH*-differentiable on* $[\vartheta_1, \vartheta_1 + \Omega(\vartheta_1, \vartheta_2)]$ *with a finite number of switching points at* $\vartheta_1 = \rho_0 < \rho_1 < ... < \rho_{n+1} = (\vartheta_1 + \Omega(\vartheta_1, \vartheta_2))$. *Then for* $x \in [\vartheta_1, \vartheta_1 + \Omega(\vartheta_1, \vartheta_2)]$*, we have*

$$H\left(\frac{1}{\Omega(\vartheta_1, \vartheta_2)} \int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)} \Im(y)dy, \Im(x)\right)$$

$$\leq \|\Im'\|_\infty \left(\frac{(x-\vartheta_1)^2 + (\vartheta_1 + \Omega(\vartheta_1, \vartheta_2) - x)^2}{2\Omega(\vartheta_1, \vartheta_2)}\right). \tag{4.1}$$

Proof. In view of the Theorem 3.10 and properties of Hausdorff metric we have

$$
\begin{aligned}
& H\left(\frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\Im(y)dy, \Im(x)\right) \\
&= H\left(\frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\Im(y)dy, \frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\Im(x)dy\right) \\
&\leq \frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)} H(\Im(y),\Im(x))dy \\
&\leq \frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)} \sup_{y\in[\vartheta_1,\vartheta_1+\Omega(\vartheta_1,\vartheta_2)]} H(\Im'(y),\{0\})|y-x|dy \\
&= \frac{1}{\Omega(\vartheta_1,\vartheta_2)} \sup_{y\in[\vartheta_1,\vartheta_1+\Omega(\vartheta_1,\vartheta_2)]} \int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)} H(\Im'(y),\{0\})|y-x|dy \\
&= \|\Im'\|_\infty \left(\frac{(x-\vartheta_1)^2+(\vartheta_1+\Omega(\vartheta_1,\vartheta_2)-x)^2}{2\Omega(\vartheta_1,\vartheta_2)}\right).
\end{aligned}
$$

Which establishes the inequality (4.1). □

Proposition 4.2. *The Theorem* 4.1 *is sharp at* $x=\vartheta_1$, *in fact attained by*

$$\Im(y)=\Omega(\vartheta_1,\vartheta_2)(y-\vartheta_1)C,$$

with $C\in\mathcal{K}_c$ *being fixed.*

Proof. Let $C=[\gamma^L,\gamma^U]$. Since $\Omega(\vartheta_1,\vartheta_2)(y-\vartheta_1)\geq 0$,

$$
\begin{aligned}
\Im(y) =& \Omega(\vartheta_1,\vartheta_2)(y-\vartheta_1)C \\
=& [\Omega(\vartheta_1,\vartheta_2)(y-\vartheta_1)\gamma^L, \Omega(\vartheta_1,\vartheta_2)(y-\vartheta_1)\gamma^U].
\end{aligned}
$$

Now from Theorem 3.7, $\Im$ is continuously gH-differentiable function and $\Im'(y)=\Omega(\vartheta_1,\vartheta_2)C$. Therefore we have

$$
\begin{aligned}
& H\left(\frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\Im(y)dy, \{0\}\right) \\
&= H\left(\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}(y-\vartheta_1)Cdy, \{0\}\right) \\
&= H\left(\left(\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}(y-\vartheta_1)dy\right)C, \{0\}\right) \\
&= H\left(\frac{\Omega^2(\vartheta_1,\vartheta_2)}{2}C, \{0\}\right) = \frac{\Omega^2(\vartheta_1,\vartheta_2)}{2}H\left(C,\{0\}\right),
\end{aligned}
$$

and

$$\left(\sup_{y\in[\vartheta_1,\vartheta_1+\Omega(\vartheta_1,\vartheta_2)]} H(\Im'(y),\{0\})\right)\left(\frac{(x-\vartheta_1)^2+(\vartheta_1+\Omega(\vartheta_1,\vartheta_2)-x)^2}{2\Omega(\vartheta_1,\vartheta_2)}\right)$$
$$=\left(\sup_{y\in[\vartheta_1,\vartheta_1+\Omega(\vartheta_1,\vartheta_2)]} H(\Omega(\vartheta_1,\vartheta_2)C,\{0\})\right)\left(\frac{\Omega^2(\vartheta_1,\vartheta_2)}{2\Omega(\vartheta_1,\vartheta_2)}\right)$$
$$=\frac{\Omega^2(\vartheta_1,\vartheta_2)}{2}H(C,\{0\}).$$

Which proves the sharpness of the theorem. □

Now, we present a result which is more general than Theorem 4.1 and consequently more general than Theorem 4.1 of [23] and Theorem 4 and 5 of [25].

