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ON LOCATION OF THE ZEROS OF POLYNOMIAL (LACUNARY TYPE)

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Abstract. The aim of this paper is to obtained some extensions and generalizations of well known result on theory of distribution of zeros and related results by relaxing the hypothesis of Eneström-Kakeya theorem for a class of Lacunary type of polynomials $p(z) := a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu, 1 \leq \mu \leq n$.

1. Introduction

Problems in different areas of science reduce the questions about zeros of complex polynomials like signal processing, communication theory, control theory and mathematical biology and many more. Recently, several significant and seemingly unrelated results relevant to theoretical computer science have benefited from taking the route to find the zeros of a complex polynomials.

If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n, then concerning the theory of distribution of zeros of polynomials, Eneström-Kakeya [3, 4, 7] proved the following result.

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Theorem 1.1. (*Eneström-Kakeya Theorem*) *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a* n^{th} *degree polynomial such that*

$$a_0 \geq a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq a_n > 0,$$

then $p(z)$ *has no zeros in* $|z| < 1$.

By applying above result on the polynomial $z^n\overline{p(1/\overline{z})}$, the following generalization of Theorem 1.1 has been described.

Theorem 1.2. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a* n^{th} *degree polynomial such that*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 \geq 0,$$

then the zeros of $p(z)$ *lies in* $|z| \leq 1$.

By relaxing the hypothesis of Theorem 1.2, Aziz and Zargar [1] proved the following result by assuming the alternating coefficients of the polynomial $p(z)$ satisfy (1.1).

Theorem 1.3. *All* n *zeros of polynomial* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a* n^{th} *such that either,* $a_n \geq a_{n-2} \geq \cdots \geq a_3 \geq a_1 > 0$ *and* $a_{n-1} \geq a_{n-3} \geq \cdots \geq a_2 \geq a_0 > 0$ *if* n *is odd, or if* n *is even* $a_n \geq a_{n-2} \geq \cdots \geq a_2 \geq a_0 > 0$ *and* $a_{n-1} \geq a_{n-3} \geq \cdots \geq a_3 \geq a_1 > 0$*, lies in the region*

$$\left| z + \frac{a_{n-1}}{a_n} \right| \leq 1 + \frac{a_{n-1}}{a_n}.$$

In this consequence, some other extensions and generalizations of Theorem 1.2 mentioned in the literature ([1]- [10]).

Govil and Rahman [2] generalized Theorem 1.2 for the polynomial with complex coefficients by considering the moduli of the coefficients to be monotonically increasing. Basically they proved the following:

Theorem 1.4. *All the zeros of a* n^{th} *degree polynomial* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *with complex coefficients such that for some real* β,

$$|\arg a_\nu - \beta| \leq \alpha \leq \pi/2, \quad 0 \leq \nu \leq n$$

and

$$|a_n| \geq |a_{n-1}| \geq |a_{n-2}| \geq \cdots \geq |a_1| \geq |a_0|,$$

lies in the disc

$$|z| \leq (\cos\alpha + \sin\alpha) + 2\frac{\sin\alpha}{|a_n|}\sum_{\nu=0}^{n-1} |a_\nu|.$$

Shah and Liman [8] also generalized the Theorem 1.4 due to Govil and Rahman [2] by proving the following results:

Theorem 1.5. *All the zeros of a n^{th} degree polynomial*

$$p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$$

with complex coefficient such that for some real β,

$$|\arg a_\nu - \beta| \leq \alpha \leq \pi/2$$

and for $k \geq 1$

$$k|a_n| \geq |a_{n-1}| \geq |a_{n-2}| \geq \cdots \geq |a_1| \geq |a_0|, \tag{1.1}$$

lies in the region

$$|z + k - 1| \leq \frac{1}{|a_n|} \left\{ (k|a_n| - |a_0|)(\cos\alpha + \sin\alpha) + |a_0| + 2\sin\alpha \sum_{\nu=0}^{n-1} |a_\nu| \right\}.$$

