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A NEW REPRESENTATION OF TRANSCENDENT FUNCTIONS OF PSEUDO-DUAL-QUATERNIONIC VARIABLES

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Abstract. In this paper, we develop and propose a modified quaternion, called the pseudo-dual-quaternion (PDQ). After identifying the differences between quaternions and PDQs, we define the PDQ and a function corresponding to the PDQ. Based on the operators provided in the PDQ, we give the form of power for the PDQ.

1. Introduction

Dual numbers were first introduced by Clifford [2] in 1873 and were later developed further during the 20th century. In a previous study [11], the dual angle was presented, which measures the actual location of two skew lines in three-dimensional (3D) space. Dual numbers extend from real numbers the combination of a new element ε with the property $\varepsilon^2 = 0$. A dual number is written as $z = a + \varepsilon b$, where a and b are the real numbers. The set of dual numbers establishes a commutative and associative algebra over the two-dimensional real numbers.

In mathematics and mechanics, in order to express the spatial rigid body displacements, Yang [13] proposed dual quaternions based on the forms of dual numbers and quaternions. Dual quaternions are constructed as quaternions by using dual numbers instead of real numbers as coefficients. To understand

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the operations of the dual quaternion, the quaternions need to be considered. A quaternion is a linear combination of the basic elements 1, i, j, and k, as introduced by Hamilton [4, 5] in 1843. Hamilton's product rule for i, j, and k is given by

$$i^2 = j^2 = k^2 = ijk = -1$$

and satisfies

$$ij = -ji = k \ , \ jk = -kj = i \quad \text{and} \quad ki = -ik = j.$$

Therefore, a dual quaternion is written as $\lambda_0 + i\lambda_1 + j\lambda_2 + k\lambda_3$, where λ_r $(r = 0, 1, 2, 3)$ are dual numbers. If we emphasize the form of dual numbers, a dual quaternion can also be represented in the form $p + \varepsilon q$, where p and q are quaternions and ε is the dual unit and commutes with every element of the algebra. As can be seen from the property of ε, unlike quaternions, dual quaternions are not a division ring. McCarthy [10] showed that rigid motions in 3D space can be represented by dual quaternions of unit length. This finding has been exploited in theoretical kinematics. Torsello [12] introduced applications to computer graphics, robotics, and computer vision. Kotelnikov [9] developed dual vectors and dual quaternions for use in the study of a method of accounting for friction in kinematic joints. Ercan and Yüce [3] generalized Euler's and De Moivre's formulas for complex numbers and quaternions to the dual quaternions. Also, the matrix representation of dual quaternions was expressed by dual quaternions. Ata and Yayli [1] considered the dual complex numbers as a generalization of complex numbers and studied a group of dual matrices by using a symplectic structure on dual quaternions. Kim and Shon [8] provided an expression of power series in dual split quaternions and the regularity of dual split quaternionic functions with differential operators on dual split quaternions. Kim [6] found the regularity of a dual quaternionic function and the polar representation of that function in Clifford analysis, using the differential operators of the dual quaternionic variables. Kim [7] extended a regular function with values in dual quaternioins, comparing it with the regularity of quaternionic functions.

Using the property of the dual unit, we propose a modified quaternion, called the pseudo-dual-quaternion (PDQ). We investigate the differences between quaternions and PDQs in terms of their algebraic and analytic properties. To determine these differences, we first define the PDQ and a function corresponding to the PDQ. We also propose addition, product, and differential operators which can be used to the PDQs. Based on these operators, we provide the form of power for the PDQs and the expression for the transcendent functions of the PDQ variables.

2. Preliminaries

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{D}_{\mathbb{P}}$ denote the set of PDQs such that

$$\mathbb{D}_{\mathbb{P}} = \{p = x_0 + ix_1 + jx_2 + kx_3 \mid x_r \in \mathbb{R} \quad (r = 0, 1, 2, 3)\}.$$

where i is the imaginary unit and j is the unit element satisfying

$$i^2 = -1,\ j^2 = 0,\ ij = k, \tag{2.1}$$

$$ij = -ji,\ jk = kj = 0 \quad \text{and} \quad ki = -ik = j, \tag{2.2}$$

which is isomorphic to $\mathbb{R}^4$. Using the properties of i and j, we have the following rules for the addition and product:

