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THE STABILITY OF A COSINE-SINE FUNCTIONAL EQUATION ON ABELIAN GROUPS

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Abstract. In this paper we establish the stability of the functional equation

$$f(x-y) = f(x)g(y) + g(x)f(y) + h(x)h(y), \quad x, y \in G,$$

where G is an abelian group.

1. Introduction

In many studies concerning functional equations related to the Cauchy equation $f(xy) = f(x)f(y)$, the main tool is a kind of stability problem inspired by the famous problem proposed in 1940 by Ulam [23]. More precisely, given a group G and a metric group H with metric d, it is asked if for every function $f : G \to H$, such that the function $(x, y) \mapsto f(xy) - f(x)f(y)$ is bounded, there

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exists a homomorphism $\chi : G \to H$ such that the function $x \mapsto d(f(x), \chi(x))$ is bounded.

The first affirmative answer to Ulam's question was given in 1941 by Hyers [14], under the assumption that G and H are Banach spaces. After Hyers's result a great number of papers on the subject have been published, generalizing Ulam's problem and Hyers's result in various directions. The interested reader should refer to [6, 7, 8, 9, 12, 13, 17, 19, 20] for a thorough account on the subject of the stability of functional equations.

In this paper we will investigate the stability problem for the trigonometric functional equation

$$f(x-y) = f(x)g(y) + g(x)f(y) + h(x)h(y),\ x, y \in G \tag{1.1}$$

on abelian groups.

Székelyhidi [22] proved the Hyers-Ulam stability for the functional equation

$$f(xy) = f(x)g(y) + g(x)f(y),\ x, y \in G$$

and cosine functional equation

$$g(xy) = g(x)g(y) - f(x)f(y),\ x, y \in G$$

on amenable group G. Chung, Choi and Kim [10] studied the Hyers-Ulam stability of

$$f(x + \sigma(y)) = f(x)g(y) - g(x)f(y),\ x, y \in G$$

where $\sigma : G \to G$ is an involution.

Recently, in [3, 4] the authors obtained the stability of the functional equations

$$f(xy) = f(x)g(y) + g(x)f(y) + h(x)h(y),\ x, y \in G,$$

$$f(x\sigma(y)) = f(x)g(y) + g(x)f(y),\ x, y \in G,$$

$$f(x\sigma(y)) = f(x)f(y) - g(x)g(y),\ x, y \in G$$

and

$$f(x\sigma(y)) = f(x)g(y) - g(x)f(y),\ x, y \in G$$

on amenable groups, where $\sigma : G \to G$ is an involutive automorphism.

The aim of the present paper is to extend the previous results to the functional equation (1.1) on abelian groups.

2. Definitions and notations

Throughout this paper $(G, +)$ denotes an abelian group with the identity element e. We denote by $\mathcal{B}(G)$ the linear space of all bounded complex-valued functions on G.

Let $\mathcal{V}$ be a linear space of complex-valued functions on G. We say that the functions $f_1, \cdots, f_n : G \to \mathbb{C}$ are linearly independent modulo $\mathcal{V}$ if

$$\lambda_1 f_1 + \cdots + \lambda_n f_n \in \mathcal{V}$$

implies that $\lambda_1 = \cdots = \lambda_n = 0$ for any $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$. We say that the linear space $\mathcal{V}$ is two-sided invariant if $f \in \mathcal{V}$ implies that the function $x \mapsto f(x+y)$ belongs to $\mathcal{V}$ for any $y \in G$.

If I is the identity map of G we say that $\mathcal{V}$ is $(-I)$-invariant if $f \in \mathcal{V}$ implies that the function $x \mapsto f(-x)$ belongs to $\mathcal{V}$. The space $\mathcal{B}(G)$ is an obvious example of a linear space of complex-valued functions on G which is two-sided invariant and $(-I)$-invariant.

Let $f : G \to \mathbb{C}$ be a function. We denote respectively by

$$f^e(x) := \frac{f(x) + f(-x)}{2},\ x \in G$$

and

$$f^o(x) := \frac{f(x) - f(-x)}{2},\ x \in G$$

the even part and the odd part of f.

3. Basic results

In this section we present some general stability properties of the functional equation (1.1). Throughout this section we let $\mathcal{V}$ denote a two-sided invariant and $(-I)$-invariant linear space of complex-valued functions on G.

Lemma 3.1. *Let $f, g, h : G \to \mathbb{C}$ be functions. Suppose that the functions*

$$x \mapsto f(x-y) - f(x)g(y) - g(x)f(y) - h(x)h(y)$$

and

$$x \mapsto f(x-y) - f(y-x) \tag{3.1}$$

belong to $\mathcal{V}$ for all $y \in G$. Then we have the following statements:

(1) $f^o \in \mathcal{V}$.