2. Lemma

Lemma 2.1. ([2]) *If $|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$ and for some $t > 0, |ta_j| \leq |a_{j-1}|$, then*

$$|ta_j - a_{j-1}| \leq \{(|ta_j| - |a_{j-1}|)\cos\alpha + (|ta_j| + |a_{j-1}|)\sin\alpha\}. \tag{2.1}$$

3. Main results

In this paper, we extend Theorem 1.3 for class of polynomials $\mathbb{P}_n$, which is defined as

$$\mathbb{P}_n := \left\{ p(z); p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu, a_0 \neq 0, 1 \leq \mu \leq n \right\}, \tag{3.1}$$

and obtain some other extensions of the classical results concerning the Eneström-Kakeya theorem.

Theorem 3.1. *Let $p \in \mathbb{P}_n$ be a n^{th} degree polynomial with complex coefficients. If $\operatorname{Re} a_j = \alpha_j, \operatorname{Im} a_j = \beta_j, 0 \leq j \leq n$ such that for some $t \geq 1, \lambda, \delta$ and $1 \leq k \leq n$, either, $t^{n-k+2}\alpha_n \geq t^{n-k}\alpha_{n-2} \geq \cdots \geq t^2\alpha_{k+2} \geq \alpha_k \leq \alpha_{k-2} \leq \cdots \leq \alpha_\mu - \lambda$ and $t^{n-k+1}\alpha_{n-1} \geq t^{n-k-1}\alpha_{n-3} \geq \cdots \geq t\alpha_{k+1} \geq \alpha_{k-1} \leq \alpha_{k-3} \leq \cdots \leq \alpha_{\mu+1} - \lambda$, if n and μ is odd, or $t^{n-k+2}\alpha_n \geq t^{n-k}\alpha_{n-2} \geq \cdots \geq t^2\alpha_{k+2} \geq \alpha_k \leq \alpha_{k-2} \leq \cdots \leq \alpha_\mu - \lambda$ and $t^{n-k+1}\alpha_{n-1} \geq t^{n-k-1}\alpha_{n-3} \geq \cdots \geq t\alpha_{k+1} \geq \alpha_{k-1} \leq$*

$\alpha_{k-3} \leq \cdots \leq \alpha_{\mu+1} - \lambda$, *if* n *and* μ *is even and* $\beta_n \leq \beta_{n-1} \leq \cdots \leq \delta + \beta_k \geq \beta_{k-1} \geq \cdots \geq \beta_\mu$, *then the* n *zeros of* $p(z)$ *lies in the disc*

$$\begin{aligned}\left|z + \frac{a_{n-1}}{a_n}\right| \quad < \quad & \frac{1}{|a_n|}\Bigg[\alpha_n + \alpha_{n-1} + (t-1)\sum_{j=k+1}^{n}(|\alpha_j| + \alpha_j) + 2(|\lambda| - \lambda) \\ & -2(\alpha_k + \alpha_{k-1}) + (\alpha_\mu + |\alpha_\mu|) + \alpha_{\mu+1} + |\alpha_0| + 3\delta + |\delta| \\ & +4\beta_k - \beta_{\mu+2} - (\beta_\mu - |\beta_\mu|) - \beta_{n-1} - \beta_n + |\beta_0|\Big].\end{aligned}$$