$$p + q = (x_0 + y_0) + i(x_1 + y_1) + j(x_2 + y_2) + k(x_3 + y_3)$$

and

$$\begin{aligned} pq \ = \ & (x_0y_0 - x_1y_1) + i(x_1y_0 + x_0y_1) + j(x_2y_0 + x_3y_1 + x_0y_2 - x_1y_3) \\ & + k(x_3y_0 - x_2y_1 + x_1y_2 + x_0y_3), \end{aligned}$$

respectively. We give a conjugate of a PDQ p as follows:

$$p^* = x_0 - ix_1 - jx_2 - kx_3.$$

We then have the norm, denoted by $\mathcal{N}(p)$, and the inverse element, denoted by p^{-1}, of a PDQ $p \in \mathbb{D}_{\mathbb{P}}$:

$$\mathcal{N}(p) := pp^* = x_0^2 + x_1^2$$

and

$$p^{-1} = \frac{p^*}{\mathcal{N}(p)} = \frac{x_0}{x_0^2 + x_1^2} - i\frac{x_1}{x_0^2 + x_1^2} - j\frac{x_2}{x_0^2 + x_1^2} - k\frac{x_3}{x_0^2 + x_1^2}, \quad (x_0,\ x_1 \neq 0),$$

respectively. For example, since j and k have $x_0 = x_1 = 0$, neither the unit element j nor k has the inverse element. The following properties are derived from each definition of the conjugate, norm, and inverse of the PDQs.

Proposition 2.1. *For p and q in $\mathbb{D}_{\mathbb{P}}$, the conjugate and norm of the PDQs satisfy the following properties: for $\alpha,\ \beta \in \mathbb{R}$,*

(i) $(p^*)^* = p$,
(ii) $(pq)^* = q^*p^*$,
(iii) $(\alpha p + \beta q)^* = \alpha p^* + \beta q^*$,
(iv) $\mathcal{N}(pq) = \mathcal{N}(p)\mathcal{N}(q)$,
(v) $\mathcal{N}(\alpha p) = \alpha^2\mathcal{N}(p)$.

Proof. The above properties are shown by the definitions of the conjugate and norm. In particular, since the product is noncommutative, in (ii), we have

$$\begin{aligned}(pq)^* &= (x_0y_0 - x_1y_1) - i(x_1y_0 + x_0y_1) - j(x_2y_0 + x_3y_1 + x_0y_2 - x_1y_3)\\ &\quad -k(x_3y_0 - x_2y_1 + x_1y_2 + x_0y_3)\end{aligned}$$

and

$$\begin{aligned}q^*p^* &= (y_0 - iy_1 - jy_2 - ky_3)(x_0 - ix_1 - jx_2 - kx_3)\\ &= (y_0x_0 - y_1x_1) - i(y_0x_1 + y_1x_0) - j(y_0x_2 + y_1x_3 + y_2x_0 - y_3x_1)\\ &\quad -k(y_0x_3 - y_1x_2 + y_2x_1 + y_3x_0).\end{aligned}$$

Hence, (ii) is satisfied. Also, in (iv), we have

$$\begin{aligned}\mathcal{N}(pq) &= (x_0y_0 - x_1y_1)^2 + (x_1y_0 + x_0y_1)^2\\ &= (x_0y_0)^2 + (x_1y_1)^2 + (x_1y_0)^2 + (x_0y_1)^2\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}(p)\mathcal{N}(q) &= (x_0^2 + x_1^2)(y_0^2 + y_1^2)\\ &= (x_0y_0)^2 + (x_0y_1)^2 + (x_1y_0)^2 + (x_1y_1)^2.\end{aligned}$$

By comparing $\mathcal{N}(pq)$ with $\mathcal{N}(p)\mathcal{N}(q)$, we obtain that (iv) is satisfied. □

Similarly, the inverse element of a PDQ has several properties.

Proposition 2.2. *For $p, q \in \mathbb{D}_{\mathbb{P}}$ and $\alpha \in \mathbb{R}$, the inverse element p^{-1} of p has the following properties:*

(1) $(p^{-1})^{-1} = p$,
(ii) $(pq)^{-1} = q^{-1}p^{-1}$,
(iii) $(\alpha p)^{-1} = \frac{1}{\alpha}p^{-1}$,
(iv) $\mathcal{N}(p^{-1}) = (\mathcal{N}(p))^{-1}$.