(2) *The following functions $\varphi_1, \varphi_2, \varphi_3 : G \times G \to \mathbb{C}$*

$$f^e(x)g^o(y) + g^e(x)f^o(y) + h^e(x)h^o(y) = \varphi_1(x,y), \tag{3.2}$$

$$g^o(x)f^e(y) + h^o(x)h^e(y) = \varphi_2(x,y), \tag{3.3}$$

$$\begin{aligned} f(x+y) - f(x-y) + 2f^o(x)g^o(y) + 2g^o(x)f^o(y) + 2h^o(x)h^o(y) \\ = \varphi_3(x,y) \end{aligned} \tag{3.4}$$

are such that the functions $x \mapsto \varphi_1(x,y)$, $x \mapsto \varphi_1(y,x)$, $x \mapsto \varphi_2(x,y)$, $x \mapsto \varphi_3(x,y)$ and $x \mapsto \varphi_3(y,x)$ belong to $\mathcal{V}$ for all $y \in G$.

Proof. By setting $y = e$ in (3.1) we get that the function $x \mapsto f(x) - f(-x)$ belongs to $\mathcal{V}$ which proves (1).

Let ψ be the function defined on $G \times G$ by

$$\psi(x,y) = f(x-y) - f(x)g(y) - g(x)f(y) - h(x)h(y). \tag{3.5}$$

From (3.5) we can verify easily that

$$\begin{aligned}\psi(x,y) = f^e(x-y) + f^o(x-y) - f^e(x)g(y) \\ - f^o(x)g(y) - g(x)f(y) - h(x)h(y).\end{aligned} \tag{3.6}$$

Now let

$$\phi(x,y) := \psi(x,y) - f^o(x-y) + f^o(x)g(y). \tag{3.7}$$

Then by using (3.6) and (3.7) we get

$$f^e(x-y) = f^e(x)g(y) + g(x)f(y) + h(x)h(y) + \phi(x,y). \tag{3.8}$$

Since f^e is an even function on the abelian group G, we have

$$f^e(x-y) = f^e(-(x-y)) = f^e((-x)-(-y)).$$

Hence, by applying (3.8) to the pair $(-x,-y)$, we obtain

$$f^e(x-y) = f^e(x)g(-y) + g(-x)f(-y) + h(-x)h(-y) + \phi(-x,-y). \tag{3.9}$$

Subtracting equation (3.9) from (3.8) we get that

$$\begin{aligned}&2f^e(x)g^o(y) + g(x)f(y) - g(-x)f(-y) + h(x)h(y) - h(-x)h(-y) \\ &= \phi(-x,-y) - \phi(x,y).\end{aligned} \tag{3.10}$$

For the pair $(-x,y)$ the identity (3.10) becomes

$$\begin{aligned}&2f^e(x)g^o(y) + g(-x)f(y) - g(x)f(-y) + h(-x)h(y) - h(x)h(-y) \\ &= \phi(x,-y) - \phi(-x,y).\end{aligned} \tag{3.11}$$

By adding (3.10) and (3.11) we obtain

$$\begin{aligned}&4f^e(x)g^o(y) + 2g^e(x)[f(y) - f(-y)] + 2h^e(x)[h(y) - h(-y)] \\ &= \phi(-x,-y) - \phi(x,y) + \phi(x,-y) - \phi(-x,y).\end{aligned}$$

Hence the identity (3.2) can be written as follows where

$$\varphi_1(x,y) := \frac{1}{4}[\phi(-x,-y) - \phi(x,y) + \phi(x,-y) - \phi(-x,y)].$$

By using (3.7) and the identity above we get, by an elementary computation, that

$$\begin{aligned}\varphi_1(x,y) = \frac{1}{4}[\psi(-x,-y) - \psi(x,y) + \psi(x,-y) \\ - \psi(-x,y) + 2f^o(x-y) - 2f^o(x+y)].\end{aligned} \tag{3.12}$$

By interchanging x and y in (3.2) we obtain

$$g^o(x)f^e(y) + f^o(x)g^e(y) + h^o(x)h^e(y) = \varphi_1(y,x),$$

and then we get

$$\varphi_2(x,y) := \varphi_1(y,x) - f^o(x)g^e(y),\ x,y \in G. \tag{3.13}$$

On the other hand, by replacing y by $-y$ in (3.5) we get that

$$\psi(x,-y) = f(x+y) - f(x)g(-y) - g(x)f(-y) - h(x)h(-y). \tag{3.14}$$

By subtracting the result of equation (3.5) from the result of equation (3.14) we obtain

$$\begin{aligned}
&f(x+y) - f(x-y))\\
&= -2f(x)g^o(y) - 2g(x)f^o(y) - 2h(x)h^o(y) + \psi(x,-y) - \psi(x,y)\\
&= -2f^e(x)g^o(y) - 2f^o(x)g^o(y) - 2g^e(x)f^o(y) - 2g^o(x)f^o(y)\\
&\quad - 2h^e(x)h^o(y) - 2h^o(x)h^o(y) + \psi(x,-y) - \psi(x,y)\\
&= -2f^o(x)g^o(y) - 2g^o(x)f^o(y) - 2h^o(x)h^o(y)\\
&\quad - 2[f^e(x)g^o(y) + g^e(x)f^o(y) + h^e(x)h^o(y)] + \psi(x,-y) - \psi(x,y)\\
&= -2f^o(x)g^o(y) - 2g^o(x)f^o(y) - 2h^o(x)h^o(y)\\
&\quad - 2\varphi_1(x,y) + \psi(x,-y) - \psi(x,y).
\end{aligned}$$