Proof. Consider the polynomial

$$\begin{aligned} f(z) &= (1 - z^2)p(z) \\ &= (1 - z^2)\left(a_0 + \sum_{j=\mu}^{n} a_\nu z^\nu\right) \\ &= -a_n z^{n+2} - a_{n-1}z^{n+1} + \sum_{j=\mu+2}^{n}(a_j - a_{j-2})z^j \\ &\quad + a_{\mu+1}z^{\mu+1} + a_\mu z^\mu + (1 - z^2)a_0 \\ &= -(a_n z + a_{n-1})z^{n+1} + \sum_{j=\mu+2}^{n}\{(\alpha_j - \alpha_{j-2})z^j + i(\beta_j - \beta_{j-2})z^j\} \\ &\quad + a_{\mu+1}z^{\mu+1} + a_\mu z^\mu + (1 - z^2)a_0 \\ &= -(a_n z + a_{n-1})z^{n+1} + \sum_{j=k+1}^{n}(t\alpha_j - \alpha_{j-2})z^j - (t-1)\sum_{j=k+1}^{n}\alpha_j z^j \\ &\quad + \sum_{j=\mu+4}^{k}(\alpha_j - \alpha_{j-2})z^j + (\alpha_{\mu+3} - \alpha_{\mu+1} + \lambda)z^{\mu+3} \\ &\quad - \lambda z^{\mu+3} + (\alpha_{\mu+2} - \alpha_\mu + \lambda)z^{\mu+2} \\ &\quad - \lambda z^{\mu+2} + i\sum_{j=k+2}^{n}(\beta_j - \beta_{j-2})z^j + i(\beta_{k+1} - (\delta + \beta_k))z^{k+1} \\ &\quad + ((\delta + \beta_k) - \beta_{k-1})z^{k+1} \\ &\quad + i(\delta + \beta_k - \beta_{k-1})z^k - i\delta z^k + i\sum_{j=\mu+2}^{k-1}(\beta_j - \beta_{j-2})z^j \\ &\quad + a_{\mu+1}z^{\mu+1} + a_\mu z^\mu + (1 - z^2)a_0. \end{aligned}$$

This implies that,

$$
\begin{aligned}
f(z)| \geq\ & |a_n z + a_{n-1}||z|^{n+1} - |z|^n \left[\sum_{j=k+1}^{n} \frac{|t\alpha_j - \alpha_{j-2}|}{|z|^{n-j}} + |t-1| \sum_{j=k+1}^{n} \frac{|\alpha_j|}{|z|^{n-j}} \right. \\
& + \sum_{j=\mu+4}^{k} \frac{|\alpha_j - \alpha_{j-2}|}{|z|^{n-j}} + \frac{|\alpha_{\mu+3} - (\alpha_{\mu+1} - \lambda)|}{|z|^{n-\mu-3}} \\
& + \frac{|\lambda|}{|z|^{n-\mu-3}} + \frac{|\alpha_{\mu+2} - (\alpha_\mu - \lambda)|}{|z|^{n-\mu-2}} + \frac{|\lambda|}{|z|^{n-\mu-2}} \\
& + \sum_{j=k+2}^{n} \frac{|\beta_j - \beta_{j-2}|}{|z|^{n-j}} + \frac{\beta_{k+1} - (\beta_k + \delta)|}{|z|^{n-k-1}} + \frac{|(\beta_k + \delta) - \beta_{k-1}|}{|z|^{n-k-1}} \\
& + \frac{|\delta + \beta_k - \beta_{k-2}|}{|z|^{n-k}} + \frac{|\delta|}{|z|^{n-k}} + \sum_{j=\mu+2}^{k-1} \frac{|\beta_j - \beta_{j-2}|}{|z|^{n-j}} \\
& \left. + \frac{|\alpha_{\mu+1}| + |\beta_{\mu+1}|}{|z|^{n-\mu-1}} + \frac{|\alpha_\mu| + |\beta_\mu|}{|z|^{n-\mu}} + \frac{|\alpha_0| + |\beta_0|}{|z|^n} + \frac{|\alpha_0| + |\beta_0|}{|z|^{n-2}} \right].
\end{aligned}
$$

