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SOLVING A VARIATIONAL INCLUSION PROBLEM WITH ITS CORRESPONDING RESOLVENT EQUATION PROBLEM INVOLVING XOR-OPERATION

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Abstract. In this paper, we consider a variational inclusion problem involving XOR-operation with its resolvent equation problem involving XOR-operation. We suggest separate iterative algorithms for solving both the problems. The existence and convergence results are proved for variational inclusion problem and for corresponding resolvent equation problem in ordered Hilbert spaces. We claim that results of this paper are new and refinement of previously known results.

1. Introduction

Many extensions and generalizations of variational inequalities came into existence to study a number of problems related to mechanics, physics, optimization and control, nonlinear programming, elasticity, basic sciences and applied sciences, etc., see for example [1, 5, 6, 8, 9] and references therein. Projection methods can not be used to solve mixed variational inequalities which contain nonlinear terms. To over come this difficulty in 1994, Hassouni and Moudafi [10] introduced and studied variational inclusions containing mixed

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variational inequalities as special cases. He used resolvent operator methods to solve variational inclusions.

The resolvent operator techniques for solving variational inclusions are useful and used to establish an equivalence between variational inclusions and resolvent equations. The resolvent equation techniques are used to develop applicable numerical techniques for solving variational inclusions and several other equivalent problems.

Li with his co-authors [11, 12, 13, 14], Ahmad with his co-authors [2, 3] considered and studied ordered variational inequalities(inclusions) using XOR-operation with some other related concepts.

The study of this paper is focused to solve a variational inclusion problem with its corresponding resolvent equation problem involving XOR-operation. We use the resolvent operator which involves XOR-operation. The results of this paper are quite new related to ordered variational inclusion problems.

2. Preliminaries

Throughout the paper, we assume $\mathcal{H}_p$ to be a real ordered positive Hilbert space with its norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, d is the metric induced by the norm $\|\cdot\|$, $2^{\mathcal{H}_p}$ (respectively, $C(\mathcal{H}_p)$) is the family of nonempty (respectively, compact) subsets of $\mathcal{H}_p$, and $\mathcal{D}(.,.)$ is the Hausdörff metric on $C(\mathcal{H}_p)$.

For the presentation of this paper, we need the following concepts and results, most of them can be found in [7, 13, 14].

Definition 2.1. A nonempty closed convex subset C of $\mathcal{H}_p$ is said to be a cone if,

(i) for any $x \in C$ and any $\lambda > 0$, $\lambda x \in C$,
(ii) if $x \in C$ and $-x \in C$, then $x = 0$.
(iii) the cone C is called normal, if there exists a constant $\lambda_N > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq \lambda_N \|y\|$, for all $x, y \in \mathcal{H}_p$,
(iv) for any $x, y \in \mathcal{H}_p$, $x \leq y$ if and only if $y - x \in C$,
(v) x and y are said to be comparable to each other, if either $x \leq y$ or $y \leq x$ holds and is denoted by $x \propto y$.

Definition 2.2. For any $x, y \in \mathcal{H}_p$, let $lub\{x, y\}$ denotes least upper bound and $glb\{x, y\}$ denotes greatest lower bound of the set $\{x, y\}$. Suppose $lub\{x, y\}$ and $glb\{x, y\}$ for the set $\{x, y\}$ exist, then some binary operations are defined below:

(i) $x \vee y = lub\{x, y\}$,

(ii) $x \wedge y = glb\{x, y\}$,
(iii) $x \oplus y = (x - y) \vee (y - x)$,
(iv) $x \odot y = (x - y) \wedge (y - x)$.

The operations $\vee, \wedge, \oplus$ and $\odot$ are called OR, AND, XOR and XNOR operations, respectively.

Lemma 2.3. *If $x \propto y$, then $lub\{x, y\}$ and $glb\{x, y\}$ exist, $x - y \propto y - x$ and $0 \leq (x - y) \vee (y - x)$.*

Lemma 2.4. *For any natural number n, $x \propto y_n$ and $y_n \to y^*$ as $n \to \infty$, then $x \propto y^*$.*

Proposition 2.5. *Let $\oplus$ be an XOR-operation and $\odot$ be an XNOR-operation. Then the following relations hold:*

(i) $x \odot x = 0$, $x \odot y = y \odot x = -(x \oplus y) = -(y \oplus x)$,
(ii) *if $x \propto 0$, then $-x \oplus 0 \leq x \leq x \oplus 0$,*
(iii) $(\lambda x) \oplus (\lambda y) = |\lambda|(x \oplus y)$,
(iv) $0 \leq x \oplus y$, *if $x \propto y$,*
(v) *if $x \propto y$, then $x \oplus y = 0$ if and only if $x = y$,*
(vi) $(x + y) \odot (u + v) \geq (x \odot u) + (y \odot v)$,
(vii) $(x + y) \odot (u + v) \geq (x \odot v) + (y \odot u)$,
(viii) *if x, y and w are comparable to each other, then $(x \oplus y) \leq (x \oplus w) + (w \oplus y)$,*
(ix) $\alpha x \oplus \beta x = |\alpha - \beta| x = (\alpha \oplus \beta) x$, *if $x \propto 0$, $\forall\ x, y, u, v, w \in \mathcal{H}_p$ and $\alpha, \beta, \lambda \in \mathbb{R}$.*

Proposition 2.6. *Let C be a normal cone in $\mathcal{H}_p$ with constant λ_N. Then for each $x, y \in \mathcal{H}_p$, the following relations hold:*

(i) $\|0 \oplus 0\| = \|0\| = 0$,
(ii) $\|x \vee y\| \leq \|x\| \vee \|y\| \leq \|x\| + \|y\|$,
(iii) $\|x \oplus y\| \leq \|x - y\| \leq \lambda_N \|x \oplus y\|$,
(iv) *if $x \propto y$, then $\|x \oplus y\| = \|x - y\|$.*

Definition 2.7. Let $A : \mathcal{H}_p \to \mathcal{H}_p$ be a single-valued mapping. Then

(i) A is said to be a comparison mapping, if for each $x, y \in \mathcal{H}_p$, $x \propto y$ then $A(x) \propto A(y)$, $x \propto A(x)$ and $y \propto A(y)$,
(ii) A is said to be strongly comparison mapping, if A is a comparison mapping and $A(x) \propto A(y)$ if and only if $x \propto y$, for any $x, y \in \mathcal{H}_p$.