Thus identity (3.4) can be written as follows:

$$\varphi_3(x,y) := -2\varphi_1(x,y) + \psi(x,-y) - \psi(x,y). \tag{3.15}$$

Since x and y are arbitrary, by using the fact that the functions $x \mapsto \psi(x,y)$, $x \mapsto f(x-y) - f(y-x)$ and f^o belong to the two-sided invariant and $(-I)$-invariant linear space $\mathcal{V}$ of complex-valued functions on G for all $y \in G$, and taking (3.12), (3.13) and (3.15) into account, we deduce the rest of the proof. $\square$

Lemma 3.2. *Let $f,g,h : G \to \mathbb{C}$ be functions. Suppose that f and h are linearly independent modulo $\mathcal{V}$, and that $h^o \notin \mathcal{V}$. If the functions*

$$x \mapsto f(x-y) - f(x)g(y) - g(x)f(y) - h(x)h(y)$$

and

$$x \mapsto f(x-y) - f(y-x)$$

belong to $\mathcal{V}$ for all $y \in G$. Then we have the following statements:

(1)

$$h^e = \gamma\, f^e \tag{3.16}$$

and

$$g^o = -\gamma\, h^o - \eta\, f^o, \tag{3.17}$$

where $\gamma, \eta \in \mathbb{C}$ are constants.

(2) *Moreover, if* $f^o \neq 0$, *then*

$$g^e = \eta\, f^e + \varphi, \tag{3.18}$$

where $\varphi \in \mathcal{V}$ *and* $\varphi(-x) = \varphi(x)$ *for all* $x \in G$.

Proof. Since f and h are linearly independent modulo $\mathcal{V}$ then $f \notin \mathcal{V}$. According to Lemma 3.1(1) we have $f^o \in \mathcal{V}$, then $f^e \notin \mathcal{V}$ and consequently $f^e \neq 0$. Therefore, there exists $y_0 \in G$ such that $f^e(y_0) \neq 0$. By setting $y = y_0$ in (3.3) we derive that there exist a constant $\gamma \in \mathbb{C}$ and a function $b_1 \in \mathcal{V}$ such that

$$g^o = -\gamma\, h^o + b_1. \tag{3.19}$$

When we substitute this in (3.3) we obtain

$$(-\gamma\, h^o(x) + b_1(x))f^e(y) + h^o(x)h^e(y) = \varphi_2(x, y),$$

which implies

$$(h^e(y) - \gamma\, f^e(y))h^o(x) = \varphi_2(x, y) - f^e(y)b_1(x).$$

So, x and y being arbitrary, we deduce that the function

$$x \mapsto (h^e(y) - \gamma\, f^e(y))h^o(x)$$

belongs to $\mathcal{V}$ for all $y \in G$. As $h^o \notin \mathcal{V}$ we get (3.16).

On the other hand we get, from (3.2), (3.16) and (3.19), that

$$\varphi_1(x, y) = f^e(x)b_1(y) + g^e(x)f^o(y) \tag{3.20}$$

for all $x, y \in G$.

If $f^o \neq 0$ then from (3.20) there exist a constant $\eta \in \mathbb{C}$ and a function $\varphi \in \mathcal{V}$ such that $g^e = \eta\, f^e + \varphi$ and $\varphi(-x) = \varphi(x)$ for all $x \in G$. This is the result (2) of Lemma 3.2. When we substitute this in the identity (3.20) we get, by a simple computation, that $\varphi_1(x, y) = f^e(x)[b_1(y) + \eta\, f^o(y)] + \varphi(x)f^o(y)$ for all $x, y \in G$. As the functions φ and $x \mapsto \varphi_1(x, y)$ belong to $\mathcal{V}$ for all $y \in G$, we deduce that the function $x \mapsto f^e(x)[b_1(y) + \eta\, f^o(y)]$ belongs to $\mathcal{V}$ for all $y \in G$. Thus, taking into account that $f^e \notin \mathcal{V}$ we infer that $b_1 = -\eta\, f^o$.