Let $|z| > 1$, so that $\frac{1}{|z|^j} < 1, j = \mu, \mu+1, \ldots, n$. Then

$$
\begin{aligned}
|f(z)| \geq\ & |a_n z + a_{n-1}||z|^{n+1} - |z|^n \left[\sum_{j=k+1}^{n} |t\alpha_j - \alpha_{j-2}| + |t-1| \sum_{j=k+1}^{n} |\alpha_j| \right. \\
& + \sum_{j=\mu+4}^{k} |\alpha_j - \alpha_{j-2}| + |(\alpha_{\mu+1} - \lambda) - \alpha_{\mu+3}| + |(\alpha_\mu - \lambda) - \alpha_{\mu+2}| \\
& + 2|\lambda| + \sum_{j=k+2}^{n} |\beta_j - \beta_{j-2}| + |\beta_{k+1} - (\delta + \beta_k)| + |(\delta + \beta_k) - \beta_{k-1}| \\
& + |\delta + \beta_k - \beta_{k-2}| + |\delta| + \sum_{j=\mu+2}^{k-1} |\beta_j - \beta_{j-2}| + |\alpha_{\mu+1}| \\
& \left. + |\beta_{\mu+1}| + |\alpha_\mu| + |\beta_\mu| + |\alpha_0| + |\beta_0| \right] \\
\geq\ & |a_n z + a_{n-1}||z|^{n+1} - |z|^n \left[\alpha_n + \alpha_{n-1} + (t-1) \sum_{j=k+1}^{n} (|\alpha_j| + \alpha_j) \right. \\
& + 2(|\lambda| - \lambda) - 2(\alpha_k + \alpha_{k-1}) + (\alpha_\mu + |\alpha_\mu|) + \alpha_{\mu+1} + |\alpha_0| \\
& \left. + 3\delta + |\delta| + 4\beta_k - \beta_{\mu+2} - (\beta_\mu - |\beta_\mu|) - \beta_{n-1} - \beta_n + |\beta_0| \right],
\end{aligned}
$$

if

$$\begin{aligned} |a_n z + a_{n-1}| &> \alpha_n + \alpha_{n-1} + (t-1)\sum_{j=k+1}^{n}(|\alpha_j| + \alpha_j) \\ &\quad +2(|\lambda| - \lambda) - 2(\alpha_k + \alpha_{k-1}) + (\alpha_\mu + |\alpha_\mu|) + \alpha_{\mu+1} + |\alpha_0| \\ &\quad +3\delta + |\delta| + 4\beta_k - \beta_{\mu+2} - (\beta_\mu - |\beta_\mu|) - \beta_{n-1} - \beta_n + |\beta_0|. \end{aligned}$$

So $f(z)$ does not vanish for

$$\begin{aligned} \left|z + \frac{a_{n-1}}{a_n}\right| &> \frac{1}{|a_n|}[\alpha_n + \alpha_{n-1} + (t-1)\sum_{j=k+1}^{n}(|\alpha_j| + \alpha_j) \\ &\quad +2(|\lambda| - \lambda) - 2(\alpha_k + \alpha_{k-1}) + (\alpha_\mu + |\alpha_\mu|) + \alpha_{\mu+1} + |\alpha_0| \\ &\quad +3\delta + |\delta| + 4\beta_k - \beta_{\mu+2} - (\beta_\mu - |\beta_\mu|) - \beta_{n-1} - \beta_n + |\beta_0|]. \end{aligned}$$

Therefore those zeros of $f(z)$ whose modulus is greater than 1 lies in

$$\begin{aligned} \left|z + \frac{a_{n-1}}{a_n}\right| &< \frac{1}{|a_n|}\Bigg[\alpha_n + \alpha_{n-1} + (t-1)\sum_{j=k+1}^{n}(|\alpha_j| + \alpha_j) \\ &\quad +2(|\lambda| - \lambda) - 2(\alpha_k + \alpha_{k-1}) + (\alpha_\mu + |\alpha_\mu|) + \alpha_{\mu+1} + |\alpha_0| \\ &\quad +3\delta + |\delta| + 4\beta_k - \beta_{\mu+2} - (\beta_\mu - |\beta_\mu|) - \beta_{n-1} - \beta_n + |\beta_0|\Bigg]. \end{aligned}$$