Theorem 4.3. *Assume that the interval valued function* $\Im:[\vartheta_1,\vartheta_1+\Omega(\vartheta_1,\vartheta_2)]\to\mathcal{K}_C$ *is continuously* gH*-differentiable on* $[\vartheta_1,\vartheta_1+\Omega(\vartheta_1,\vartheta_2)]$ *with a finite number of switching points at* $\vartheta_1=\rho_0<\rho_1<\ldots<\rho_{n+1}=\vartheta_1+\Omega(\vartheta_1,\vartheta_2)$. *Then, for* $x\in[\vartheta_1,\vartheta_1+\Omega(\vartheta_1,\vartheta_2)]$, *we have*

$$H\left(\frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\Im(y)dy,\left(\Im(x)(1-t)+\frac{\Im(\vartheta_1)+\Im(\vartheta_1+\Omega(\vartheta_1,\vartheta_2))}{2}t\right)\right)$$
$$\leq\|\Im'\|_\infty\left(\frac{(x-\vartheta_1)^2+(\vartheta_1+\Omega(\vartheta_1,\vartheta_2)-x)^2}{2\Omega(\vartheta_1,\vartheta_2)}(1-t)+\frac{\Omega(\vartheta_1,\vartheta_2)}{2}t\right), \tag{4.2}$$

where $t\in[0,1]$.

Proof. By using the properties of Pompeiu-Hausdorff metric and Proposition 3.4, we have

$$H\left(\frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\Im(y)dy,\left(\Im(x)(1-t)+\frac{\Im(\vartheta_1)+\Im(\vartheta_1+\Omega(\vartheta_1,\vartheta_2))}{2}t\right)\right)$$
$$=H\left(\frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\Im(y)dy,\right.$$
$$\left.\frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\left(\Im(x)(1-t)+\frac{\Im(\vartheta_1)+\Im(\vartheta_1+\Omega(\vartheta_1,\vartheta_2))}{2}t\right)dy\right)$$
$$=H\left(\frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}(\Im(y)(1-t)+\Im(y)t)dy,\right.$$
$$\left.\frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\left(\Im(x)(1-t)+\frac{\Im(\vartheta_1)+\Im(\vartheta_1+\Omega(\vartheta_1,\vartheta_2))}{2}t\right)dy\right)$$

$$= (1-t)H\left(\frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\Im(y)dy, \frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\Im(x)dy\right)$$
$$+ tH\left(\frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\Im(y)dy,\right.$$
$$\left.\frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\left(\frac{\Im(\vartheta_1)+\Im(\vartheta_1+\Omega(\vartheta_1,\vartheta_2))}{2}\right)dy\right).$$

Now

$$H\left(\frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\Im(y)dy,\right.$$
$$\left.\frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\left(\frac{\Im(\vartheta_1)+\Im(\vartheta_1+\Omega(\vartheta_1,\vartheta_2))}{2}\right)dy\right)$$
$$=H\left(\frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\left(\frac{\Im(y)}{2}+\frac{\Im(y)}{2}\right)dy,\right.$$
$$\left.\frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\left(\frac{\Im(\vartheta_1)}{2}+\frac{\Im(\vartheta_1+\Omega(\vartheta_1,\vartheta_2))}{2}\right)dy\right)$$
$$=\frac{1}{2}H\left(\frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\Im(y)dy, \frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\Im(\vartheta_1)dy\right)$$
$$+\frac{1}{2}H\left(\frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\Im(y)dy,\right.$$
$$\left.\frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\Im(\vartheta_1+\Omega(\vartheta_1,\vartheta_2))dy\right).$$