If $f^o = 0$ then we get from (3.20), and noticing that $f^e \notin \mathcal{V}$, that $b_1 = 0$. Hence, in both cases we have $b_1 = -\eta\, f^o$. By substituting this back into (3.19) we obtain (3.17). This completes the proof. □

Proposition 3.3. *Let* $m : G \to \mathbb{C}$ *be a nonzero multiplicative function such that* $m(-x) = m(x)$ *for all* $x \in G$. *Then the solutions* $f, h : G \to \mathbb{C}$ *of the functional equation*

$$f(x + y) = f(x)m(y) + m(x)f(y) + h(x)h(y), \;\; x, y \in G \tag{3.21}$$

such that $f(-x) = f(x)$*,* $h(-x) = -h(x)$ *for all* $x \in G$ *and* $h \neq 0$ *are the pairs*

$$f = \frac{1}{2}a^2 m \ \text{ and } \ h = a\,m,$$

where $a : G \to \mathbb{C}$ *is a nonzero additive function.*

Proof. It is simple to check that the indicated functions are solutions of the functional equation. It is thus left to show that any solutions $f, h : G \to \mathbb{C}$ can be written in the indicated forms. Replacing y by $-y$ in (3.21) yields the functional equation

$$f(x-y) = f(x)m(y) + m(x)f(y) - h(x)h(y),$$

because f and m are even functions, and h is an odd function. By (3.21) we get that

$$f(x+y) + f(x-y) = 2f(x)m(y) + 2m(x)f(y).$$

Notice that $m(x) \neq 0$ for all $x \in G$, because m is a nonzero multiplicative function on the group G. Moreover since $m(-x) = -m(x)$ for all $x \in G$ we have

$$m(x+y) = m(x-y) = m(x)m(y)$$

for all $x, y \in G$. Thus, by dividing both sides of (3.21) by $m(x+y)$ we get that $F := f/m$ satisfies the classical quadratic functional equation

$$F(x+y) + F(x-y) = 2F(x) + 2F(y).$$

Hence from [21, Theorem 13.13] we derive that F has the form $F(x) = Q(x,x)$, $x \in G$, where $Q : G \times G \to \mathbb{C}$ is a symmetric, bi-additive map. Hence

$$f(x) = Q(x,x)m(x) \tag{3.22}$$

for all $x \in G$. Substituting this in (3.21) and dividing both sides by $m(x+y) = m(x)m(y)$, and using that Q is a symmetric, bi-additive map we derive that

$$2Q(x,y) = H(x)H(y) \tag{3.23}$$

for all $x, y \in G$ with $H := h/m$. Since, H is a nonzero function on G, because h is, we get that there exists $y_0 \in G$ such that $H(y_0) \neq 0$. Hence, by setting $y = y_0$ in the last identity and dividing both sides by $H(y_0)$, and taking into account that Q is bi-additive, we deduce that $H = a$, where $a : G \to \mathbb{C}$ is additive. So $h = a\,m$.

Notice that a is nonzero. On the other hand, by replacing H by a in (3.23) and setting $x = y$ we deduce that $Q(x,x) = \frac{1}{2}a^2(x)$ for all $x \in G$. When we substitute this in (3.22) we get that $f = \frac{1}{2}a^2 m$. This completes the proof. □

Proposition 3.4. *Let $f, g, h : G \to \mathbb{C}$ be functions. Suppose that f and h are linearly independent modulo $\mathcal{B}(G)$. If the function*

$$(x, y) \mapsto f(x + y) - f(x)g(y) - g(x)f(y) - h(x)h(y)$$

is bounded then we obtain one of the following possibilities:

(1)

$$\begin{cases} f &= -\lambda^2 f_0 + \lambda^2 b, \\ g &= \frac{1+\rho^2}{2} f_0 + \rho\, g_0 + \frac{1-\rho^2}{2} b, \\ h &= \lambda\, \rho\, f_0 + \lambda\, g_0 - \lambda\, \rho\, b, \end{cases}$$

where $b : G \to \mathbb{C}$ is a bounded function, $\rho \in \mathbb{C}$, $\lambda \in \mathbb{C} \setminus \{0\}$ are constants and $f_0, g_0 : G \to \mathbb{C}$ are functions satisfying the cosine functional equation

$$f_0(x + y) = f_0(x)f_0(y) - g_0(x)g_0(y),\ x, y \in G;$$

(2)

$$\begin{cases} f &= \lambda^2 M + a\, m + b, \\ g &= \beta\lambda\, (1 - \frac{1}{2}\beta\lambda\,)M + (1 - \beta\lambda\,)m - \frac{1}{2}\beta^2 a\, m - \frac{1}{2}\beta^2 b, \\ h &= \lambda\, (1 - \beta\lambda\,)M - \lambda\, m - \beta a\, m - \beta b, \end{cases}$$

where $m : G \to \mathbb{C}$ is a nonzero bounded multiplicative function, $M : G \to \mathbb{C}$ is a non bounded multiplicative function, $a : G \to \mathbb{C}$ is a nonzero additive function, $b : G \to \mathbb{C}$ is a bounded function and $\beta \in \mathbb{C}$, $\lambda \in \mathbb{C} \setminus \{0\}$ are constants;