Definition 2.8. A single-valued mapping $A : \mathcal{H}_p \to \mathcal{H}_p$ is said to be β-ordered compression mapping, if A is a comparison mapping and

$$A(x) \oplus A(y) \leq \beta(x \oplus y), \text{ for } 0 < \beta < 1.$$

Definition 2.9. Let $T : \mathcal{H}_p \to C(\mathcal{H}_p)$ be a set-valued mapping. Let $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a set-valued mapping. Let $f, g,\ A : \mathcal{H}_p \to \mathcal{H}_p$ be single-valued mappings. Then

(i) T is said to be $\mathcal{D}$-Lipschitz-type-continuous, if for any $x, y \in \mathcal{H}_p$, $x \propto y$, there exists a constant $\lambda_T > 0$ such that

$$\mathcal{D}(T(x), T(y)) \leq \lambda_T \|x \oplus y\|.$$

(ii) M is said to be a comparison mapping, if for any $v_x \in M(x)$, $x \propto v_x$, and if $x \propto y$, then for $v_x \in M(x)$ and $v_y \in M(y)$, $v_x \propto v_y$, for all $x, y \in \mathcal{H}_p$,

(iii) a comparison mapping M is said to be α-non-ordinary difference mapping, if for each $x, y \in \mathcal{H}_p$, $v_x \in M(x)$ and $v_y \in M(y)$ such that

$$(v_x \oplus v_y) \oplus \alpha(x \oplus y) = 0.$$

(iv) the comparison mapping M is said to be α-non-ordinary difference mapping with respect to A, if

$$(v_x \oplus v_y) \oplus \alpha(A(x) \oplus A(y)) = 0.$$

(v) the comparison mapping M is said to be θ-ordered rectangular, if there exists a constant $\theta > 0$, for any $x, y \in \mathcal{H}_p$, there exist $v_x \in M(x)$ and $v_y \in M(y)$ such that

$$\langle v_x \odot v_y, -(x \oplus y)\rangle \geq \theta\|x \oplus y\|^2, \text{ for all } x, y \in \mathcal{H}_p,$$

holds.

(vi) a comparison mapping M is said to be λ-XOR-ordered strongly monotone mapping, if $x \propto y$, then there exists a constant $\lambda > 0$ such that

$$\lambda(v_x \oplus v_y) \geq x \oplus y, \text{ for all } x, y \in \mathcal{H}_p, v_x \in M(x), v_y \in M(y).$$

(vii) the comparison mapping M is said to be θ-ordered rectangular with respect to f and g, if

$$\langle v_x \odot v_y, -[(f(x), g(x)) \oplus (f(y), g(y))]\rangle \geq \theta\|(f(x), g(x)) \oplus (f(y), g(y)\|^2,$$

for all $x, y \in \mathcal{H}_p$,

(viii) λ-XOR-ordered strongly with respect to f and g, if

$$\lambda(v_x \oplus v_y) \geq [(f(x), g(x)) \oplus (f(y), g(y))],$$

for all $x, y \in \mathcal{H}_p, v_x \in M(x), v_y \in M(y)$.

Definition 2.10. Let $A, f, g : \mathcal{H}p \to \mathcal{H}_p$ be the single-valued mappings. The mapping A is called a β-ordered compression mapping with respect to f and g, if A is a comparison mapping and

$$(A(x) \oplus A(y)) \leq \beta_2[(f(x), g(x)) \oplus (f(y), g(y))],\ 0 < \beta_2 < 1, \text{ for all } x, y \in \mathcal{H}_p.$$

Definition 2.11. Let $A, f, g : \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings and $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a set-valued mapping and M is a XOR-α-non-ordinary difference mapping if M is an α-non-ordinary difference mapping and $[A \oplus \lambda M(f,g)](\mathcal{H}_p) = \mathcal{H}_p$, for $\lambda, \alpha > 0$.

Definition 2.12. Let $A, f, g : \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings, $M : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a set-valued mapping and M is a XOR-α-non-ordinary difference mapping. Then the resolvent operator $\mathcal{J}^A_{\lambda, M(f,g)}$ associated with A, f and g is defined as:

$$\mathcal{J}^A_{\lambda, M(f,g)}(x) = [A \oplus \lambda M(f,g)]^{-1}(x), \text{ for all } x \in \mathcal{H}_p \text{ and } \alpha, \lambda > 0.$$

Proposition 2.13. *Let $f, g : \mathcal{H}_p \to \mathcal{H}_p$ be the one-one single-valued mappings and $A : \mathcal{H}_p \to \mathcal{H}_p$ be a β-ordered compression mapping with respect to f and g. Let $M : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a set-valued θ-ordered rectangular mapping with respect to f and g. Then the resolvent operator $\mathcal{J}^A_{\lambda, M(f,g)} : \mathcal{H}_p \to \mathcal{H}_p$ is single-valued, for $\theta\lambda > \beta$ and $\lambda > 0$.*

Proof. For any $u \in \mathcal{H}_p$ and a constant $\lambda > 0$, let $x, y \in [A \oplus \lambda M(f,g)]^{-1}(u)$, Then, let

$$v_x = \frac{1}{\lambda}(u \oplus A(x)) \in M(f(x), g(x))$$

and

$$v_y = \frac{1}{\lambda}(u \oplus A(y)) \in M(f(y), g(y)).$$

Using (i) and (ii) of Proposition 2.5, we have

$$\begin{aligned} v_x \odot v_y &= \frac{1}{\lambda}(u \oplus A(x)) \odot \frac{1}{\lambda}(u \oplus A(y)) \\ &= \frac{1}{\lambda}[(u \oplus A(x)) \odot (u \oplus A(y))] \\ &= -\frac{1}{\lambda}[(u \oplus A(x)) \oplus (u \oplus A(y))] \\ &= -\frac{1}{\lambda}[(u \oplus u) \oplus (A(x) \oplus A(y))] \\ &= -\frac{1}{\lambda}[0 \oplus (A(x) \oplus A(y))] \\ &\leq -\frac{1}{\lambda}[A(x) \oplus A(y)]. \end{aligned}$$

Then, we have

$$v_x \odot v_y \leq -\frac{1}{\lambda}[A(x) \oplus A(y)]. \tag{2.1}$$

Since M is θ-ordered rectangular mapping with respect to f and g. A is β-ordered compression mapping with respect to f and g and using (2.1), we have

$$\begin{aligned}\theta\|(f(x),&g(x)) \oplus (f(y),g(y))\|^2 \\ &\leq \langle v_x \odot v_y, -[(f(x),g(x)) \oplus (f(y),g(y))]\rangle \\ &\leq \langle -\frac{1}{\lambda}[A(x) \oplus A(y)], -[(f(x),g(x)) \oplus (f(y),g(y))]\rangle \\ &\leq \frac{1}{\lambda}\langle A(x) \oplus A(y), [(f(x),g(x)) \oplus (f(y),g(y))]\rangle \\ &\leq \frac{\beta}{\lambda}\langle (f(x),g(x)) \oplus (f(y),g(y)), (f(x),g(x)) \oplus (f(y),g(y))\rangle \\ &= \frac{\beta}{\lambda}\|(f(x),g(x)) \oplus (f(y),g(y))\|^2,\end{aligned}$$

that is,

$$\theta\|(f(x),g(x)) \oplus (f(y),g(y))\|^2 \leq \frac{\beta}{\lambda}\|(f(x),g(x)) \oplus (f(y),g(y))\|^2.$$