Now using Theorem 4.1 we obtain for $t\in[0,1]$

$$H\left(\frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\Im(y)dy, \left(\Im(x)(1-t)+\frac{\Im(\vartheta_1)+\Im(\vartheta_1+\Omega(\vartheta_1,\vartheta_2))}{2}t\right)\right)$$
$$\le (1-t)\|\Im'\|_\infty\left(\frac{(x-\vartheta_1)^2+(\vartheta_1+\Omega(\vartheta_1,\vartheta_2)-x)^2}{2\Omega(\vartheta_1,\vartheta_2)}\right)$$
$$+\frac{1}{2}t\left(\|\Im'\|_\infty\frac{\Omega^2(\vartheta_1,\vartheta_2)}{2\Omega(\vartheta_1,\vartheta_2)}+\|\Im'\|_\infty\frac{\Omega^2(\vartheta_1,\vartheta_2)}{2\Omega(\vartheta_1,\vartheta_2)}\right)$$
$$\le \|\Im'\|_\infty\left(\frac{(x-\vartheta_1)^2+(\vartheta_1+\Omega(\vartheta_1,\vartheta_2)-x)^2}{2\Omega(\vartheta_1,\vartheta_2)}(1-t)+\frac{\Omega(\vartheta_1,\vartheta_2)}{2}t\right).$$

Which establishes the result. □

Remark 4.4. We would like to remark the following:

(1) If we put $t = 0$ in Theorem 4.3, then Theorem 4.1 is established.

(2) If we put $\Omega(\vartheta_1, \vartheta_2) = \vartheta_2 - \vartheta_1$, then from inequality (4.2), we have

$$H\left(\frac{1}{\vartheta_2 - \vartheta_1}\int_{\vartheta_1}^{\vartheta_2} \Im(y)dy, \left(\Im(x)(1-t) + \frac{\Im(\vartheta_1) + \Im(\vartheta_2)}{2}t\right)\right) \leq \|\Im'\|_\infty \left(\frac{(x-\vartheta_1)^2 + (\vartheta_2 - x)^2}{2\Omega(\vartheta_1, \vartheta_2)}(1-t) + \frac{(\vartheta_2 - \vartheta_1)}{2}t\right), \tag{4.3}$$

which is Theorem 5 of Chalco-Cano et al. [25].

(3) Now if we put $t = 0$ in inequality (4.3), we get Theorem 4 of Chalco-Cano [25].

(4) If we put $\Im^L = \Im^U = \Im$ in inequality (4.2) we arrive at

$$\left|\frac{1}{\Omega(\vartheta_1, \vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)} \Im(y)dy, \left(\Im(x)(1-t) + \frac{\Im(\vartheta_1) + \Im(\vartheta_1 + \Omega(\vartheta_1, \vartheta_2))}{2}t\right)\right| \leq \|\Im'\|_\infty \left(\frac{(x-\vartheta_1)^2 + (\vartheta_1 + \Omega(\vartheta_1, \vartheta_2) - x)^2}{2\Omega(\vartheta_1, \vartheta_2)}(1-t) + \frac{\Omega(\vartheta_1, \vartheta_2)}{2}t\right). \tag{4.4}$$

Which is generalization of classical Ostrowski inequality (1.1).

(5) If we put $\Omega(\vartheta_1, \vartheta_2) = \vartheta_2 - \vartheta_1$ in (4.4) we get inequality (8) of Chalco-Cano et al. [25].

We would like to remark that the inequality (4.1) is valid for any continuously gH-differentiable interval-valued function defined on $[\vartheta_1, \vartheta_1 + \Omega(\vartheta_1, \vartheta_2)]$ with a finite number of switching points (see illustration in section 6). From the example we can see that $\Im$ is continuously gH-differentiable and (4.1) is valid but the endpoint functions are not necessarily differentiable. Noting this, we present another result where the endpoint functions are differentiable and we omit the existence of finite number of switching points of $\Im$.