(3)

$$\begin{cases} f &= \frac{1}{2} a^2\, m + \frac{1}{2} a_1\, m + b, \\ g &= -\frac{1}{4}\beta^2 a^2\, m + \beta a\, m - \frac{1}{4}\beta^2 a_1\, m + m - \frac{1}{2}\beta^2 b, \\ h &= -\frac{1}{2}\beta a^2\, m + a\, m - \frac{1}{2}\beta a_1\, m - \beta\, b, \end{cases}$$

where $m : G \to \mathbb{C}$ is a nonzero bounded multiplicative function, $a, a_1 : G \to \mathbb{C}$ are additive functions such that a is nonzero, $b : G \to \mathbb{C}$ is a bounded function and $\beta \in \mathbb{C}$ is a constant;

(4) $f(x + y) = f(x)m(y) + m(x)f(y) + (a(x)m(x) + b(x))(a(y)m(y) + b(y))$ *for all $x, y \in G$,*

$$g = -\frac{1}{2}\beta^2 f + (1 + \beta a)m + \beta b$$

and

$$h = -\beta f + a\, m + b,$$

where $m : G \to \mathbb{C}$ is a nonzero bounded multiplicative function, $a : G \to \mathbb{C}$ is a nonzero additive function, $b : G \to \mathbb{C}$ is a bounded function and $\beta \in \mathbb{C}$ is a constant;

(5) $f(x+y) = f(x)g(y) + g(x)f(y) + h(x)h(y)$ *for all* $x, y \in G$.

Proof. We proceed as in the proof of [4, Lemma 3.4]. □

4. Stability of equation (1.1)

In this section we prove the main result of the present paper.

Theorem 4.1. *$f, g, h : G \to \mathbb{C}$ be functions. The function*

$$(x,y) \mapsto f(x-y) - f(x)g(y) - g(x)f(y) - h(x)h(y)$$

is bounded if and only if one of the following assertions holds:

(1) $f = 0$, *g is arbitrary and* $h \in \mathcal{B}(G)$;

(2) $f, g, h \in \mathcal{B}(G)$;

(3)
$$\begin{cases} f &= \alpha\, m - \alpha\, b, \\ g &= \frac{1-\alpha\lambda^2}{2} m + \frac{1+\alpha\lambda^2}{2} b - \lambda\,\varphi, \\ h &= \alpha\lambda\, m - \alpha\lambda\, b + \varphi, \end{cases}$$

where $m : G \to \mathbb{C}$ is a multiplicative function such that $m(-x) = m(x)$ for all $x \in G$ or $m \in \mathcal{B}(G)$, $b, \varphi : G \to \mathbb{C}$ are bounded functions and $\alpha \in \mathbb{C} \setminus \{0\}, \lambda \in \mathbb{C}$ are constants;

(4)
$$\begin{cases} f &= f_0, \\ g &= -\frac{\lambda^2}{2} f_0 + g_0 - \lambda\, b, \\ h &= \lambda\, f_0 + b, \end{cases}$$

where $b : G \to \mathbb{C}$ is a bounded function, $\lambda \in \mathbb{C}$ is a constant and $f_0, g_0 : G \to \mathbb{C}$ are functions satisfying the functional equation

$$f_0(x-y) = f_0(x)g_0(y) + g_0(x)f_0(y), \;\; x, y \in G;$$

(5)
$$\begin{cases} f &= -\lambda^2 f_0 + \lambda^2 b, \\ g &= \frac{1+\rho^2}{2} f_0 + \rho\, g_0 + \frac{1-\rho^2}{2} b, \\ h &= \lambda\,\rho\, f_0 + \lambda\, g_0 - \lambda\,\rho\, b, \end{cases}$$

where $b : G \to \mathbb{C}$ is a bounded function, $\rho \in \mathbb{C}$, $\lambda \in \mathbb{C} \setminus \{0\}$ are constants and $f_0, g_0 : G \to \mathbb{C}$ are functions satisfying the cosine functional equation

$$f_0(x+y) = f_0(x)f_0(y) - g_0(x)g_0(y), \ x, y \in G,$$

such that $f_0(-x) = f_0(x)$ and $g_0(-x) = g_0(x)$ for all $x \in G$;

(6)
$$\begin{cases} f &= \lambda^2 f_0 - \lambda^2 b, \\ g &= \frac{1}{2} f_o + \frac{1}{2} b, \\ h &= \lambda\, g_0, \end{cases}$$

where $b : G \to \mathbb{C}$ is a bounded function, $\lambda \in \mathbb{C} \setminus \{0\}$ is a constant and $f_0, g_0 : G \to \mathbb{C}$ are functions satisfying the cosine functional equation

$$f_0(x+y) = f_0(x)f_0(y) - g_0(x)g_0(y), \ x, y \in G,$$

such that $f_0(-x) = f_0(x)$ and $g_0(-x) = -g_0(x)$ for all $x \in G$;

(7)
$$\begin{cases} f &= \frac{1}{2} a^2\, m + b, \\ g &= m, \\ h &= -ia\, m, \end{cases}$$

where $m : G \to \mathbb{C}$ is a nonzero bounded multiplicative function, $a : G \to \mathbb{C}$ is a nonzero additive function and $b : G \to \mathbb{C}$ is a bounded function such that $m(-x) = m(x)$ and $b(-x) = -b(x)$ for all $x \in G$;