Hence we have

$$\Big(\theta - \frac{\beta}{\lambda}\Big)\|(f(x),g(x)) \oplus (f(y),g(y))\|^2 \leq 0,$$

for $\theta\lambda > \beta$, which implies that

$$\|(f(x),g(x)) \oplus (f(y),g(y))\| = 0.$$

It follows that

$$(f(x),g(x)) \oplus (f(y),g(y)) = 0,$$

it implies that

$$(f(x),g(x)) = (f(y),g(y)).$$

Therefore, we have

$$f(x) = f(y),\ g(x) = g(y).$$

Since f and g are one-one mappings, it follows that, $x = y$. Hence the resolvent operator $\mathcal{J}^A_{\lambda,M(f,g)}$ is single-valued, for $\theta\lambda > \beta$. □

Proposition 2.14. *Let the set-valued mapping $M : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be an λ-XOR-ordered strongly monotone with respect to f, g. Let $A : \mathcal{H}_p \to \mathcal{H}_p$ be a strongly comparison mapping with respect to $\mathcal{J}^H_{\lambda,M(f,g)}$ and suppose that $(f(x),g(x)) \oplus (f(y),g(y)) \propto x \oplus y$. Then the resolvent operator $\mathcal{J}^A_{\lambda,M(f,g)}$: $\mathcal{H}_p \to \mathcal{H}_p$ is a comparison mapping.*

Proof. For any $x, y \in \mathcal{H}_p$, let

$$v_x^* = \frac{1}{\lambda}[x \oplus A(\mathcal{J}_{\lambda,M(f,g)}^A(x))] \in M(f(\mathcal{J}_{\lambda,M(f,g)}^A(x)), g(\mathcal{J}_{\lambda,M(f,g)}^A(x))) \tag{2.2}$$

and

$$v_y^* = \frac{1}{\lambda}[y \oplus A(\mathcal{J}_{\lambda,M(f,g)}^A(y))] \in M(f(\mathcal{J}_{\lambda,M(f,g)}^A(y)), g(\mathcal{J}_{\lambda,M(f,g)}^A(y))). \tag{2.3}$$

Since M is a λ-XOR-ordered strongly monotone mapping with respect to f and g, $(f(x), g(x)) \oplus (f(y), g(y)) \propto x \oplus y$, using (2.2), (2.3), we have

$$\begin{aligned}(f(x), g(x)) \oplus (f(y), g(y)) &\leq \lambda(v_x^* \oplus v_y^*) \\ &\leq \Big[(x \oplus A(\mathcal{J}_{\lambda,M(f,g)}^A(x))) \oplus (y \oplus A(\mathcal{J}_{\lambda,M(f,g)}^A(y)))\Big] \\ &\leq (x \oplus y) \oplus \Big[A(\mathcal{J}_{\lambda,M(f,g)}^A(x)) \oplus A(\mathcal{J}_{\lambda,M(f,g)}^A(y))\Big].\end{aligned}$$

Hence we have

$$\begin{aligned}0 \leq \Big[(f(x), g(x)) \oplus (f(y), g(y)) \oplus (x \oplus y)\Big] \\ \oplus \Big[A(\mathcal{J}_{\lambda,M(f,g)}^A(x)) \oplus A(\mathcal{J}_{\lambda,M(f,g)}^A(y))\Big],\end{aligned}$$

it implies that

$$0 \leq \Big[(x \oplus y) \oplus (x \oplus y)\Big] \oplus \Big[A(\mathcal{J}_{\lambda,M(f,g)}^A(x)) \oplus A(\mathcal{J}_{\lambda,M(f,g)}^A(y))\Big].$$

That is,

$$0 \leq \Big[A(\mathcal{J}_{\lambda,M(f,g)}^A(x)) \oplus A(\mathcal{J}_{\lambda,M(f,g)}^A(y))\Big].$$

it means that

$$\begin{aligned}0 \leq \Big[A(\mathcal{J}_{\lambda,M(f,g)}^A(x)) - A(\mathcal{J}_{\lambda,M(f,g)}^A(y))\Big] \\ \vee \Big[A(\mathcal{J}_{\lambda,M(f,g)}^A(y)) - A(\mathcal{J}_{\lambda,M(f,g)}^A(x))\Big].\end{aligned}$$

Therefore, it implies that either

$$0 \leq \Big[A(\mathcal{J}_{\lambda,M(f,g)}^A(x)) - A(\mathcal{J}_{\lambda,M(f,g)}^A(y))\Big]$$

or

$$0 \leq \Big[A(\mathcal{J}_{\lambda,M(f,g)}^A(y)) - A(\mathcal{J}_{\lambda,M(f,g)}^A(x))\Big].$$

Thus, we have

$$A(\mathcal{J}_{\lambda,M(f,g)}^A(x)) \geq A(\mathcal{J}_{\lambda,M(f,g)}^A(y))$$

or

$$A(\mathcal{J}_{\lambda,M(f,g)}^A(y)) \geq A(\mathcal{J}_{\lambda,M(f,g)}^A(x)).$$

It follows that

$$A(\mathcal{J}^A_{\lambda,M(f,g)}(x)) \propto A(\mathcal{J}^A_{\lambda,M(f,g)}(y)).$$

Since A is a strongly comparison mapping with respect to $\mathcal{J}^A_{\lambda,M(f,g)}$. Therefore $\mathcal{J}^A_{\lambda,M(f,g)}(x) \propto \mathcal{J}^A_{\lambda,M(f,g)}(y)$, that is, the resolvent operator $\mathcal{J}^A_{\lambda,M(f,g)}$ is a comparison mapping. □

Proposition 2.15. *Let $M : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a λ-XOR-α-non-ordinary difference mapping with respect to A and $\mathcal{J}^A_{\lambda,M(f,g)}$ and $A : \mathcal{H}_p \to \mathcal{H}_p$ be a strongly comparison mapping such that*

$$A(\mathcal{J}^A_{\lambda,M(f,g)}(x)) \oplus (A(\mathcal{J}^A_{\lambda,M(f,g)}(y)) \propto \mathcal{J}^A_{\lambda,M(f,g)}(x) \oplus \mathcal{J}^A_{\lambda,M(f,g)}(y).$$

Then the following condition holds for $\alpha\lambda > \mu, \mu \geq 1$,

$$\mathcal{J}^A_{\lambda,M(f,g)}(x) \oplus \mathcal{J}^A_{\lambda,M(f,g)}(y) \leq \frac{\mu}{(\alpha\lambda \oplus \mu)}(x \oplus y), \ \text{for all } x, y \in \mathcal{H}_p,$$

that is, the resolvent operator $\mathcal{J}^A_{\lambda,M(f,g)}$ is Lipschitz-type-continuous.

