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SOLVING FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS BY USING NUMERICAL TECHNIQUES

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Abstract. This paper mainly focuses on numerical techniques based on the Adomian Decomposition Method (ADM) and Direct Homotopy Analysis Method (DHAM) for solving Fredholm integro-differential equations of the second kind. The reliability of the methods and reduction in the size of the computational work give this methods wider applicability. Convergence analysis of the exact solution of the proposed methods will be established. Moreover, we proved the uniqueness of the solution. To illustrate the methods, an example is presented.

1. Introduction

In this paper, we consider Fredholm integro-differential equation of the form:

$$\sum_{j=0}^{k} p_j(x)u^{(j)}(x) = f(x) + \lambda \int_a^b K(x,t)G(u(t))dt \tag{1.1}$$

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with the initial conditions

$$u^{(r)}(a) = b_r, \quad r = 0, 1, 2, \cdots, (k-1), \tag{1.2}$$

where $u^{(j)}(x)$ is the j^{th} derivative of the unknown function $u(x)$ that will be determined, $K(x,t)$ is the kernel of the equation, $f(x)$ and $p_j(x)$ are analytic functions, G is nonlinear function of u, a, b, λ, and b_r are real finite constants.

The Fredholm integro-differential equations arise in many scientific applications. It was also shown that these equations can be derived from boundary value problems. Erik Ivar Fredholm (1866–1927) is best remembered for his work on integral equations and spectral theory [6, 7, 8, 9, 10, 11].

In recent years, many authors focus on the development of numerical and analytical techniques for integro-differential equations. For instance, we can remember the following works: Abbasbandy and Elyas [1] studied some applications on variational iteration method for solving system of nonlinear Volterra integro-differential equations, Alao et al. [2] used Adomian decomposition and variational iteration methods for solving integro-differential equations, Behzadi et al. [3] solved some class of nonlinear Volterra-Fredholm integro-differential equations by homotopy analysis method, to the antisymmetric flow over a stretching sheet by Ariel et al. [4], to the Helmholtz equation and fifth-order KdV equation by Rafei and Ganji [25], for the thin film flow of a fourth grade fluid down a vertical cylinder by Siddiqui et al. [26], to the nonlinear Volterra-Fredholm integral equations by Hamoud and Ghadle [12, 13], to integro-differential equation [5, 14], to system of Fredholm integral equations [24]. Moreover, many methods for solving integro-differential equations have been studied by several authors [15, 16, 17, 18, 19, 20, 21, 22].

The main objective of the present paper is to study the behavior of the solution that can be formally determined by semi-analytical approximated methods as the ADM and DHAM. Moreover, we proved the uniqueness results of the Fredholm integro-differential equation (1.1), with the initial conditions (1.2).

2. Adomian decomposition method (ADM)

Now, we can rewrite Eq.(1.1) in the form:

$$p_k(x)u^k(x) + \sum_{j=0}^{k-1} p_j(x)u^j(x) = f(x) + \lambda \int_a^b K(x,t)G(u(t))dt. \tag{2.1}$$

Then

$$u^k(x) = \frac{f(x)}{p_k(x)} + \lambda \int_a^b \frac{K(x,t)}{p_k(x)} G(u(t))dt - \sum_{j=0}^{k-1} \frac{p_j(x)}{p_k(x)} u^j(x).$$

To obtain the approximate solution, we integrating (k)-times in the interval $[a, x]$ with respect to x we obtain,

$$\begin{aligned} u(x) &= L^{-1}\left(\frac{f(x)}{p_k(x)}\right) + \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r + \lambda L^{-1}\left(\int_a^b \frac{K(x,t)}{p_k(x)} G(u(t))dt\right) \\ &\quad - \sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)} u_n^{(j)}(x)\right), \end{aligned} \tag{2.2}$$

where L^{-1} is the multiple integration operator given as follows:

$$L^{-1}(\cdot) = \int_a^b \int_a^b \cdots \int_a^b (\cdot) dx dx \cdots dx \quad (k - times).$$