Proof. The above properties can be expressed by the definition of the inverse element of a PDQ. In particular, in (iv), we have

$$\begin{aligned}\mathcal{N}(p^{-1}) &= \left(\frac{x_0}{x_0^2 + x_1^2}\right)^2 + \left(\frac{x_1}{x_0^2 + x_1^2}\right)^2\\ &= \frac{x_0^2}{(x_0^2 + x_1^2)^2} + \frac{x_1^2}{(x_0^2 + x_1^2)^2} = \frac{1}{x_0^2 + x_1^2}\end{aligned}$$

and

$$(\mathcal{N}(p))^{-1} = (x_0^2 + x_1^2)^{-1} = \frac{(x_0^2 + x_1^2) - i0 - j0 - k0}{(x_0^2 + x_1^2)^2 + 0^2} = \frac{1}{x_0^2 + x_1^2}.$$

Since the above two equations are same, we obtain that (iv) is satisfied. □

3. Differential operators for transcendent functions of PDQ variables

We attempt to expand the various number systems through several expressions about a transcendent function of the PDQ variables. We also try to supply the non-expandable properties of dual quaternions. For example, in dual quaternions, if we use the general definition of the conjugation in complex analysis, there are no inverse elements and many constraints exist when representing the conjugation of differential operators. However, a PDQ is based on the property of dual quaternions, and is defined by an inverse element and norm, from which a form of transcendent functions can be obtained.

A pseudo-dual-quaternion p can be written as $p = x_0 + \overrightarrow{\mathbf{x}}$, where x_0 is the scalar part and $\overrightarrow{\mathbf{x}}$ is the vector part of p, induced by

$$x_0 = \frac{p + p^*}{2} \quad \text{and} \quad \overrightarrow{\mathbf{x}} = \frac{p - p^*}{2}.$$

In particular, if $x_0 = 0$, then p is termed a pure PDQ. For two pure PDQs, $\overrightarrow{\mathbf{x}} = ix_1 + jx_2 + kx_3$ and $\overrightarrow{\mathbf{y}} = iy_1 + jy_2 + ky_3$, their product is:

$$\begin{aligned} \overrightarrow{\mathbf{x}}\overrightarrow{\mathbf{y}} &= (ix_1 + jx_2 + kx_3)(iy_1 + jy_2 + ky_3) \\ &= -x_1y_1 + j(x_3y_1 - x_1y_3) + k(x_1y_2 - x_2y_1). \end{aligned}$$

Let $\cdot_D$ be the inner product and $\times_D$ be the cross product in PDQs such that

$$\overrightarrow{\mathbf{x}} \cdot_D \overrightarrow{\mathbf{y}} = x_1y_1$$

and

$$\overrightarrow{\mathbf{x}} \times_D \overrightarrow{\mathbf{y}} = j(x_3y_1 - x_1y_3) + k(x_1y_2 - x_2y_1),$$

respectively. For example, we note that

$$\overrightarrow{\mathbf{x}} \cdot_D \overrightarrow{\mathbf{x}} = x_1^2 \quad \text{and} \quad \overrightarrow{\mathbf{x}} \times_D \overrightarrow{\mathbf{x}} = 0.$$

Hence, we can write that

$$\overrightarrow{\mathbf{x}}\overrightarrow{\mathbf{y}} = -\overrightarrow{\mathbf{x}} \cdot_D \overrightarrow{\mathbf{y}} + \overrightarrow{\mathbf{x}} \times_D \overrightarrow{\mathbf{y}}.$$

Thus, for two PDQs p and q, their product is

$$pq = x_0y_0 - \overrightarrow{\mathbf{x}} \cdot_D \overrightarrow{\mathbf{y}} + x_0\overrightarrow{\mathbf{y}} + \overrightarrow{\mathbf{x}}y_0 + \overrightarrow{\mathbf{x}} \times_D \overrightarrow{\mathbf{y}}.$$