Proof. Let v_x^* and v_y^* are same as in (2.2) and (2.3). Then

$$\begin{aligned} v_x^* \oplus v_y^* &= \Big[\frac{1}{\lambda}(x \oplus A(\mathcal{J}^A_{\lambda,M(f,g)}(x))) \oplus \frac{1}{\lambda}(y \oplus A(\mathcal{J}^A_{\lambda,M(f,g)}(y)))\Big] \\ &= \frac{1}{\lambda}\Big[(x \oplus y) \oplus (A(\mathcal{J}^A_{\lambda,M(f,g)}(x))) \oplus (A(\mathcal{J}^A_{\lambda,M(f,g)}(y)))\Big] \\ &\leq \frac{\mu}{\lambda}\Big[(x \oplus y) \oplus (A(\mathcal{J}^A_{\lambda,M(f,g)}(x))) \oplus (A(\mathcal{J}^A_{\lambda,M(f,g)}(y)))\Big], \quad (2.4) \end{aligned}$$

for $\mu \geq 1$. Since M is an α-non-ordinary difference mapping with respect to A and $\mathcal{J}^A_{\lambda,M(f,g)}$ using (2.4), we have

$$\alpha\Big[A(\mathcal{J}^A_{\lambda,M(f,g)}(x)) \oplus A(\mathcal{J}^A_{\lambda,M(f,g)}(y))\Big] = v_x^* \oplus v_y^*,$$

and hence

$$\begin{aligned} &\alpha\Big[A(\mathcal{J}^A_{\lambda,M(f,g)}(x)) \oplus A(\mathcal{J}^A_{\lambda,M(f,g)}(y))\Big] \\ &\quad \leq \frac{\mu}{\lambda}\Big[(x \oplus y) \oplus (A(\mathcal{J}^A_{\lambda,M(f,g)}(x)) \oplus (A(\mathcal{J}^A_{\lambda,M(f,g)}(y)))\Big]. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\frac{\alpha\lambda}{\mu}\Big[A(\mathcal{J}^A_{\lambda,M(f,g)}(x)) \oplus A(\mathcal{J}^A_{\lambda,M(f,g)}(y))\Big] \\ &\quad \leq \Big[(x \oplus y) \oplus (A(\mathcal{J}^A_{\lambda,M(f,g)}(x)) \oplus (A(\mathcal{J}^A_{\lambda,M(f,g)}(y)))\Big], \end{aligned}$$

it follows that

$$\Big[\frac{\alpha\lambda}{\mu} \oplus 1\Big]\Big[A(\mathcal{J}^A_{\lambda,M(f,g)}(x)) \oplus A(\mathcal{J}^A_{\lambda,M(f,g)}(y))\Big] \leq (x \oplus y).$$

Since

$$A(\mathcal{J}^A_{\lambda,M(f,g)}(x)) \oplus A(\mathcal{J}^A_{\lambda,M(f,g)}(y)) \propto \mathcal{J}^A_{\lambda,M(f,g)}(x) \oplus \mathcal{J}^A_{\lambda,M(f,g)}(y),$$

we have

$$\begin{aligned}
&\Big(\frac{\alpha\lambda}{\mu} \oplus 1\Big)\Big[\mathcal{J}^A_{\lambda,M(f,g)}(x) \oplus \mathcal{J}^A_{\lambda,M(f,g)}(y)\Big] \\
&\quad \leq \Big(\frac{\alpha\lambda}{\mu} \oplus 1\Big)\Big[A(\mathcal{J}^A_{\lambda,M(f,g)}(x)) \oplus A(\mathcal{J}^A_{\lambda,M(f,g)}(y))\Big] \\
&\quad \leq (x \oplus y).
\end{aligned}$$

It follows that,

$$\mathcal{J}^A_{\lambda,M(f,g)}(x) \oplus \mathcal{J}^A_{\lambda,M(f,g)}(y) \quad \leq \quad \Big(\frac{\mu}{\alpha\lambda \oplus \mu}\Big)(x \oplus y).$$

Thus the resolvent operator $\mathcal{J}^A_{\lambda,M(f,g)}$ is Lipschitz-type-continuous. □

Remark 2.16. The Proposition 2.13, Proposition 2.14 and Proposition 2.15 are proved by taking M to be a bi-mapping with respect to the mappings f and g. Already existing concepts related to XOR-operation are generalized for bi-mapping M with respect to the mappings f and g.

3. Formulation of the problem and existence of solutions

Let $\mathcal{H}_p$ be a real ordered positive Hilbert space. Let $P : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$, $f, g : \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings, $T, S : \mathcal{H}_p \to C(\mathcal{H}_p)$ and $M : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be the set-valued mappings. We consider the following problem: Find $x \in \mathcal{H}_p$, $a \in T(x)$ and $b \in S(x)$ such that

$$0 \in P(a, b) \oplus M(f(x), g(x)). \tag{3.1}$$

We call problem (3.1) as variational inclusion problem involving XOR-operation.

Below we mention some special cases of problem (3.1).

(i) If $M(f(x), g(x)) = M(x)$, then the problem (3.1) reduces to the problem of finding $x \in \mathcal{H}_p$, $a \in T(x), b \in S(x)$ such that

$$0 \in P(a, b) \oplus M(x). \tag{3.2}$$

Problem (3.2) is studied by Ahmad et.al [4].

(ii) If T is single-valued, $S \equiv 0$, $P(a, b) = P(x)$ and $M(f(x), g(x)) = M(x)$, then problem (3.1) becomes the problem of finding $x \in \mathcal{H}_p$ such that

$$0 \in P(x) \oplus M(x). \tag{3.3}$$

Problem (3.3) is studied by I.Ahmad et.al [3].

(iii) If $P \equiv 0$, $M(f(x), g(x)) = M(x)$, then from the problem (3.1) we can obtain the problem of finding $x \in \mathcal{H}_p$ such that

$$0 \in M(x). \tag{3.4}$$

Problem (3.4) is introduced and studied by Li [13].

We remark that for suitable choices of operators involved in the formulation of Problem (3.1), one can obtain many previously studied problems studied by Li et al. [11, 12, 13, 14] and Ahmad et al. [2, 3].

The following lemma is a fixed point formulation of problem (3.1), which can be proved easily by using the definition of resolvent operator defined by (2.12).

Lemma 3.1. *The variational inclusion problem* (3.1) *involving XOR-operation has a solution $x \in \mathcal{H}_p$, $a \in T(x)$ and $b \in S(x)$ such that*

$$x = \mathcal{J}^A_{\lambda, M(f,g)}[\lambda P(a, b) \oplus A(x)],$$

where $\lambda > 0$ is a constant.

Based on Lemma 3.1, we construct the following algorithm for finding the solution of problem (3.1).

Algorithm 3.2. Let $A, f, g : \mathcal{H}_p \to \mathcal{H}_p$, $P : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be single-valued mappings, and $T, S : \mathcal{H}_p \to C(\mathcal{H}_p)$, $M : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be the set-valued mappings.