Theorem 4.5. *Assume that the end point functions $\Im^L$ and $\Im^U$ of an interval valued function $\Im : [\vartheta_1, \vartheta_1 + \Omega(\vartheta_1, \vartheta_2)] \to \mathcal{K}_C$ are continuously differentiable. Then for $x \in [\vartheta_1, \vartheta_1 + \Omega(\vartheta_1, \vartheta_2)]$, we have*

$$H\left(\frac{1}{\Omega(\vartheta_1, \vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)} \Im(y)dy, \Im(x)\right) \leq \|\Im'\|_\infty \left(\frac{(x-\vartheta_1)^2 + (\vartheta_1 + \Omega(\vartheta_1, \vartheta_2) - x)^2}{2\Omega(\vartheta_1, \vartheta_2)}\right). \tag{4.5}$$

Proof.

$$
\begin{aligned}
&H\left(\frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}[\Im^L(y),\Im^U(y)]dy,[\Im^L(x),\Im^U(x)]\right)\\
&= H\Bigg(\left[\frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\Im^L(y)dy,\frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\Im^U(y)dy\right],\\
&\qquad [\Im^L(x),\Im^U(x)]dy\Bigg)\\
&= \max\left\{\left|\frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\Im^L(y)dy,\Im^L(x))\right|,\right.\\
&\qquad \left.\left|\frac{1}{\Omega(\vartheta_1,\vartheta_2)}\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\Im^U(y)dy,\Im^U(x))\right|\right\}\\
&\leq \max\left\{\|\Im^{L\prime}\|_\infty\left(\frac{(x-\vartheta_1)^2+(\vartheta_1+\Omega(\vartheta_1,\vartheta_2)-x)^2}{2\Omega(\vartheta_1,\vartheta_2)}\right),\right.\\
&\qquad \left.\|\Im^{U\prime}\|_\infty\left(\frac{(x-\vartheta_1)^2+(\vartheta_1+\Omega(\vartheta_1,\vartheta_2)-x)^2}{2\Omega(\vartheta_1,\vartheta_2)}\right)\right\}\\
&= \left(\frac{(x-\vartheta_1)^2+(\vartheta_1+\Omega(\vartheta_1,\vartheta_2)-x)^2}{2\Omega(\vartheta_1,\vartheta_2)}\right)\max\left\{\|\Im^{L\prime}\|_\infty,\|\Im^{U\prime}\|_\infty\right\}\\
&= \|\Im'\|_\infty\left(\frac{(x-\vartheta_1)^2+(\vartheta_1+\Omega(\vartheta_1,\vartheta_2)-x)^2}{2\Omega(\vartheta_1,\vartheta_2)}\right).
\end{aligned}
$$

This completes the proof. □

In the following result another generalization of Ostrowski type inequality is established.

Theorem 4.6. *Assume that the interval valued function $\Im : [\vartheta_1,\vartheta_1+\Omega(\vartheta_1,\vartheta_2)] \to \mathcal{K}_C$ is gH-differentiable in $[\vartheta_1,\vartheta_1+\Omega(\vartheta_1,\vartheta_2)]$ such that the end point functions $\Im^L$ and $\Im^U$ are continuously differentiable. Let*

$$h_1 : (\ \vartheta_1,\vartheta_1+\Omega(\vartheta_1,\vartheta_2)\ ] \to [\vartheta_1,\vartheta_1+\Omega(\vartheta_1,\vartheta_2)]$$

defined by $h_1(x) \leq x$ and

$$h_2 : [\ \vartheta_1,\vartheta_1+\Omega(\vartheta_1,\vartheta_2)\ ) \to [\vartheta_1,\vartheta_1+\Omega(\vartheta_1,\vartheta_2)]$$

defined by $h_2(x) \geq x$. Then for all $x \in [\vartheta_1,\vartheta_1+\Omega(\vartheta_1,\vartheta_2)]$, we have

$$H\left(\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\Im(y)dy,(h_2(x)-h_1(x))\Im(x)+(\vartheta_1+\Omega(\vartheta_1,\vartheta_2)\right.$$

$$-h_2(x))\Im(\vartheta_1+\Omega(\vartheta_1,\vartheta_2))+(h_1(x)-\vartheta_1)\Im(\vartheta_1)\Bigg)$$

$$\leq \|\Im'\|_\infty\left(\frac{1}{2}\left\{\frac{\Omega^2(\vartheta_1,\vartheta_2)}{2}+\left((x-\vartheta_1)-\frac{\Omega(\vartheta_1,\vartheta_2)}{2}\right)^2\right\}\right.$$

$$\left.+\left(h_1(x)+\frac{\vartheta_1+x}{2}\right)^2+\left(h_2(x)-\frac{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)+x}{2}\right)^2\right). \tag{4.6}$$