Now we apply ADM

$$G(u(t)) = \sum_{n=0}^{\infty} A_n, \tag{2.3}$$

where A_n; $n \geq 0$ are the Adomian polynomials determined formally as follows:

$$A_n = \frac{1}{n!}\Big[\frac{d^n}{d\mu^n} G(\sum_{i=0}^{\infty} \mu^i u_i)\Big]\Big|_{\mu=0}. \tag{2.4}$$

The Adomian polynomials were introduced in [16] as:

$$\begin{aligned} A_0 &= G(u_0); \\ A_1 &= u_1 G^{'}(u_0); \\ A_2 &= u_2 G^{'}(u_0) + \frac{1}{2!} u_1^2 G^{''}(u_0); \\ A_3 &= u_3 G^{'}(u_0) + u_1 u_2 G^{''}(u_0) + \frac{1}{3!} u_1^3 G^{'''}(u_0), \cdots . \end{aligned}$$

The standard decomposition technique represents the solution of u as the following series:

$$u = \sum_{i=0}^{\infty} u_i. \tag{2.5}$$

By substituting (2.3) and (2.5) in Eq. (2.2) we have

$$\begin{aligned}\sum_{i=0}^{\infty} u_i(x) &= L^{-1}\left(\frac{f(x)}{p_k(x)}\right) + \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r \\ &+\lambda \sum_{i=0}^{\infty} L^{-1}\left(\int_a^b \frac{K(x,t)}{p_k(x)} A_i(t)dt\right) \\ &-\sum_{i=0}^{\infty}\sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)} u_i^{(j)}(x)\right).\end{aligned}$$

The components $u_0, u_1, u_2, \cdots$ are usually determined recursively by

$$\begin{aligned}u_0 &= L^{-1}\left(\frac{f(x)}{p_k(x)}\right) + \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r, \\ u_1 &= \lambda L^{-1}\left(\int_a^b \frac{K(x,t)}{p_k(x)} A_0(t)dt\right) - \sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)} u_0^{(j)}(x)\right), \\ u_{n+1} &= \lambda L^{-1}\left(\int_a^b \frac{K(x,t)}{p_k(x)} A_n(t)dt\right) - \sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)} u_n^{(j)}(x)\right), \ n \geq 1.\end{aligned}$$

Then, $u(x) = \sum_{i=0}^{n} u_i$ as the approximate solution.

3. Direct homotopy analysis method (DHAM)

Consider Fredholm integro-differential equation (1.1) and substitute the kernel $K(x,t) = g(x)h(t)$ we obtain

$$\sum_{j=0}^{k} p_j(x)u^{(j)}(x) = f(x) + \lambda g(x)\int_a^b h(t)G(u(t))dt.$$

To obtain the approximate solution, we integrating (k)-times in the interval $[a,x]$ with respect to x we obtain,

$$\begin{aligned}u(x) &= L^{-1}\left(\frac{f(x)}{p_k(x)}\right) + \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r + \lambda L^{-1}\left(\frac{g(x)}{p_k(x)}\int_a^b h(t)G(u(t))dt\right) \\ &-\sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)} u_n^{(j)}(x)\right),\end{aligned} \tag{3.1}$$

Setting

$$\begin{aligned} Q &= \int_a^b h(t)G(u(t))dt \\ F &= L^{-1}\left(\frac{f(x)}{p_k(x)}\right) + \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r - \sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)} u_n^{(j)}(x)\right), \end{aligned} \tag{3.2}$$

then, Eq. (3.1) can be written as

$$u(x) = F(x) + \lambda L^{-1}\left(\frac{g(x)}{p_k(x)} Q\right).$$

We define the nonlinear homotopy operator as [23]:

$$N[u(x)] = u(x) - F(x) - \lambda L^{-1}\left(\frac{g(x)}{p_k(x)} Q\right).$$

The corresponding mth-order deformation equation is as follows:

$$L[u_m(x) - \chi_m u_{m-1}(x)] = BH(x) R_m(\overrightarrow{u_{m-1}(x)}),$$

where

$$R_m(\overrightarrow{u_{m-1}(x)}) = u_{m-1}(x) - F(x)(1-\chi_m) - \lambda L^{-1}\left(\frac{g(x)}{p_k(x)} Q\right) \tag{3.3}$$