Given $x_0 \in \mathcal{H}_p$, $a_0 \in T(x_0)$, $b_0 \in S(x_0)$ and using Lemma 3.1, let

$$x_1 = (1-\alpha)x_0 + \alpha[\mathcal{J}^A_{\lambda, M(f(x_0), g(x_0))}(\lambda P(a_0, b_0) \oplus A(x_0))].$$

Since $a_0 \in T(x_0)$, $b_0 \in S(x_0)$, by Nadler [15], there exists $a_1 \in T(x_1)$, $b_1 \in S(x_1)$ and using Proposition 2.6, we have

$$\begin{aligned} \|a_0 \oplus a_1\| &\leq \|a_0 - a_1\| \leq \mathcal{D}(T(x_0), T(x_1)), \\ \|b_0 \oplus b_1\| &\leq \|b_0 - b_1\| \leq \mathcal{D}(S(x_0), S(x_1)). \end{aligned}$$

Let

$$x_2 = (1-\alpha)x_1 + \alpha[\mathcal{J}^A_{\lambda, M(f(x_1), g(x_1))}(\lambda P(a_1, b_1) \oplus A(x_1))].$$

Again by Nadler [15], there exists $a_2 \in T(x_2)$ and $b_2 \in S(x_2)$ such that

$$\begin{aligned} \|a_1 \oplus a_2\| &\leq \|a_1 - a_2\| \leq \mathcal{D}(T(x_1), T(x_2)), \\ \|b_1 \oplus b_2\| &\leq \|b_1 - b_2\| \leq \mathcal{D}(S(x_1), S(x_2)). \end{aligned}$$

Continuing the above process inductively, we compute

$$
\begin{aligned}
x_{n+1} &= (1-\alpha)x_n + \alpha[\mathcal{J}^A_{\lambda,M(f(x_n),g(x_n))}(\lambda P(a_n,b_n)\oplus A(x_n))], && (3.5)\\
a_{n+1} &\in T(x_{n+1});\ \|a_{n+1}\oplus a_n\| \le \|a_1 - a_2\| \le \mathcal{D}(T(x_{n+1}),T(x_n)), && (3.6)\\
b_{n+1} &\in S(x_{n+1});\ \|b_{n+1}\oplus b_n\| \le \|b_1 - b_2\| \le \mathcal{D}(S(x_{n+1}),S(x_n)), && (3.7)
\end{aligned}
$$

where $\lambda > 0$, $\alpha \in [0,1]$ and $n = 0,1,2,\ldots$.

Now, we prove the following existence and convergence results for variational inclusion problem involving XOR-operation (3.1).

Theorem 3.3. *Let $\mathcal{H}_p$ be a real ordered positive Hilbert space and $C \subseteq \mathcal{H}_p$ be a normal cone with constant λ_N. Let $A, f, g : \mathcal{H}_p \to \mathcal{H}_p$, $P : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the single-valued mappings and $T, S : \mathcal{H}_p \to C(\mathcal{H}_p)$, $M : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be the set-valued mappings such that A is a β-ordered compression mapping with respect to f and g, strongly comparison mapping with respect to $\mathcal{J}^A_{\lambda,M(f,g)}$, M is a θ-ordered rectangular mapping with respect f and g, λ-XOR-ordered strongly monotone with respect to f and g, XOR-α-non-ordinary difference mapping with respect to A and $\mathcal{J}^A_{\lambda,M(f,g)}$, T is δ_T-$\mathcal{D}$-Lipschitz-type-continuous and S is a δ_S-$\mathcal{D}$-Lipschitz-type-continuous mapping,f and g are one-one mappings. If $(f(x), g(x)) \oplus (f(y), g(y)) \propto (x \oplus y)$, $A(\mathcal{J}^A_{\lambda,M(f,g)}(x)) \oplus (A\mathcal{J}^A_{\lambda,M(f,g)}(y)) \propto \mathcal{J}^A_{\lambda,M(f,g)}(x)) \oplus \mathcal{J}^A_{\lambda,M(f,g)}(y)$, $x_{n+1} \propto x_n$, n=0,1,2,.... and if the following conditions are satisfied:*

$$
0 < \Big[\lambda_N[(1-\alpha)+\alpha\xi] + \lambda_N\alpha\theta'|\lambda|(\beta'_p\delta_T + \beta''_p\delta_S) + \lambda_N\alpha\theta'\beta\Big] < 1, \quad (3.8)
$$

and

$$
\|\mathcal{J}^A_{\lambda,M(f(x_n),g(x_n))}(x) \oplus \mathcal{J}^A_{\lambda,M(f(x_{n-1}),g(x_{n-1}))}(x)\| \le \xi\|x_n \oplus x_{n-1}\|, \quad (3.9)
$$

$$
\text{where } \theta' = \frac{\mu}{\alpha\lambda \oplus \mu},
$$

then the variational inclusion problem involving XOR-operation (3.1) *is solvable.*

Proof. From Algorithm 3.2 and Proposition 2.5, we evaluate

$$
\begin{aligned}
0 \le\ & x_{n+1} \oplus x_n\\
=\ & \Big[(1-\alpha)x_n + \alpha\Big(\mathcal{J}^A_{\lambda,M(f(x_n),g(x_n))}[\lambda P(a_n,b_n)\oplus A(x_n)]\Big)\Big]\\
& \oplus\Big[(1-\alpha)x_{n-1} + \alpha\Big(\mathcal{J}^A_{\lambda,M(f(x_{n-1}),g(x_{n-1}))}[\lambda P(a_{n-1},b_{n-1})\oplus A(x_{n-1})]\Big)\Big]\\
=\ & (1-\alpha)(x_n \oplus x_{n-1}) + \alpha\Big(\mathcal{J}^A_{\lambda,M(f(x_n),g(x_n))}[\lambda P(a_n,b_n)\oplus A(x_n)]\\
& \oplus\mathcal{J}^A_{\lambda,M(f(x_{n-1}),g(x_{n-1}))}[\lambda P(a_{n-1},b_{n-1})\oplus A(x_{n-1})]\Big). \qquad (3.10)
\end{aligned}
$$