(8) *$f(x-y) = f(x)g(y) + g(x)f(y) + h(x)h(y)$ for all $x, y \in G$;*

(9)
$$\begin{cases} f &= F_0 + \varphi, \\ g &= -\frac{1}{2}\delta^2 F_0 + G_0 + \delta\, H_0 - \rho\, \varphi, \\ h &= -\delta\, F_0 + H_0 - \delta\, \varphi, \end{cases}$$

where $\rho \in \mathbb{C}$, $\delta \in \mathbb{C} \setminus \{0\}$ are constants and the functions $F_0, G_0, H_0 : G \to \mathbb{C}$ are of the forms (6)-(7) under the same constraints, with $F_0(-x) = F_0(x)$, $G_0(-x) = G_0(x)$, $H_0(-x) = -H_0(x)$, $\varphi(-x) = -\varphi(x)$ for all $x \in G$, such that

(i) *$b(-x) = b(x)$ for all $x \in G$ and $\rho = \frac{1+\lambda\delta^2}{2\lambda^2}$ if F_0, G_0 and H_0 are of the form (6),*

(ii) *$b = 0$ and $\rho = \frac{1}{2}\delta^2$ if F_0, G_0 and H_0 are of the form (7).*

Proof. To study the stability of the functional equation (1.1) we will discuss two cases according to whether f and h are linearly independent modulo $\mathcal{B}(G)$

or not.

Case A: f and h are linearly dependent modulo $\mathcal{B}(G)$. We split the discussion into the cases $h \in \mathcal{B}(G)$ and $h \notin \mathcal{B}(G)$.

Subcase A.1: $h \in \mathcal{B}(G)$. Then the function

$$(x, y) \mapsto f(x-y) - f(x)g(y) - g(x)f(y)$$

is bounded. Since the group G is abelian it is an amenable group. So, according to [3, Theorem 3.3], we have of the following assertions:

(1) $f = 0$, g is arbitrary and $h \in \mathcal{B}(G)$. The result occurs in (1) of Theorem 4.1.
(2) $f, g, h \in \mathcal{B}(G)$. The result occurs in (2) of Theorem 4.1.
(3) $f = a\,m + b$ and $g = m$, where $a : G \to \mathbb{C}$ is an additive function, $m : G \to \mathbb{C}$ is a bounded multiplicative function and $b : G \to \mathbb{C}$ is a bounded function such that $m(-x) = m(x)$ and $a(-x) = a(x)$ for all $x \in G$. Then $2a(x) = a(x) + a(-x) = a(x - x) = a(e) = 0$ for all $x \in G$. Hence $a(x) = 0$ for all $x \in G$. We deduce that $f, g, h \in \mathcal{B}(G)$. This is the result (2) of Theorem 4.1.
(4) $f = \alpha\, m - \alpha\, b$, $g = \frac{1}{2}\, m + \frac{1}{2}\, b$, where $\alpha \in \mathbb{C}\setminus\{0\}$ is a constant, $b : G \to \mathbb{C}$ is a bounded function and $m : G \to \mathbb{C}$ is a multiplicative function such that $m(-x) = m(x)$ for all $x \in G$ or $m \in \mathcal{B}(G)$. This is the result (3) of Theorem 4.1 for $\lambda = 0$.
(5) $f(x - y)) = f(x)g(y) + g(x)f(y)$ for all $x, y \in G$. Therefore, taking into account that $h \in \mathcal{B}(G)$, we obtain the result (4) of Theorem 4.1 for $\lambda = 0$.

Subcase A.2: $h \notin \mathcal{B}(G)$. Then $f \notin \mathcal{B}(G)$. Indeed if $f \in \mathcal{B}(G)$ then the functions $x \mapsto f(x)g(y)$ and $x \mapsto f(x - y)$ belong to $\mathcal{B}(G)$ for all $y \in G$. As the function $x \mapsto \psi(x, y)$ belongs to $\mathcal{B}(G)$ for all $y \in G$ we get that the function $x \mapsto g(x)f(y) + h(x)h(y)$ belongs to $\mathcal{B}(G)$ for all $y \in G$. So, taking into account that $h \notin \mathcal{B}(G)$, we get that there exist a constant $\alpha \in \mathbb{C} \setminus \{0\}$ and a function $k \in \mathcal{B}(G)$ such that

$$h = \alpha g + k. \tag{4.1}$$

Substituting (4.1) in (3.5) we get, by an elementary computation, that

$$\psi(x, y) = f(x - y) - k(x)k(y) - g(x)[f(y) + \alpha\, h(y)] - g(y)[f(x) + \alpha k(x)]$$

for all $x, y \in G$. It follows that the function $x \mapsto g(x)[f(y) + \alpha h(y)]$ belongs to $\mathcal{B}(G)$ for all $y \in G$, so that $h = -\frac{1}{\alpha} f$ or $g \in \mathcal{B}(G)$. Hence, taking (4.1) into account, we get that $h \in \mathcal{B}(G)$, which contradicts the assumption on h. We deduce that $f \notin \mathcal{B}(G)$. Since f and h are linearly dependent modulo $\mathcal{B}(G)$ we deduce that there exist a constant $\lambda \in \mathbb{C} \setminus \{0\}$ and a function $\varphi \in \mathcal{V}$ such that