and

$$\chi_m = \begin{cases} 1, & m > 1, \\ 0, & m \leq 1. \end{cases}$$

Choosing the auxiliary linear operator $L[u] = u$, we obtain

$$\begin{aligned} u_0(x) &: \quad \text{Choosing initial guess,} \\ u_1(x) &= BH(x)\Big[u_0(x) - L^{-1}\left(\frac{f(x)}{p_k(x)}\right) - \sum_{r=0}^{k-1} \frac{1}{r!}(x-a)^r b_r \\ &\quad -\lambda L^{-1}\left(\frac{g(x)}{p_k(x)} \int_a^b h(t)G(u_0(t))dt\right) + \sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)} u_0^{(j)}(x)\right)\Big], \\ u_m(x) &= \chi_m u_{m-1}(x) + BH(x)\Big[u_{m-1}(x) \\ &\quad -\lambda L^{-1}\left(\frac{g(x)}{p_k(x)} \int_a^b h(t)G(u_{m-1}(t))dt\right) \\ &\quad +\sum_{j=0}^{k-1} L^{-1}\left(\frac{p_j(x)}{p_k(x)} u_{m-1}^{(j)}(x)\right)\Big], m > 1, \end{aligned} \tag{3.4}$$

with auxiliary function $H(x)$ and auxiliary parameter B. Then, $u(x) = \sum_{i=0}^{m} u_i$ as the approximate solution.

4. Uniqueness results

In this section, we shall give the uniqueness results of Eq. (1.1), with the initial conditions (1.2) and prove it. We can be written equation (1.1) in the form of:

$$\begin{aligned} u(x) &= L^{-1}\Big[\frac{f(x)}{p_k(x)}\Big] + \sum_{r=0}^{k-1} \frac{(x-a)^r}{r!} b_r + \lambda_1 L^{-1}\Big[\int_a^b \frac{1}{p_k(x)} K(x,t) G(u_n(t))dt\Big] \\ & \quad - L^{-1}\Big[\sum_{j=0}^{k-1} \frac{p_j(x)}{p_k(x)} u^{(j)}(x)\Big], \end{aligned}$$

and

$$L^{-1}\Big[\int_a^b \frac{1}{p_k(x)} K(x,t) G(u_n(t))dt\Big] = \int_a^b \frac{(x-t)^k}{k! p_k(x)} K(x,t) G(u_n(t))dt,$$

$$\sum_{j=0}^{k-1} L^{-1}\Big[\frac{p_j(x)}{p_k(x)}\Big] u^{(j)}(x) = \sum_{j=0}^{k-1} \int_a^b \frac{(x-t)^{k-1} p_j(t)}{k-1! p_k(t)} u^{(j)}(t)dt.$$

We set,

$$\Psi(x) = L^{-1}\Big[\frac{f(x)}{p_k(x)}\Big] + \sum_{r=0}^{k-1} \frac{(x-a)^r}{r!} b_r.$$

Before starting and proving the main results, we introduce the following hypotheses:

(H1): There exist two constants α and $\gamma_j > 0$, $j = 0, 1, \cdots, k$ such that, for any $u_1, u_2 \in C(J, \mathbb{R})$

$$|G(u_1)) - G(u_2))| \leq \alpha |u_1 - u_2|$$

and

$$\left|D^j(u_1) - D^j(u_2)\right| \leq \gamma_j |u_1 - u_2|,$$

we suppose that the nonlinear terms $G(u(x))$ and $D^j(u) = (\frac{d^j}{dx^j})u(x) = \sum_{i=0}^{\infty} \gamma_{i_j}$, ($D^j$ is a derivative operator), $j = 0, 1, \cdots, k$, are Lipschitz continuous.