Using Lipschitz-type-continuity of the resolvent operator $\mathcal{J}^A_{\lambda,M(f,g)}$, Proposition 2.6 and (3.9) we have

$$
\begin{aligned}
&\|x_{n+1} \oplus x_n\| \\
&\leq \lambda_N \Big\|(1-\alpha)(x_n \oplus x_{n-1}) + \alpha\Big[\mathcal{J}^A_{\lambda,M(f(x_n),g(x_n))}[\lambda P(a_n,b_n) \oplus A(x_n)] \\
&\quad \oplus \mathcal{J}^A_{\lambda,M(f(x_{n-1}),g(x_{n-1}))}[\lambda P(a_{n-1},b_{n-1}) \oplus A(x_{n-1})]\Big]\Big\| \\
&= \lambda_N \Big\|(1-\alpha)(x_n \oplus x_{n-1}) + \alpha\Big[\mathcal{J}^A_{\lambda,M(f(x_n),g(x_n))}[\lambda P(a_n,b_n) \oplus A(x_n)] \\
&\quad \oplus \mathcal{J}^A_{\lambda,M(f(x_n),g(x_n))}[\lambda P(a_{n-1},b_{n-1}) \oplus A(x_{n-1})] \\
&\quad \oplus \mathcal{J}^A_{\lambda,M(f(x_n),g(x_n))}[\lambda P(a_{n-1},b_{n-1}) \oplus A(x_{n-1})] \\
&\quad \oplus \mathcal{J}^A_{\lambda,M(f(x_{n-1}),g(x_{n-1}))}[\lambda P(a_{n-1},b_{n-1}) \oplus A(x_{n-1})]\Big]\Big\| \\
&\leq \lambda_N(1-\alpha)\|x_n \oplus x_{n-1}\| + \lambda_N\alpha\big\|\mathcal{J}^A_{\lambda,M(f(x_n),g(x_n))}[\lambda P(a_n,b_n) \oplus A(x_n)] \\
&\quad \oplus \mathcal{J}^A_{\lambda,M(f(x_n),g(x_n))}[\lambda P(a_{n-1},b_{n-1}) \oplus A(x_{n-1})]\big\| \\
&\quad + \lambda_N\alpha\big\|\mathcal{J}^A_{\lambda,M(f(x_n),g(x_n))}[\lambda P(a_{n-1},b_{n-1}) \oplus A(x_{n-1})] \\
&\quad \oplus \mathcal{J}^A_{\lambda,M(f(x_{n-1}),g(x_{n-1}))}[\lambda P(a_{n-1},b_{n-1}) \oplus A(x_{n-1})]\big\| \\
&\leq \lambda_N(1-\alpha)\|x_n \oplus x_{n-1}\| + \lambda_N\alpha\theta^{'}|\lambda|\big\|[P(a_n,b_n) \oplus P(a_{n-1},b_{n-1})] \\
&\quad \oplus [A(x_n) \oplus A(x_{n-1})]\big\| + \lambda_N\alpha\xi\|x_n \oplus x_{n-1}\| \\
&\leq \lambda_N[(1-\alpha)+\alpha\xi]\|x_n \oplus x_{n-1}\| + \lambda_N\alpha\theta^{'}|\lambda|\big\|P(a_n,b_n) \oplus P(a_{n-1},b_{n-1})\big\| \\
&\quad + \lambda_N\alpha\theta^{'}|\lambda|\big\|A(x_n) \oplus A(x_{n-1})\big\|. \qquad (3.11)
\end{aligned}
$$

Since P is $\beta_p^{'}$-strongly compression mapping in the first argument and $\beta_p^{''}$-strongly compression mapping in the second argument, T is δ_T-$\mathcal{D}$-Lipschitz-type-continuous, S is δ_S-$\mathcal{D}$-Lipschitz-type-continuous and using Proposition 2.6, we have

$$
\begin{aligned}
\|P(a_n,b_n) \oplus P(a_{n-1},b_{n-1})\| &= \|P(a_n,b_n) \oplus P(a_{n-1},b_n) \\
&\quad \oplus P(a_{n-1},b_n) \oplus P(a_{n-1},b_{n-1})\| \\
&\leq \|(P(a_n,b_n) \oplus P(a_{n-1},b_n)) \\
&\quad - (P(a_{n-1},b_n) \oplus P(a_{n-1},b_{n-1}))\| \\
&\leq \|P(a_n,b_n) \oplus P(a_{n-1},b_n)\| \\
&\quad + \|P(a_{n-1},b_n) \oplus P(a_{n-1},b_{n-1})\| \\
&\leq \beta_p^{'}\|a_n \oplus a_{n-1}\| + \beta_p^{''}\|b_n \oplus b_{n-1}\|
\end{aligned}
$$

$$\begin{aligned}
&\leq \beta_p^{'}\mathcal{D}(T(x_n), T(x_{n-1})) + \beta_p^{''}\mathcal{D}(S(x_n), S(x_{n-1})) \\
&\leq \beta_p^{'}\delta_T\|x_n - x_{n-1}\| + \beta_p^{''}\delta_S\|x_n - x_{n-1}\| \\
&= (\beta_p^{'}\delta_T + \beta_p^{''}\delta_S)\|x_n - x_{n-1}\|. \qquad (3.12)
\end{aligned}$$

Since A is β-ordered compression mapping and using Proposition 2.6, we have

$$\begin{aligned}
\|A(x_n) \oplus A(x_{n-1})\| &\leq \|\beta[(x_n) \oplus (x_{n-1})]\| \\
&= \beta\|(x_n) \oplus (x_{n-1})\| \\
&\leq \beta\|x_n - x_{n-1}\|. \qquad (3.13)
\end{aligned}$$

Using (3.12), (3.13), (3.11) becomes

$$\begin{aligned}
\|x_{n+1} \oplus x_n\| &\leq \lambda_N[(1-\alpha) + \alpha\xi]\|x_n \oplus x_{n-1}\| \\
&\quad + \lambda_N\alpha\theta^{'}|\lambda|(\beta_p^{'}\delta_T + \beta_p^{''}\delta_S)\|x_n - x_{n-1}\| \\
&\quad + \lambda_N\alpha\theta^{'}\beta\|x_n - x_{n-1}\| \\
&\leq \lambda_N[(1-\alpha) + \alpha\xi]\|x_n - x_{n-1}\| \\
&\quad + \lambda_N\alpha\theta^{'}|\lambda|(\beta_p^{'}\delta_T + \beta_p^{''}\delta_S)\|x_n - x_{n-1}\| \\
&\quad + \lambda_N\alpha\theta^{'}\beta\|x_n - x_{n-1}\| \\
&\leq \gamma(\theta)\|x_n - x_{n-1}\|. \qquad (3.14)
\end{aligned}$$

where $\gamma(\theta) = \left[\lambda_N[(1-\alpha) + \alpha\xi] + \lambda_N\alpha\theta^{'}|\lambda|(\beta_p^{'}\delta_T + \beta_p^{''}\delta_S) + \lambda_N\alpha\theta^{'}\beta\right]$ and $\theta^{'} = \left(\frac{\mu}{\alpha\lambda\oplus\mu}\right)$. By condition (3.8) it is clear that $0 < \gamma(\theta) < 1$, thus $\{x_n\}$ is a Cauchy sequence in $\mathcal{H}_p$. Since $\mathcal{H}_p$ is a complete space, there exists an x in $\mathcal{H}_p$ such that $x_n \to 0$, as $n \to \infty$.