But those zeros of $f(z)$ whose modulus is less than or equal to 1 already satisfy above inequality. Hence all the zeros of $f(z)$ and those of $p(z)$ lies in

$$\begin{aligned} \left|z + \frac{a_{n-1}}{a_n}\right| &\leq \frac{1}{|a_n|}\Bigg[\alpha_n + \alpha_{n-1} + (t-1)\sum_{j=k+1}^{n}(|\alpha_j| + \alpha_j) + 2(|\lambda| - \lambda) \\ &\quad -2(\alpha_k + \alpha_{k-1}) + (\alpha_\mu + |\alpha_\mu|) + \alpha_{\mu+1} + |\alpha_0| \\ &\quad +3\delta + |\delta| + 4\beta_k - \beta_{\mu+2} - (\beta_\mu - |\beta_\mu|) - \beta_{n-1} - \beta_n + |\beta_0|\Bigg]. \end{aligned}$$

This completes the proof. □

For taking $t = 1$ and $\lambda = 0$, we have the following result.

Corollary 3.2. *Let $p \in \mathbb{P}_n$ be a n^{th} degree polynomial with complex coefficients. If $\operatorname{Re} a_j = \alpha_j, \operatorname{Im} a_j = \beta_j, 0 \leq j \leq n$ such that for some $t \geq 1, \lambda, \delta$ and $1 \leq k \leq n$, either, $\alpha_n \geq \alpha_{n-2} \geq \cdots \geq \alpha_{k+2} \geq \alpha_k \leq \alpha_{k-2} \leq \cdots \leq \alpha_\mu$ and $\alpha_{n-1} \geq \alpha_{n-3} \geq \cdots \geq \alpha_{k+1} \geq \alpha_{k-1} \leq \alpha_{k-3} \leq \cdots \leq \alpha_{\mu+1}$, if n and μ is odd, or $\alpha_n \geq \alpha_{n-2} \geq \cdots \geq \alpha_{k+2} \geq \alpha_k \leq \alpha_{k-2} \leq \cdots \leq \alpha_\mu$ and $\alpha_{n-1} \geq \alpha_{n-3} \geq \cdots \geq \alpha_{k+1} \geq \alpha_{k-1} \leq \alpha_{k-3} \leq \cdots \leq \alpha_{\mu+1}$, if n and μ is even and $\beta_n \leq \beta_{n-1} \leq \cdots \leq \delta + \beta_k \geq \beta_{k-1} \geq \cdots \geq \beta_\mu$, then the n zeros of*

$p(z)$ lies in the disc

$$\left|z+\frac{a_{n-1}}{a_n}\right| < \frac{1}{|a_n|}\left[\alpha_n+\alpha_{n-1}-2(\alpha_k+\alpha_{k-1})+(\alpha_\mu+|\alpha_\mu|)+\alpha_{\mu+1}+|\alpha_0|\right.$$
$$\left.+3\delta+|\delta|+4\beta_k-\beta_{\mu+2}-(\beta_\mu-|\beta_\mu|)-\beta_{n-1}-\beta_n+|\beta_0|\right].$$

Remark 3.3. We also have some direct extension of Theorem 1.3 for the class of polynomials $\mathbb{P}_n$ by taking all imaginary part is zero, i.e. $\beta_j=0$, for all j=0, $\mu,\mu+1,\cdots,n$ and some other extended generalization of Theorem 3.1 by taking $\delta=1,\lambda=0$ and $t=1$ respectively.

Remark 3.4. On taking $\beta_i=0, 0\le i\le n, k=0$ and assuming $\alpha_i=a_i>0, 0\le i\le n$ in Corollary 3.2, Theorem 1.3 has been obtained.

Next, we extend and generalized the Theorem 1.5 for the class of polynomials $\mathbb{P}_n$. Basically we prove the following.