Proof. From Theorem 47 of [19] and properties of Hausdorff metric, we have

$$H\left(\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\Im(y)dy,(h_2(x)-h_1(x))\Im(x)\right.$$

$$\left.+(\vartheta_1+\Omega(\vartheta_1,\vartheta_2)-h_2(x))\Im(\vartheta_1+\Omega(\vartheta_1,\vartheta_2))+(h_1(x)-\vartheta_1)\Im(\vartheta_1)\right)$$

$$=H\left(\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}[\Im^L(y),\Im^U(y)]dy,\ (h_2(x)-h_1(x))\,[\Im^L(x),\Im^U(x)]\right.$$

$$+(\vartheta_1+\Omega(\vartheta_1,\vartheta_2)-h_2(x))[\Im^L(\vartheta_1+\Omega(\vartheta_1,\vartheta_2)),\Im^U(\vartheta_1+\Omega(\vartheta_1,\vartheta_2))]$$

$$\left.+(h_1(x)-\vartheta_1)[\Im^L(\vartheta_1),\Im^U(\vartheta_1)]\right)$$

$$=\max\left\{\left|\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\Im^L(y)dy-(h_2(x)-h_1(x))\Im^L(x)\ +(\vartheta_1+\Omega(\vartheta_1,\vartheta_2))\right.\right.$$

$$\left.\Im^L(\vartheta_1+\Omega(\vartheta_1,\vartheta_2))+(h_1(x)-\vartheta_1)\Im^L(\vartheta_1)\right|,$$

$$\left|\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\Im^U(y)dy-(h_2(x)-h_1(x))\Im^U(x)+(\vartheta_1+\Omega(\vartheta_1,\vartheta_2))\right.$$

$$\left.\left.\Im^U(\vartheta_1+\Omega(\vartheta_1,\vartheta_2))+(h_1(x)-\vartheta_1)\Im^U(\vartheta_1)\right|\right\}$$

$$\leq\max\left\{\|\Im^{L'}\|_\infty\left\{\frac{1}{2}\left[\frac{\Omega^2(\vartheta_1,\vartheta_2)}{2}+\left((x-\vartheta_1)-\frac{\Omega(\vartheta_1,\vartheta_2)}{2}\right)^2\right]\right.\right.$$

$$\left.+\left(h_1(x)+\frac{\vartheta_1+x}{2}\right)^2+\left(h_2(x)-\frac{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)+x}{2}\right)^2\right\},$$

$$
\begin{aligned}
&\|\Im^{U'}\|_\infty \left\{\frac{1}{2}\left[\frac{\Omega^2(\vartheta_1,\vartheta_2)}{2}+\left((x-\vartheta_1)-\frac{\Omega(\vartheta_1,\vartheta_2)}{2}\right)^2\right]\right.\\
&\left.\left.+\left(h_1(x)+\frac{\vartheta_1+x}{2}\right)^2+\left(h_2(x)-\frac{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)+x}{2}\right)^2\right\}\right\}\\
&=\max\left\{\|\Im^{L'}\|_\infty,\|\Im^{U'}\|_\infty\right\}\left\{\frac{1}{2}\left[\frac{\Omega^2(\vartheta_1,\vartheta_2)}{2}+\left((x-\vartheta_1)-\frac{\Omega(\vartheta_1,\vartheta_2)}{2}\right)^2\right]\right.\\
&\left.+\left(h_1(x)+\frac{\vartheta_1+x}{2}\right)^2+\left(h_2(x)-\frac{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)+x}{2}\right)^2\right\}\\
&=\|\Im'\|_\infty\left\{\frac{1}{2}\left[\frac{\Omega^2(\vartheta_1,\vartheta_2)}{2}+\left((x-\vartheta_1)-\frac{\Omega(\vartheta_1,\vartheta_2)}{2}\right)^2\right]+\left(h_1(x)+\frac{\vartheta_1+x}{2}\right)^2\right.\\
&\left.+\left(h_2(x)-\frac{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)+x}{2}\right)^2\right\}.
\end{aligned}
$$