From (3.6)and (3.7) of Algorithm 3.2, it follows that

$$\|a_{n+1} \oplus a_n\| \leq \|a_{n+1} - a_n\| \leq \mathcal{D}(T(x_{n+1}), T(x_n)) \leq \delta_T\|x_{n+1} - x_n\|, \quad (3.15)$$

and

$$\|b_{n+1} \oplus b_n\| \leq \|b_{n+1} - b_n\| \leq \mathcal{D}(S(x_{n+1}), S(x_n)) \leq \delta_S\|x_{n+1} - x_n\|. \quad (3.16)$$

It is clear from (3.15) and (3.16) that $\{a_n\}$ and $\{b_n\}$ are also a Cauchy sequences in $\mathcal{H}_p$ and so that there exist a and b in $\mathcal{H}_p$ such that $a_n \to a$ and $b_n \to b$, as $n \to \infty$. By using the continuity of operators P, f, g, A, M, T, S and $\mathcal{J}^A_{\lambda,M(f,g)}$, it follows that

$$x = \mathcal{J}^A_{\lambda,M(f,g)}[\lambda P(a,b) \oplus A(x)].$$

By Lemma 3.1, we conclude that (x, a, b) where $x \in \mathcal{H}_p$, $a \in T(x)$ and $b \in S(x)$ is a solution of variational inclusion problem involving XOR-operation (3.1).

□

Remark 3.4. For suitable choices of operators involved in the formulation of the Problem 3.1, Algorithm 3.2 and Theorem 3.3, one can obtain many previously known results of [2, 3, 4, 12, 13, 14], etc..

4. Resolvent Equation Problem

In connection with variational inclusion problem involving XOR-operation (3.1), we consider the following resolvent equation problem involving XOR-operation: Find $x, z \in \mathcal{H}_p$, $a \in T(x)$ and $b \in S(x)$ such that

$$P(a,b) \oplus \lambda^{-1}\mathcal{R}^A_{\lambda,M(f,g)}(z) = 0, \tag{4.1}$$

where $\mathcal{R}^A_{\lambda,M(f,g)} = [I \oplus A(\mathcal{R}^A_{\lambda,M(f,g)})]$ and $\mathcal{J}^A_{\lambda,M(f,g)} = [A \oplus \lambda M(f,g)]^{-1}$, $\lambda > 0$ is a constant. Problem (4.1) is called resolvent equation problem involving XOR-operation.

Now we discuss the equivalence between variational inclusion problem involving XOR-operation (3.1) and resolvent equation problem involving XOR-operation (4.1).

Proposition 4.1. *The variational inclusion problem involving XOR-operation* (3.1) *has a solution* $x \in \mathcal{H}_p$, $a \in T(x)$, $b \in S(x)$ *if and only if resolvent equation problem involving XOR-operation has a solution* $x, z \in \mathcal{H}_p$, $a \in T(x)$, $b \in S(x)$, *where*

$$x = \mathcal{J}^A_{\lambda,M(f,g)}(z) \tag{4.2}$$

and

$$z = \lambda P(a,b) \oplus A(x), \tag{4.3}$$

$\lambda > 0$ *is a constant.*

Proof. Let $x \in \mathcal{H}_p$, $a \in T(x)$, $b \in S(x)$ be a solution of variational inclusion problem involving XOR-operation (3.1). Then by the Lemma 3.1, it is a solution of the following equation:

$$x = \mathcal{J}^A_{\lambda,M(f,g)}[\lambda P(a,b) \oplus A(x)].$$

It follows from (4.3) that, $z = \lambda P(a,b) \oplus A(x)$, thus we have

$$x = \mathcal{J}^A_{\lambda,M(f,g)}(z).$$

By using the fact that $\mathcal{R}^A_{\lambda,M(f,g)} = [I \oplus A(\mathcal{J}^A_{\lambda,M(f,g)})]$, we have

$$\begin{aligned} z &= \lambda P(a,b) \oplus A(x), \\ z \oplus A(\mathcal{J}^A_{\lambda,M(f,g)}(z)) &= \lambda P(a,b) \oplus A(x) \oplus A(x), \end{aligned}$$

that is,

$$\begin{aligned} z \oplus A(\mathcal{J}^A_{\lambda,M(f,g)})(z) &= \lambda P(a,b), \\ \Big[I \oplus A(\mathcal{J}^A_{\lambda,M(f,g)})\Big](z) &= \lambda P(a,b), \end{aligned}$$

thus we have $P(a,b) \oplus \lambda^{-1}\mathcal{R}^A_{\lambda,M(f,g)}(z) = 0$, i.e the required resolvent equation problem involving XOR-operation (4.1).

Conversely, suppose that $x, z \in \mathcal{H}_p$, $a \in T(x)$, $b \in S(x)$ is a solution of resolvent equation problem involving XOR-operation (4.1), that is, we have

$$\lambda P(a,b) \oplus \mathcal{R}^A_{\lambda,M(f,g)}(z) = 0,$$

implies that

$$\begin{aligned} \lambda P(a,b) &= \mathcal{R}^A_{\lambda,M(f,g)}(z) \\ &= [I \oplus A(\mathcal{J}^A_{\lambda,M(f,g)})](z) \\ &= z \oplus A(\mathcal{J}^A_{\lambda,M(f,g)}(z)) \\ &= \lambda P(a,b) \oplus A(x) \oplus A[\mathcal{J}^A_{\lambda,M(f,g)}(\lambda P(a,b) \oplus A(x))]. \end{aligned}$$

That is, we have

$$A(x) = A[\mathcal{J}^A_{\lambda,M(f,g)}(\lambda P(a,b) \oplus A(x))].$$

Since A is one-one, we have

$$x = \mathcal{J}^A_{\lambda,M(f,g)}[\lambda P(a,b) \oplus A(x)].$$

Thus by Lemma 3.1, it follows that $x \in \mathcal{H}_p$, $a \in T(x)$, $b \in S(x)$ is a solution of variational inclusion problem involving XOR-operation (3.1). □

Based on Proposition 4.1, we suggest the following iterative algorithm for computing the solution of resolvent equation problem involving XOR-operation (4.1).