The scalar product of two PDQs $p = x_0 + \overrightarrow{\mathbf{x}}$ and $q = y_0 + \overrightarrow{\mathbf{y}}$, is defined as

$$< p, q >_D = x_0y_0 + x_1y_1,$$

that is, $< p, q >_D$ is the scalar part of pq^*. The above expressions provide a metric in the four-dimensional (4D) space with base i and j, satisfying $i^2 = -1$

and $j^2 = 0$, respectively. Using the scalar product, we can obtain an angle θ between two PDQs p and q such that

$$\cos\theta = \frac{< p, q >_D}{\sqrt{\mathcal{N}(p)}\sqrt{\mathcal{N}(q)}}.$$

Also, every PDQ $p = x_0 + ix_1 + jx_2 + kx_3$ can be written in the form

$$p = r(\cos\theta + \vec{v}\sin\theta), \quad -\pi < \theta \leq \pi,$$

which is termed the polar form of a PDQ, where $r = \sqrt{\mathcal{N}(p)} = \sqrt{x_0^2 + x_1^2}$, $\cos\theta = \frac{1}{r}x_0$, $\sin\theta = \frac{1}{r}x_1$ and

$$\vec{v} = \frac{\overrightarrow{\mathbf{x}}}{x_1}, \quad x_1 \neq 0. \tag{3.1}$$

We can express θ as the angle between $p \in \mathbb{D}_{\mathbb{P}}$ and the real axis. The notion of $\vec{v}\sin\theta$ is a projection of p onto the 3D space (with bases i and j) of pure PDQs. Since

$$\vec{v}^2 = \frac{\overrightarrow{\mathbf{x}}}{x_1}\frac{\overrightarrow{\mathbf{x}}}{x_1} = \frac{(-x_1^2)}{x_1^2} = -1,$$

we have

$$\vec{v}^n = \begin{cases} (-1)^m, & n = 2m, \\ (-1)^m\vec{v}, & n = 2m+1. \end{cases}$$

Using the expression of the Maclaurin series for $\cos\theta$ and $\sin\theta$, that is,

$$\cos\theta = \sum_{t=0}^{\infty}\frac{(-1)^t\theta^{2t}}{(2t)!} \quad \text{and} \quad \sin\theta = \sum_{t=0}^{\infty}\frac{(-1)^t\theta^{2t+1}}{(2t+1)!}\vec{v},$$

respectively, we describe $e^{\vec{v}\theta}$ such that

$$\begin{aligned} e^{\vec{v}\theta} &= 1 + (\vec{v}\theta) + \frac{(\vec{v}\theta)^2}{2!} + \frac{(\vec{v}\theta)^3}{3!} + \frac{(\vec{v}\theta)^4}{4!} + \cdots \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + \vec{v}\Big(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\Big). \end{aligned}$$

Thus, we obtain Euler's formula for PDQs such that

$$\begin{aligned} e^{\vec{v}\theta} &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + \vec{v}\Big(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\Big) \\ &= \cos\theta + \vec{v}\sin\theta, \end{aligned}$$

for $\theta \in \mathbb{R}$.

Theorem 3.1. *Let $p = \cos\theta + \vec{v}\sin\theta$ be a unit PDQ. We then have*

$$p^n = \cos n\theta + \vec{v}\sin n\theta, \tag{3.2}$$

for every $n \in \mathbb{Z}$, being the set of integers. Also, the formula holds for $n < 0$ such that

$$p^{-n} = \cos n\theta - \vec{v} \sin n\theta.$$

Proof. Using mathematical induction, we first consider the equation (3.2) when $n = 1$. Then, we have $p = \cos\theta + \vec{v}\sin\theta$. Suppose that for $n = m$, the equation (3.2) is satisfied. Then, by trigonometric addition formula for sine and cosine,

$$\begin{aligned} p^{m+1} &= p^m\, p = (\cos m\theta + \vec{v}\sin m\theta)(\cos\theta + \vec{v}\sin\theta) \\ &= (\cos m\theta\cos\theta - \sin m\theta\sin\theta) + \vec{v}(\cos m\theta\sin\theta + \sin m\theta\cos\theta) \\ &= \cos((m+1)\theta) + \vec{v}\sin((m+1)\theta). \end{aligned}$$