Theorem 3.5. *Let $p\in\mathbb{P}_n$ be a n^{th} degree polynomial with complex coefficients such that for some real β,*

$$|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2},\quad 1\le j\le n,$$

and for some real δ and $0\le k\le n$, either

$$|a_n|\ge|a_{n-2}|\ge|a_{n-4}|\ge\cdots\ge\delta|a_k|\le|a_{k-2}|\le\cdots\le|a_\mu|\quad and$$
$$|a_{n-1}|\ge|a_{n-3}|\ge|a_{n-5}|\ge\cdots\ge\delta|a_{k+1}|\le|a_{k-1}|\le\cdots\le\cdots\le|a_{\mu+1}|$$

if n and μ is even, or

$$|a_n|\ge|a_{n-2}|\ge|a_{n-4}|\ge\cdots\ge\delta|a_k|\le|a_{k-2}|\le\cdots\le\cdots\le|a_\mu|\quad and$$
$$|a_{n-1}|\ge|a_{n-3}|\ge|a_{n-5}|\ge\cdots\ge\delta|a_{k+1}|\le|a_{k-1}|\le\cdots\le\cdots\le|a_{\mu+1}|$$

if n and μ is odd, then all zeros of $p(z)$ lies in the disc

$$\left|z+\frac{a_{n-1}}{a_n}\right| \le \frac{1}{|a_n|}\{(|a_n|+|a_{n-1}|-(|\delta|-1)|a_k|-(|\delta|-1)|a_{k+1}|$$
$$-|a_{k-2}|-|a_{k-1}|+|a_{\mu+1}|+|a_\mu|)\cos\alpha$$
$$+|\delta-1|(|a_k|+|a_{k+1}|)+|a_{\mu+1}|+|a_\mu|$$
$$+(|a_n|+|a_{n-1}|+2\sum_{j=\mu}^{n-2}|a_j|-|a_1|-|a_0|$$
$$+(|\delta|-1)(|a_{k+1}|+|a_k|)\ -|a_{\mu+1}-|a_\mu||)\sin\alpha\}.$$

Proof. Consider the polynomial

$$\begin{aligned}
f(z) &= (1-z^2)p(z) \\
&= (1-z^2)\left(a_0+\sum_{j=\mu}^{n} a_\nu z^\nu\right) \\
&= -a_n z^{n+2} - a_{n-1}z^{n+1} + \sum_{j=\mu+2}^{n}(a_j-a_{j-2})z^j \\
&\quad +a_{\mu+1}z^{\mu+1} + a_\mu z^\mu + (1-z^2)a_0 \\
&= -a_n z^{n+2} - a_{n-1}z^{n+1} + \sum_{j=\mu+2}^{k-1}(a_j-a_{j-2})z^j \\
&\quad +(a_k-a_{k-2})z^k + (a_{k+1}-a_{k-1})z^{k+1} \\
&\quad + \sum_{j=k+2}^{n}(a_j-a_{j-2})z^j + a_{\mu+1}z^{\mu+1} + a_\mu z^\mu + (1-z^2)a_0 \\
&= -a_n z^{n+2} - a_{n-1}z^{n+1} + \sum_{j=\mu+2}^{k-1}(a_j-a_{j-2})z^j \\
&\quad +(\delta a_k-a_{k-2})z^k + (1-\delta)a_k z^k \\
&\quad +(\delta a_{k+1}-a_{k-1})z^{k+1} + (1-\delta)z^{k+1} \\
&\quad + \sum_{j=k+2}^{n}(a_j-a_{j-2})z^j + a_{\mu+1}z^{\mu+1} + a_\mu z^\mu + (1-z^2)a_0,
\end{aligned}$$