Algorithm 4.2. *Using the same arguments as in Algorithm* 3.1*, for* $z_0, x_0 \in \mathcal{H}_p$*,* $a_0 \in T(x_0)$*,* $b_0 \in S(x_0)$*, we compute the sequences* $\{z_n\}$*,* $\{x_n\}$ *and* $\{b_n\}$ *by the following iterative schemes:*

$$x_{n+1} = \mathcal{J}^A_{\lambda,M(f(x_n),g(x_n))}(z_{n+1}), \tag{4.4}$$

$$a_{n+1} \in T(x_{n+1}) : \|a_{n+1} \oplus a_n\| \le \|a_{n+1} - a_n\| \le \mathcal{D}(T(x_{n+1}), T(x_n)), \tag{4.5}$$

$$b_{n+1} \in S(x_{n+1}) : \|b_{n+1} \oplus b_n\| \le \|b_{n+1} - b_n\| \le \mathcal{D}(S(x_{n+1}), S(x_n)) \tag{4.6}$$

and

$$z_{n+1} = \lambda P(a_n, b_n) \oplus A(x_n), \tag{4.7}$$

where $n = 0, 1, 2, 3, \ldots$ *and* $\lambda > 0$ *is a constant.*

Proposition 4.3. *If all the mappings and conditions are same as in Theorem 3.3 except condition* (3.8) *and if the following conditions are satisfied:*

$$0 < \gamma(\Theta^{'}) < 1, \tag{4.8}$$

where

$$\gamma(\Theta^{'}) = \big[|\lambda|(\beta_{p^{'}}\delta_T + \beta_{p^{''}}\delta_S) + \beta\big]\Big[\frac{\theta^{'}}{1-\xi}\Big], \tag{4.9}$$

then the resolvent equation problem involving XOR*-operation* (4.1) *has a solution* (z, x, a, b)*,* $z, x \in \mathcal{H}_p$*,* $a \in T(x)$*,* $b \in S(x)$*. Moreover, the sequences* $\{z_n\}$*,* $\{x_n\}$*,* $\{a_n\}$ *and* $\{b_n\}$ *generated by the Algorithm 4.2 converge strongly to* z, x, a *and* b*, the solution of resolvent equation problem involving XOR-operation* (4.1)*.*

Proof. From Algorithm 4.2, we have

$$\begin{aligned}\|z_{n+1} \oplus z_n\| &= \|[\lambda P(a_n, b_n) \oplus A(x_n)] \oplus [\lambda P(a_{n-1}, b_{n-1}) \oplus A(x_{n-1})]\| \\ &= \|[\lambda P(a_n, b_n) \oplus \lambda P(a_{n-1}, b_{n-1})] \oplus [A(x_n \oplus A(x_{n-1}))]\| \\ &\le \|[\lambda P(a_n, b_n) \oplus \lambda P(a_{n-1}, b_{n-1})] - [A(x_n \oplus A(x_{n-1}))]\| \\ &\le |\lambda|\|P(a_n, b_n) \oplus P(a_{n-1}, b_{n-1})\| + \|A(x_n) \oplus A(x_{n-1})\|.\end{aligned} \tag{4.10}$$

Using (3.12), (3.13), (4.10) becomes

$$\|z_{n+1} \oplus z_n\| \le |\lambda|(\beta_{p^{'}}\delta_T + \beta_{p^{''}}\delta_S)\|x_n - x_{n-1}\| + \beta\|x_n - x_{n-1}\|. \tag{4.11}$$

As $z_{n+1} \propto z_n$, n=0,1,2,.., (4.11) becomes

$$\|z_{n+1} - z_n\| \le [|\lambda|(\beta_{p^{'}}\delta_T + \beta_{p^{''}}\delta_S) + \beta]\|x_n - x_{n-1}\|. \tag{4.12}$$

From (4.4), Proposition (2.15) and condition (3.9) we have

$$\begin{aligned}\|x_n \oplus x_{n-1}\| &= \big\|\mathcal{J}^A_{\lambda,M(f(x_n),g(x_n))}(z_n) \oplus \mathcal{J}^A_{\lambda,M(f(x_{n-1}),g(x_{n-1}))}(z_{n-1})\big\| \\ &= \big\|\mathcal{J}^A_{\lambda,M(f(x_n),g(x_n))}(z_n) \oplus \mathcal{J}^A_{\lambda,M(f(x_n),g(x_n))}(z_{n-1}) \\ &\quad \oplus \mathcal{J}^A_{\lambda,M(f(x_n),g(x_n))})(z_{n-1}) \oplus \mathcal{J}^A_{\lambda,M(f(x_{n-1}),g(x_{n-1}))}(z_{n-1})\big\| \\ &\le \big\|\mathcal{J}^A_{\lambda,M(f(x_n),g(x_n))}(z_n) \oplus \mathcal{J}^A_{\lambda,M(f(x_n),g(x_n))}(z_{n-1})\big\| \\ &\quad + \big\|\mathcal{J}^A_{\lambda,M(f(x_n),g(x_n))})(z_{n-1}) \oplus \mathcal{J}^A_{\lambda,M(f(x_{n-1}),g(x_{n-1}))}(z_{n-1})\big\| \\ &\le \theta^{'}\|z_n - z_{n-1}\| + \xi\|x_n - x_{n-1}\|.\end{aligned} \tag{4.13}$$

As $x_n \propto x_{n-1}$, n=0,1,2,... .,we have

$$\|x_n - x_{n-1}\| \le \theta^{'}\|z_n - z_{n-1}\| + \xi\|x_n - x_{n-1}\|,$$

which implies that

$$(1-\xi)\|x_n - x_{n-1}\| \le \theta^{'}\|z_n - z_{n-1}\|.$$

Hence we have

$$\|x_n - x_{n-1}\| \leq \frac{\theta^{'}}{(1-\xi)}\|z_n - z_{n-1}\|. \tag{4.14}$$

Combining (4.12) and (4.14), we have

$$\|z_{n+1} - z_n\| \leq \left[|\lambda|({\beta_p}'\delta_T + {\beta_p}''\delta_S) + \beta\right]\left[\frac{\theta^{'}}{1-\xi}\right]\|z_n - z_{n-1}\|, \tag{4.15}$$

that is,

$$\|z_{n+1} - z_n\| \leq \gamma(\Theta^{'})\|z_n - z_{n-1}\|. \tag{4.16}$$

Since $\gamma(\Theta^{'}) < 1$ by (4.8), it follows that $\{z_n\}$ is a Cauchy sequence in $\mathcal{H}_p$, so there exist $z \in \mathcal{H}_p$ such that $z_n \to z$ as $n \to \infty$. Also it follows from (4.14) that $\{x_n\}$ is a Cauchy sequence in $\mathcal{H}_p$, thus there exist $x \in \mathcal{H}_p$ such that $x_n \to x$ as $n \to \infty$. It follows from (3.15) and (3.16) that $\{a_n\}$ and $\{b_n\}$ are also Cauchy sequences in $\mathcal{H}_p$ and so that there exist a and b in $\mathcal{H}_p$ such that $a_n \to a$, $b_n \to b$, as $n \to \infty$.

By using the continuity of the operations $P, f, g, A, M, T, S, \mathcal{J}^A_{\lambda,M(f,g)}$ and $\mathcal{R}^A_{\lambda,M(f,g)}$, we have

$$z = \lambda P(a,b) \oplus A(x). \tag{4.17}$$

Using the same argument as in the proof of Theorem 3.3, we claim that $z, x \in \mathcal{H}_p$, $a \in T(x)$, $b \in S(x)$ is a solution of resolvent equation problem involving XOR-operation (4.1). □

Remark 4.4. We remark that the concept of resolvent equation with XOR-operation is quite new and appeared first time in the literature.

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