Thus, the equation (3.2) is satisfied for positive integers n. Furthermore, from the definition of p^{-1}, we have

$$p^{-1} = \frac{p^*}{\mathcal{N}(q)} = \frac{\cos\theta + (-\vec{v})\sin\theta}{r^2} = \cos\theta - \vec{v}\sin\theta.$$

Applying the same description process of equation (3.2) to the computation process to show equation (3.2), we find that (3.2) is satisfied. □

We may form the sequence of powers, for a PDQ number P with the form $p = x_0 + \overrightarrow{\mathbf{x}}$. That is, we have

$$\begin{aligned} p^2 &= (x_0^2 - x_1^2) + \overrightarrow{\mathbf{x}}2x_0, \\ p^3 &= (x_0^3 - 3x_0x_1^2) + \overrightarrow{\mathbf{x}}(3x_0^2 - x_1^2), \\ p^4 &= (x_0^4 - 6x_0^2x_1^2 + x_1^4) + \overrightarrow{\mathbf{x}}(4x_0^3 - 4x_0x_1^2), \\ &\vdots \\ p^n &= \sum_{\substack{r=0 \\ r:even}}^{n} (-1)^{\left[\frac{r}{2}\right]} \binom{n}{r} x_0^{n-r}x_1^r + \overrightarrow{\mathbf{x}} \sum_{\substack{r=1 \\ r:odd}}^{n} (-1)^{\left[\frac{r}{2}\right]} \binom{n}{r} x_0^{n-r}x_1^{r-1}, \end{aligned}$$

where $\left[\frac{r}{2}\right]$ is the greatest integer less than or equal to $\frac{r}{2}$. Thus, we have

$$\begin{aligned} \frac{p^n}{n!} &= \sum_{m=0}^{n} \frac{(-1)^m}{(2m)!(n-2m)!} x_0^{n-2m} x_1^{2m} \\ &\quad + \overrightarrow{\mathbf{x}} \sum_{m=0}^{n} \frac{(-1)^m}{(2m+1)!(n-2m-1)!} x_0^{n-2m-1} x_1^{2m}. \end{aligned} \tag{3.3}$$

From the above formulas, we obtain an expression of transcendent functions in PDQs, using the form described by $\overrightarrow{\mathbf{x}}$.

Theorem 3.2. *For $p = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{D}_{\mathbb{P}}$, we write the exponential function such that*

$$e^p = e^{x_0}\{\cos(x_1) + \vec{v}\sin(x_1)\}.$$

Proof. Some evident simplifications are sufficient to arrange the terms in a more familiar way, since by definition of the exponential function

$$e^p = 1 + p + \frac{p^2}{2!} + \frac{p^3}{3!} + \frac{p^4}{4!} + \frac{p^5}{5!} + \cdots . \tag{3.4}$$

Referring to equation (3.3), we expand equation (3.4):

$$\begin{aligned} e^p &= 1 + p + \frac{p^2}{2!} + \frac{p^3}{3!} + \frac{p^4}{4!} + \frac{p^5}{5!} + \cdots \\ &= \Big(1 + x_0 + \frac{1}{2!}x_0^2 - \frac{1}{2!}x_1^2 + \frac{1}{3!}x_0^3 - \frac{1}{3!}3x_0x_1^2 + \frac{1}{4!}x_0^4 - \frac{1}{4!}6x_0^2x_1^2 \\ &\quad + \frac{1}{4!}x_1^4 + \frac{1}{5!}x_0^5 - \frac{1}{5!}10x_0^3x_1^2 + \frac{1}{5!}5x_0x_1^4 + \cdots\Big) \\ &\quad + \overrightarrow{\mathbf{x}}\Big(1 + \frac{1}{2!}2x_0 + \frac{1}{3!}3x_0^2 - \frac{1}{3!}x_1^2 + \frac{1}{4!}4x_0^3 - \frac{1}{4!}4x_0x_1^2 \\ &\quad + \frac{1}{5!}5x_0^4 - \frac{1}{5!}10x_0^2x_1^2 + \frac{1}{5!}x_1^4 + \cdots\Big), \end{aligned}$$

where $\overrightarrow{\mathbf{x}} = ix_1 + jx_2 + kx_3$ and $p = x_0 + \overrightarrow{\mathbf{x}}$. Since we have