(H2): we suppose that for all $a \leq t \leq x \leq b$, and $j = 0, 1, \cdots, k$,

$$\left|\frac{\lambda(x-t)^k K(x,t)}{k! p_k(x)}\right| \leq \theta_1, \qquad \left|\frac{\lambda(x-t)^k K(x,t)}{k!}\right| \leq \theta_2,$$

$$\left|\frac{(x-t)^{k-1}p_j(t)}{(k-1)!p_k(t)}\right| \leq \theta_3, \qquad \left|\frac{(x-t)^{k-1}p_j(t)}{(k-1)!}\right| \leq \theta_4,$$

(H3): There exist three functions $\theta_3^*, \theta_4^*,$ and $\gamma^* \in C(D, \mathbb{R}^+)$, the set of all positive function continuous on $D = \{(x,t) \in \mathbb{R}\times\mathbb{R} : 0 \leq t \leq x \leq 1\}$ such that

$$\theta_3^* = \max|\theta_3|\,, \ \theta_4^* = \max|\theta_4|\,, \text{ and } \gamma^* = \max|\gamma_j|\,.$$

(H4): $\Psi(x)$ is bounded function for all x in $J = [a, b]$.

Theorem 4.1. *Assume that (H1)–(H4) hold. If*

$$0 < \psi = (\alpha\theta_1 + k\gamma^*\theta_3^*)(b-a) < 1, \tag{4.1}$$

Then there exists a unique solution $u(x) \in C(J)$ to Eqs. (1.1) – (1.2).

Proof. Let u_1 and u_2 be two different solutions of Eqs. (1.1) – (1.2). Then

$$\begin{aligned}
\Big|u_1 - u_2\Big| &= \Big|\int_a^b \frac{\lambda(x-t)^k K(x,t)}{p_k(x)k!}[G(u_1) - G(u_2))]dt \\
&\quad -\sum_{j=0}^{k-1}\int_a^b \frac{(x-t)^{k-1}p_j(t)}{p_k(t)(k-1)!}[D^j(u_1) - D^j(u_2))]dt\Big| \\
&\leq \int_a^b \Big|\frac{\lambda(x-t)^k K(x,t)}{p_k(x)k!}\Big|\Big|G(u_1) - G(u_2))\Big|dt \\
&\quad -\sum_{j=0}^{k-1}\int_a^b \Big|\frac{(x-t)^{k-1}p_j(t)}{p_k(t)(k-1)!}\Big|\Big|D^j(u_1) - D^j(u_2))\Big|dt \\
&\leq (\alpha\theta_1 + k\gamma^*\theta_3^*)(b-a)|u_1 - u_2|,
\end{aligned}$$

we get $(1-\psi)|u_1 - u_2| \leq 0$. Since $0 < \psi < 1$, so $|u_1 - u_2| = 0$. therefore, $u_1 = u_2$ and the proof is completed. □

5. Illustrative example

In this section, we present the numerical techniques based on ADM and DHAM to solve Fredholm integro-differential equations. To show the efficiency of the present methods for our problem in comparison with the exact solution we report absolute error.

Example 5.1. Consider the following Fredholm integro-differential equation.

$$u'(x) = e^x(1+x) - x + \int_0^1 xu(t)dt,$$

with the initial condition

$$u(0) = 0,$$

and the the exact solution is $u(x) = xe^x$.

TABLE 1. Numerical Results of the Example 5.1

x	Exact	ADM	DHAM
0.1	0.1103782	0.1103782	0.1105170
0.2	0.2442805	0.2437249	0.2442805
0.3	0.4049576	0.4037076	0.4049576
0.4	0.5967298	0.5945076	0.5967298
0.5	0.8243606	0.8208884	0.8233606
0.6	1.0932712	1.0882712	1.0932712
0.7	1.4096268	1.4028213	1.4096268
0.8	1.7804327	1.7715438	1.7804327
0.9	2.2136428	2.2023928	2.2136428

6. Discussion and conclusion

We discussed the ADM and DHAM for solving Fredholm integro-differential equations of the second kind. To assess the accuracy of each method, the test example with known exact solution are used. In this work, the above methods have been successfully employed to obtain the approximate solution of a Fredholm integro-differential equation. The results show that these methods are very efficient, convenient and can be adapted to fit a larger class of problems. The comparison reveals that although the numerical results of these methods are similar approximately, Table 1. shows that the numerical results obtained with DHAM coincide with the exact solutions.

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