$$h = \lambda f + \varphi. \tag{4.2}$$

When we substitute (4.2) in (3.5) we obtain by an elementary computation

$$\psi(x, y) + \varphi(x)\varphi(y) = f(x - y) - f(x)\phi(y) - \phi(x)f(y) \tag{4.3}$$

for all $x, y \in G$, where

$$\phi := g + \frac{\lambda^2}{2} f + \lambda \varphi. \tag{4.4}$$

Since the functions ψ and φ are bounded we derive from (4.3) that the function $(x, y) \mapsto f(x - y) - f(x)\phi(y) - \phi(x)f(y)$ is also bounded. Hence, according to [3, Theorem 3.3] and taking (4.2) into account and that $h \notin \mathcal{B}(G)$, we have one of the following possibilities:

(1) $f = a\,m + b$ and $\phi = m$, where $a : G \to \mathbb{C}$ is an additive function, $m : G \to \mathbb{C}$ is a bounded multiplicative function and $b : G \to \mathbb{C}$ is a bounded function such that $m(-x) = m(x)$ and $a(-x) = a(x)$ for all $x \in G$. As in Case A.1(3) we prove that the result (2) of Theorem 4.1 holds.

(2) $f = \alpha\, m - \alpha\, b$, $\phi = \frac{1}{2}m + \frac{1}{2}b$, where $\alpha \in \mathbb{C} \setminus \{0\}$ is a constant, $b : G \to \mathbb{C}$ is a bounded function and $m : G \to \mathbb{C}$ is a multiplicative function such that $m(-x) = m(x)$ for all $x \in G$ or $m \in \mathcal{B}(G)$. So, by using (4.4) and (4.2) we get that

$$g = \frac{1}{2}m + \frac{1}{2}b - \frac{\lambda^2}{2}(\alpha m - \alpha b) - \lambda\,\varphi = \frac{1 - \alpha\lambda^2}{2}m + \frac{1 + \alpha\lambda^2}{2}b - \lambda\,\varphi$$

and $h = \alpha\lambda\, m - \alpha\lambda\, b + \varphi$. The result occurs in (3) of Theorem 4.1.

(3) $f(x - y) = f(x)\phi(y) + \phi(x)f(y)$ for all $x, y \in G$. By putting $f_0 := f$ and $g_0 := \phi$ we get the result (4) of Theorem 4.1.

Case B: f and h are linearly independent modulo $\mathcal{B}(G)$. Then $f \notin \mathcal{B}(G)$. Moreover, according to Lemma 3.1(1), we have $f^o \in \mathcal{B}(G)$ and then $f^e \neq 0$. It follows from (3.3), with φ_2 satisfying the same constraint in Lemma 3.1, that if $h^o \in \mathcal{B}(G)$ then $g^o \in \mathcal{B}(G)$. So we will discuss the following subcases: $h^o \in \mathcal{B}(G)$ and $h^o \notin \mathcal{B}(G)$.

Subcase B.1: $h^o \in \mathcal{B}(G)$. Let $x, y \in G$ be arbitrary. From (3.5) we get, by using (3.2) and (3.3), that

$$\begin{aligned} f^e(x-y) &= [f^e(x)+f^o(x)][g^e(y)+g^o(y)] \\ &\quad + [g^e(x)+g^o(x)][f^e(y)+f^o(y)] \\ &\quad + [h^e(x)+h^o(x)][h^e(y)+h^o(y)] \\ &\quad - f^o(x-y)+\psi(x,y) \\ &= f^e(x)g^e(y)+g^e(x)f^e(y)+h^e(x)h^e(y) \\ &\quad + [f^e(x)g^o(y)+g^e(x)f^o(y)+h^e(x)h^o(y)] \\ &\quad + [f^o(x)g^e(y)+g^o(x)f^e(y)+h^o(x)h^e(y)] \\ &\quad + f^o(x)g^o(y)+g^o(x)f^o(y)+h^o(x)h^o(y) \\ &\quad - f^o(x-y)+\psi(x,y) \\ &= f^e(x)g^e(y)+g^e(x)f^e(y)+h^e(x)h^e(y) \\ &\quad + f^o(x)g^o(y)+g^o(x)f^o(y)+h^o(x)h^o(y) \\ &\quad - f^o(x-y)+\varphi_1(x,y)+\varphi_1(y,x)+\psi(x,y). \end{aligned}$$