It establishes the inequality. □

Remark 4.7. As a consequence of Theorem 4.6 we have the special case for $h_1(x)=\frac{\vartheta_1+x}{2}$, $h_2(x)=\frac{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)+x}{2}$ and for all $x\in[\vartheta_1,\vartheta_1+\Omega(\vartheta_1,\vartheta_2)]$,

$$
\begin{aligned}
&H\left(\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\Im(y)dy,\frac{\Omega^2(\vartheta_1,\vartheta_2)}{2}\left[\Im(x)+\frac{x-\vartheta_1}{\Omega(\vartheta_1,\vartheta_2)}\Im(\vartheta_1)\right.\right.\\
&\qquad\left.\left.+\frac{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)-x}{\Omega(\vartheta_1,\vartheta_2)}\Im(\vartheta_1+\Omega(\vartheta_1,\vartheta_2))\right]\right)\\
&\le\frac{1}{2}\|\Im'\|_\infty\left[\left(\frac{\Omega(\vartheta_1,\vartheta_2)}{2}\right)^2+\left(x-\frac{2\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}{2}\right)^2\right].
\end{aligned}
$$

Now, if $x=\vartheta_1+\frac{\Omega(\vartheta_1,\vartheta_2)}{2}$ we get even more accurate formula from above inequality. In fact, we have

$$
\begin{aligned}
&H\left(\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)}\Im(y)dy,\frac{\Omega(\vartheta_1,\vartheta_2)}{2}\left[\Im\left(\vartheta_1+\frac{\Omega(\vartheta_1,\vartheta_2)}{2}\right)+\frac{\Im(\vartheta_1)+\Im(\Omega(\vartheta_1,\vartheta_2))}{2}\right]\right)\\
&\le\frac{\Omega^2(\vartheta_1,\vartheta_2)}{8}\|\Im'\|_\infty.
\end{aligned}
$$

5. Possible applications: Error estimation to quadrature rules

Assume that $A = [\lambda^L, \lambda^U] \in \mathcal{K}_c$. Then the function $\Im$ defined by

$$\begin{aligned}\Im(x) =& [\Im^L(x), \Im^U(x)]\\ =& A\mathcal{F}(x)\\ =& [\lambda^L, \lambda^U]\mathcal{F}(x)\\ =& \begin{cases} [\lambda^L\mathcal{F}(x), \lambda^U\mathcal{F}(x)], & \mathcal{F}(x) \geq 0,\\ [\lambda^U\mathcal{F}(x), \lambda^L\mathcal{F}(x)], & \mathcal{F}(x) \leq 0.\end{cases}\end{aligned}$$

It is known that if the functions $\Im^L(x)$ and $\Im^U(x)$ are differentiable at $x = c$, then so is $\mathcal{F}(x)$ at c. However the converse is not true. Again, if $\Im(x)$ is gH-differentiable at c, then $\Im^L(x)$ and $\Im^U(x)$ may not be differentiable at c. This depicts that the properties of the functions $\Im^L(x)$ and $\Im^U(x)$ are not necessarily inherited from $\mathcal{F}$.

Now, in order to obtain $\int_a^b \Im(x) = [\int_a^b \Im^L(x), \int_a^b \Im^U(x)]$, we will approach by obtaining approximates of $\int_a^b \Im^L(x)$ and $\int_a^b \Im^U(x)$ by using known classical quadrature rules. Since $\Im^L(x)$ and $\Im^U(x)$ are not necessarily differentiable, therefore the classical results of error estimation involving differentiability are useless. However, for any interval-valued function $\Im(x)$, we can extend the quadrature rules like Reimann, Trapezoidal and Simpsom types for real functions to obtain an approximation of $\int_a^b \Im(x)$. For this, let $I = [\vartheta_1, \vartheta_1 + \Omega(\vartheta_1, \vartheta_2)]$ be a real interval, $P_l : \vartheta_1 = \rho_0 < \rho_1 < ... < \rho_n = \vartheta_1 + \Omega(\vartheta_1, \vartheta_2)$ be its partition and $\alpha \in R^n$ be a vector such that $\alpha_j \in [\rho_{j-1}, \rho_j], j = 1, 2, ..., n$, then Q_R (quadrature rule of Reimann type), Q_S (quadrature rule of Simpson type) and Q_T (quadrature rule of Trapezoidal type) are respectively given by