$$\begin{cases} e^{x_0} = \displaystyle\sum_{r=0}^{\infty} \frac{x_0^r}{r!} = 1 + x_0 + \frac{x_0^2}{2!} + \frac{x_0^3}{3!} + \frac{x_0^4}{4!} + \frac{x_0^5}{5!} + \cdots , \\ \cos(x_1) = \displaystyle\sum_{r=0}^{\infty} \frac{(-1)^r}{(2r)!}x_1^{2r} = 1 - \frac{x_1^2}{2!} + \frac{x_1^4}{4!} + \cdots , \\ \sin(x_1) = \displaystyle\sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)!}x_1^{2r+1} = x_1 - \frac{x_1^3}{3!} + \frac{x_1^5}{5!} + \cdots , \end{cases}$$

we can observe that $e^{x_0}\cos(x_1)$ is equivalent to the scalar part of e^p and $\frac{e^{x_0}\sin(x_1)}{x_1}$ is equivalent to the vector part of e^p. Thus, we can write

$$\begin{aligned} e^p &= \Big(1 + x_0 + \frac{x_0^2}{2!} + \frac{x_0^3}{3!} + \frac{x_0^4}{4!} + \frac{x_0^5}{5!} + \cdots\Big)\Big(1 - \frac{x_1^2}{2!} + \frac{x_1^4}{4!} + \cdots\Big) \\ &\quad + \overrightarrow{\mathbf{x}}\Big(1 + x_0 + \frac{x_0^2}{2!} + \frac{x_0^3}{3!} + \frac{x_0^4}{4!} + \frac{x_0^5}{5!} + \cdots\Big)\Big(1 - \frac{x_1^2}{3!} + \frac{x_1^4}{5!} + \cdots\Big) \\ &= e^{x_0}\cos(x_1) + \overrightarrow{\mathbf{x}}\frac{e^{x_0}\sin(x_1)}{x_1} \\ &= e^{x_0}\Big(\cos(x_1) + \overrightarrow{\mathbf{x}}\frac{\sin(x_1)}{x_1}\Big). \end{aligned}$$

Since we denote $\vec{v} = \frac{\vec{\mathbf{x}}}{x_1}$ (see (3.1)), the final expression $e^{x_0}(\cos(x_1) + \vec{v}\sin(x_1))$ for e^p is obtained. □

Consider the form of the differential operator for the PDQ and the formula for the function of the PDQ variable. Let a function $f : \mathbb{D}_{\mathbb{P}} \to \mathbb{D}_{\mathbb{P}}$ be denoted by

$$f(p) = f(x_0 + ix_1 + jx_2 + kx_3) = u_0 + iu_1 + ju_2 + ku_3,$$

where $u_r = u_r(x_0, x_1, x_2, x_3) : \mathbb{R}^4 \to \mathbb{R}$ are real-valued functions $(r = 0, 1, 2, 3)$.

Consider the Cauchy-Fueter operators such that

$$D^* = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}$$

and then, we have

$$\begin{aligned} D^* f \quad = \quad & \frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} + i\Big(\frac{\partial u_0}{\partial x_1} + \frac{\partial u_1}{\partial x_0}\Big) \\ & + j\Big(\frac{\partial u_0}{\partial x_2} + \frac{\partial u_1}{\partial x_3} + \frac{\partial u_2}{\partial x_0} - \frac{\partial u_3}{\partial x_1}\Big) \\ & + k\Big(\frac{\partial u_3}{\partial x_0} + \frac{\partial u_0}{\partial x_3} - \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\Big). \end{aligned}$$

Similarly, the conjugate of the Cauchy-Fueter operators is

$$D = \frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1} - j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3}$$

and then, we have

$$\begin{aligned} Df \quad = \quad & \frac{\partial u_0}{\partial x_0} + \frac{\partial u_1}{\partial x_1} + i\Big(\frac{\partial u_1}{\partial x_0} - \frac{\partial u_0}{\partial x_1}\Big) \\ & + j\Big(\frac{\partial u_2}{\partial x_0} - \frac{\partial u_0}{\partial x_2} + \frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\Big) \\ & + k\Big(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} + \frac{\partial u_3}{\partial x_0} - \frac{\partial u_0}{\partial x_3}\Big). \end{aligned}$$

Thus, we obtain the following result:

$$Tf = \frac{1}{2}(D^* f + Df) = \Big(\frac{\partial u_0}{\partial x_0} + i\frac{\partial u_1}{\partial x_0} + j\frac{\partial u_2}{\partial x_0} + k\frac{\partial u_3}{\partial x_0}\Big) = \frac{\partial f}{\partial x_0}. \tag{3.5}$$

We consider a form of the differentiated exponential function in the PDQs.