that is,

$$\begin{aligned}
|f(z)| &\geq |z|^n\left[|z||a_n z+a_{n-1}| - \left\{\sum_{j=\mu+2}^{k-1}\frac{|a_j-a_{j-2}|}{|z|^{n-j}} + \frac{|\delta a_k-a_{k-2}|}{|z|^{n-k}}\right.\right. \\
&\quad +\frac{|1-\delta||a_k|}{|z|^{n-k}} + \frac{|\delta a_{k+1}-a_{k-1}|}{|z|^{n-k-1}} + \frac{|1-\delta||a_{k+1}|}{|z|^{n-k-1}} \\
&\quad \left.\left. + \sum_{j=k+2}^{n}\frac{|a_j-a_{j-2}|}{|z|^{n-j}} + \frac{|a_{\mu+1}|}{|z|^{n-\mu-1}} + \frac{|a_\mu|}{|z|^{n-\mu}} + \frac{|a_0|}{|z|^{n-2}} + \frac{|a_0|}{|z|^{n}}\right\}\right].
\end{aligned}$$

For $|z| > 1$,

$$\begin{aligned}|f(z)| &> |z|^n\left[|z||a_n z + a_{n-1}| - \left\{\sum_{j=\mu+2}^{k-1}|a_j - a_{j-2}| + |\delta a_k - a_{k-2}|\right.\right.\\ &\quad +|1-\delta||a_k| + |\delta a_{k+1} - a_{k-1}| + |1-\delta||a_{k+1}|\\ &\quad \left.\left.+\sum_{j=k+2}^{n}|a_j - a_{j-2}| + |a_{\mu+1}| + |a_\mu| + 2|a_0|\right\}\right].\end{aligned}$$

Now, using the Lemma 2.1, we get

$$\begin{aligned}|f(z)| &\geq |z|^n\left[|z||a_n z + a_{n-1}| - \{((|\delta|+1)|a_k| + (|\delta|+1)|a_{k+1}|\right.\\ &\quad -|a_n| - |a_{n-1}| + |a_{\mu+1}| + |a_\mu|)\cos\alpha\\ &\quad +|\delta - 1|(|a_k| + |a_{k+1}|) + |a_{\mu+1}| + |a_\mu| + 2|a_0|\\ &\quad +(|a_n| + |a_{n-1}| + 2\sum_{j=\mu+2}^{n-2}|a_j| + |a_{\mu+1}| + |a_\mu|\\ &\quad \left.+(|\delta| - 1)(|a_{k+1}| + |a_k|))\sin\alpha\}\right]\\ &> 0,\end{aligned}$$

if

$$\begin{aligned}|z + \frac{a_{n-1}}{a_n}| > \frac{1}{|a_n|}&\{((|\delta|+1)|a_k| + (|\delta|+1)|a_{k+1}|\\ &- |a_n| - |a_{n-1}| + |a_{\mu+1}| + |a_\mu|)\cos\alpha\\ &+ |\delta - 1|(|a_k| + |a_{k+1}|) + |a_{\mu+1}| + |a_\mu| + 2|a_0| + (|a_n| + |a_{n-1}|\\ &+ 2\sum_{j=\mu+2}^{n-2}|a_j| + |a_{\mu+1}| + |a_\mu| + (|\delta| - 1)(|a_{k+1}| + |a_k|))\sin\alpha\},\end{aligned}$$

then all the zeros of $f(z)$, i.e., $p(z)$ lies in the disc

$$\begin{aligned}|z + \frac{a_{n-1}}{a_n} \leq \frac{1}{|a_n|}&\{((|\delta|+1)|a_k| + (|\delta|+1)|a_{k+1}| - |a_n| - |a_{n-1}| + |a_{\mu+1}|\\ &+ |a_\mu|)\cos\alpha + |\delta - 1|(|a_k| + |a_{k+1}|) + |a_{\mu+1}| + |a_\mu| + 2|a_0|\\ &+ (|a_n| + |a_{n-1}| + 2\sum_{j=\mu+2}^{n-2}|a_j| + |a_{\mu+1}| + |a_\mu|\\ &+ (|\delta| - 1)(|a_{k+1}| + |a_k|))\sin\alpha\}.\end{aligned}$$

This completes the proof of Theorem 3.5. □

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