$$Q_R(\Im, I_l, \alpha) = \sum_{j=1}^{n} \Im(\alpha_j) l_j, \tag{5.1}$$

$$Q_S(\Im, I_l, \alpha) = \frac{1}{6}\sum_{j=1}^{n} \{\Im(\rho_{j-1}) + 4\Im(\alpha_j) + \Im(\rho_j)\} l_j^2 \tag{5.2}$$

and

$$Q_T(\Im, I_l, \alpha) = \frac{1}{2}\sum_{j=1}^{n} \{\Im(\rho_j) + \Im(\rho_{j-1})\} l_j. \tag{5.3}$$

To obtain an error estimation for quadrature rules, we present the following results.

Corollary 5.1. *Let P_l be a partition of of the interval $[\vartheta_1, \vartheta_1 + \Omega(\vartheta_1, \vartheta_2)]$ with $l_j = \rho_j - \rho_{j-1}$ and a vector $\alpha \in R^n$ satisfying $\alpha_j \in [\rho_{j-1}, \rho_j], j = 1, 2, ..., n$.*

(1) *Assume that the hypothesis of Theorem 4.1 holds, then we have*

$$\begin{aligned} R_R(\Im, I_l, \alpha) =& H\left(\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)} \Im(y)dy, Q_R(\Im, I_l, \alpha)\right) \\ \leq& \frac{1}{4}\|\Im'\|_\infty \sum_{j=1}^{n} l_j^2, \end{aligned}$$

where Q_R is the quadrature rule of Reimann type given by (5.1).

(2) *Assume that the hypothesis of Theorem 4.3 holds, then we have*

$$\begin{aligned} R_S(\Im, I_l, \alpha) =& H\left(\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)} \Im(y)dy, Q_S(\Im, I_l, \alpha)\right) \\ \leq& \frac{1}{3}\|\Im'\|_\infty \sum_{j=1}^{n} l_j^2, \end{aligned}$$

where Q_S is the quadrature rule of Simpson type given by (5.2).

(3) *Assume that the hypothesis of Theorem 4.3 holds, then we have*

$$\begin{aligned} R_T(\Im, I_l, \alpha) =& H\left(\int_{\vartheta_1}^{\vartheta_1+\Omega(\vartheta_1,\vartheta_2)} \Im(y)dy, Q_T(\Im, I_l, \alpha)\right) \\ \leq& \frac{1}{2}\|\Im'\|_\infty \sum_{j=1}^{n} l_j^2, \end{aligned}$$

where Q_T is the quadrature rule of Trapezoidal type given by (5.3).

Proof. (1) Since the hypothesis of Theorem 4.1 hold. Then using the properties of Hausdorff metric and integrals, we see that

$$\begin{aligned} R_R(\Im, I_l, \alpha) =& H\left(\sum_{j=1}^{n} \int_{\rho_{j-1}}^{\rho_j} \Im(y)dy, \sum_{j=1}^{n} \Im(\alpha_j)l_j\right) \\ =& \sum_{j=1}^{n} H\left(\int_{\rho_{j-1}}^{\rho_j} \Im(y)dy, \Im(\alpha_j)l_j\right) \\ \leq& \sum_{j=1}^{n} \|\Im'\|_\infty \left[\frac{l_j^2}{4} + \left(\alpha_j - \frac{\rho_{j-1}+\rho_j}{2}\right)^2\right] \end{aligned}$$

$$=\|\Im'\|_\infty \sum_{j=1}^{n}\left[\frac{l_j^2}{4}+\left(\alpha_j-\frac{\rho_{j-1}+\rho_j}{2}\right)^2\right].$$

Now if $\alpha_j = \frac{\rho_{j-1}+\rho_j}{2}, \forall j$, then

$$R_R(\Im, I_l, \alpha) = \frac{1}{4}\|\Im'\|_\infty \sum_{j=1}^{n} l_j^2.$$

which proves (1).