Theorem 3.3. *Using differential operator T for the exponential function $f(p) = e^p$ in PDQs, we have $T(e^p) = e^p$.*

Proof. From equation (3.5), since $\frac{\partial e^{x_0}}{\partial x_0} = e^{x_0}$, we have

$$\frac{\partial f}{\partial x_0} = e^{x_0}(\cos(x_1) + \vec{v}\sin(x_1)) = e^p.$$

Thus, we obtain $T(e^p) = e^p$. □

Similarly, we have

$$\begin{aligned}
Sf &= \frac{1}{2}(D^* f - Df) \\
&= -\frac{\partial u_1}{\partial x_1} + i\frac{\partial u_0}{\partial x_1} + j\Big(\frac{\partial u_0}{\partial x_2} + \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\Big) + k\Big(\frac{\partial u_0}{\partial x_3} - \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\Big) \\
&= i\Big(\frac{\partial u_0}{\partial x_1} + i\frac{\partial u_1}{\partial x_1} + j\frac{\partial u_2}{\partial x_1} + k\frac{\partial u_3}{\partial x_1}\Big) + j\Big(\frac{\partial u_0}{\partial x_2} + \frac{\partial u_1}{\partial x_3}\Big) + k\Big(\frac{\partial u_0}{\partial x_3} - \frac{\partial u_1}{\partial x_2}\Big) \\
&= i\frac{\partial f}{\partial x_1} + j\Big(\frac{\partial u_0}{\partial x_2} + i\frac{\partial u_1}{\partial x_2}\Big) + k\Big(\frac{\partial u_0}{\partial x_3} + i\frac{\partial u_1}{\partial x_3}\Big) \\
&= i\frac{\partial f}{\partial x_1} + j\frac{\partial f_1}{\partial x_2} + k\frac{\partial f_1}{\partial x_3}, \qquad (3.6)
\end{aligned}$$

where $f_1 : \mathbb{R}^4 \to \mathbb{C}$ is a complex-valued function by $f_1 = u_0 + iu_1$. From the properties (2.1) and (2.2) of i, j, and k, the function Sf is also written as

$$Sf = i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} + k\frac{\partial f}{\partial x_3}.$$

We also consider the form of the differentiated exponential function in PDQs for obtaining the vector part of the differentiated functions in PDQs.

Theorem 3.4. *Using the differential operator S for the exponential function $f(p) = e^p$ in the PDQs, we have*

$$S(e^p) = i\vec{v}e^p - R, \qquad (3.7)$$

where $R = e^{x_0}\sin(x_1) \in \mathbb{R}$.

Proof. In order to be formed the result (3.7), when equation (3.6) is applied to $f(p) = e^p$, the following calculations are performed:

$$\begin{aligned}
S(f) &= i\frac{\partial}{\partial x_1}e^{x_0}(\cos(x_1) + \vec{v}\sin(x_1)) + j\frac{\partial f_1}{\partial x_2} + k\frac{\partial f_1}{\partial x_3} \\
&= ie^{x_0}\Big\{\frac{\partial}{\partial x_1}\cos(x_1) + (\frac{\partial}{\partial x_1}\vec{v})\sin(x_1) + \vec{v}(\frac{\partial}{\partial x_1}\sin(x_1))\Big\} \\
&= ie^{x_0}\Big\{-\sin(x_1) + i\sin(x_1) + \vec{v}\cos(x_1)\Big\} \\
&= ie^{x_0}\vec{v}\Big\{\vec{v}\sin(x_1) - \vec{v}i\sin(x_1) + \cos(x_1)\Big\} \\
&= i\vec{v}e^p - e^{x_0}\sin(x_1).
\end{aligned}$$

Thus, we obtain the equation

$$S(e^p) = i\vec{v}e^p - R,$$

where $R = e^{x_0} \sin(x_1) \in \mathbb{R}$. □

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