(2) From Theorem 4.3 with $l = 1/3$, we have

$$\begin{aligned}
&R_S(\Im, I_l, \alpha)\\
&= H\left(\sum_{j=1}^{n}\int_{\rho_{j-1}}^{\rho_j}\Im(y)dy, \frac{1}{6}\sum_{j=1}^{n}\left\{\Im(\rho_{j-1})+4\Im\left(\frac{\rho_j+\rho_{j-1}}{2}\right)+\Im(\rho_j)\right\} l_j^2\right)\\
&= \sum_{j=1}^{n} H\left(\int_{\rho_{j-1}}^{\rho_j}\Im(y)dy, \frac{1}{6}\left\{\Im(\rho_{j-1})+4\Im\left(\frac{\rho_j+\rho_{j-1}}{2}\right)+\Im(\rho_j)\right\} l_j^2\right)\\
&\leq \sum_{j=1}^{n}\|\Im'\|_\infty\left[\frac{1}{3}\{(\alpha_j-\rho_{j-1})^2+(\alpha_j-\rho_j)^2\}+l_j\right]\\
&= \frac{1}{6}\|\Im'\|_\infty\sum_{j=1}^{n}\left[2\{(\alpha_j-\rho_{j-1})^2+(\alpha_j-\rho_j)^2\}+l_j\right].
\end{aligned}$$

Now if $\alpha_j = \frac{\rho_{j-1}+\rho_j}{2}$, for alll j, then

$$\begin{aligned}
R_S(\Im, I_l, \alpha) &= \frac{1}{6}\|\Im'\|_\infty\sum_{j=1}^{n}\left[2\left\{\frac{l_j^2}{4}+\frac{l_j^2}{4}\right\}+l_j\right]\\
&= \frac{1}{3}\|\Im'\|_\infty\sum_{j=1}^{n} l_j^2,
\end{aligned}$$

which proves (2).

(3) Put $t = 1$ in Theorem 4.3 and follow the same procedure as in the proof of the part (2), the result follows immediately. □

6. Numerical example

Let $\Im(x) = [-2, 2]\sin(2x)$ be interval valued function such that $x \in [0, \pi]$. Then we have

$$\Im(x) = \begin{cases} [-2\sin(2x), 2\sin(2x)] \text{ if } 0 \le x \le \frac{\pi}{4}; \\ [2\sin(2x), -2\sin(2x)] \text{ if } \frac{\pi}{4} \le x \le \frac{3\pi}{4}; \\ [-2\sin(2x), 2\sin(2x)] \text{ if } \frac{3\pi}{4} \le x \le \pi. \end{cases}$$

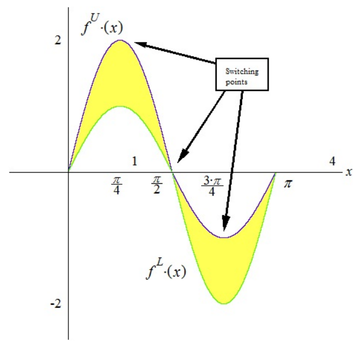

FIGURE 1

Now, $\Im'(x) = [2, 4]\cos(2x)$ and $\|\Im'\|_\infty = 4$. However, from Figure 1 it is easy to see that $\Im$ has three switching points (viz. $\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$) in $(0, \pi)$. Further, if we consider that $\Omega(\vartheta_1, \vartheta_2) = \vartheta_2 - \vartheta_1$, then from left hand side of inequality (4.1) we have

$$\begin{aligned} H\left(\frac{1}{\pi}\int_0^\pi [1,2]\sin(2x)dx, \Im\left(\frac{\pi}{2}\right)\right) =& H\left(\frac{1}{\pi}[-2,2], \{0\}\right) \\ =& \frac{2}{\pi} \end{aligned}$$

and from right hand side of inequality (4.1) we have

$$4\left(\frac{\left(\frac{\pi}{2}\right)^2 + \left(\frac{\pi}{2}\right)^2}{2\pi}\right) = \pi.$$

Therefore the inequality (4.1) is verified.

7. CONCLUSIONS

In this paper, we have discussed the Ostrowski type inequalities for generalized Hukukara differentiable interval valued functions. An error estimation to quadrature rule of Riemann type, Simpson type and Trapezoidal type is also presented for interval-valued functions. It will be interesting to extend the results of this paper under fuzzy environments [5, 13].

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