Thus, x and y being arbitrary, by using the fact that the functions f^o, g^o, h^o and ψ are bounded, and taking (3.12) into account, we deduce from the identity above that the function $(x,y) \mapsto f^e(x-y)-f^e(x)g^e(y)-g^e(x)f^e(y)-h^e(x)h^e(y)$ is bounded, so is the function $(x,y) \mapsto f^e(x+y)-f^e(x)g^e(y)-g^e(x)f^e(y)-h^e(x)h^e(y)$. Moreover since the functions f and h are linearly independent modulo $\mathcal{B}(G)$ and $f^o, h^o \in \mathcal{B}(G)$ we get that f^e and h^e are linearly independent. Hence, according to Proposition 3.4 we are lead to one of the following possibilities:

(1)

$$\begin{cases} f^e &= -\lambda^2 f_0 + \lambda^2 b, \\ g^e &= \frac{1+\rho^2}{2} f_0 + \rho\, g_0 + \frac{1-\rho^2}{2} b, \\ h^e &= \lambda\,\rho\, f_0 + \lambda\, g_0 - \lambda\,\rho\, b, \end{cases}$$

where $b : G \to \mathbb{C}$ is a bounded function, $\rho \in \mathbb{C}$, $\lambda \in \mathbb{C} \setminus \{0\}$ are constants and $f_0, g_0 : G \to \mathbb{C}$ are functions satisfying the cosine functional equation

$$f_0(x+y) = f_0(x)f_0(y) - g_0(x)g_0(y), \; x, y \in G.$$

Notice that $f_0 \notin \mathcal{B}(G)$ because $f^e = -\lambda^2 f_0 + \lambda^2 b$, $f^e \notin \mathcal{B}(G)$ and $b \in \mathcal{B}(G)$. Since f^e and h^e are linearly independent modulo $\mathcal{B}(G)$ so are the functions f_0 and g_0. Indeed, if not then there exist a constant $\alpha \in \mathbb{C}$ and a function $\varphi \in \mathcal{B}(G)$ such that $g_0 = \alpha f_0 + \varphi$. Hence

$$h^e = \lambda\,\rho\, f_0 + \lambda\,(\alpha f_0 + \varphi) - \lambda\,\rho\, b = \lambda\,(\rho+\alpha) f_0 + b_1,$$

where $b_1 := \lambda\,\varphi - \lambda\,\rho\,b$ belongs to $\mathcal{B}(G)$. Then

$$\lambda\,h^e + (\rho+\alpha)f^e = \lambda\,b_1 + \lambda^2(\rho+\alpha)b,$$

which implies that the function $\lambda\,h^e + (\rho+\alpha)f^e$ belongs to $\mathcal{B}(G)$. This contradicts the fact that f^e and h^e are linearly independent modulo $\mathcal{B}(G)$ because $\lambda \neq 0$. Hence f_0 and g_0 are linearly independent modulo $\mathcal{B}(G)$.

On the other hand let $\psi_1 := f^o$, $\psi_2 := g^o$ and $\psi_3 := h^o$. The identity (3.2) implies

$$\begin{aligned}\varphi_1(x,y) &= (-\lambda^2 f_0(x) + \lambda^2 b(x))\psi_2(y)\\ &\quad + (\frac{1+\rho^2}{2} f_0(x) + \rho\,g_0(x) + \frac{1-\rho^2}{2} b(x))\psi_1(y)\\ &\quad + (\lambda\,\rho\,f_0(x) + \lambda\,g_0(x) - \lambda\,\rho\,b(x))\psi_3(y)\\ &= f_0(x)[-\lambda^2\psi_2(y) + \frac{1+\rho^2}{2}\psi_1(y) + \lambda\,\rho\,\psi_3(y)]\\ &\quad + g_0(x)[\rho\,\psi_1(y) + \lambda\,\psi_3(y)]\\ &\quad + b(x)[\lambda^2\psi_2(y) + \frac{1-\rho^2}{2}\psi_1(y) - \lambda\,\rho\,\psi_3(y)],\end{aligned}$$

for all $x, y \in G$. So, taking (3.12) into account and that the functions ψ, b, ψ_1, ψ_2 and ψ_3 are bounded, we deduce from the identity above that the function

$$x \mapsto f_0(x)[-\lambda^2\,\psi_2(y) + \frac{1+\rho^2}{2}\psi_1(y) + \lambda\,\rho\,\psi_3(y)] + g_0(x)[\rho\,\psi_1(y) + \lambda\,\psi_3(y)]$$

belongs to $\mathcal{B}(G)$ for all $y \in G$. Since f_0 and g_0 are linearly independent modulo $\mathcal{B}(G)$ we get that

$$-\lambda^2\psi_2(y) + \frac{1+\rho^2}{2}\psi_1(y) + \lambda\,\rho\,\psi_3(y) = 0$$

and

$$\rho\,\psi_1(y) + \lambda\,\psi_3(y) = 0$$

for all $y \in G$, from which we derive by a small computation that $\psi_2 = \frac{1-\rho^2}{2\lambda^2}\psi_1$ and $\psi_3 = -\frac{\rho}{\lambda}\psi_1$. As $f = f^e + f^o = f^e + \psi_1$, $g = g^e + g^o = g^e + \psi_2 = g^e + \frac{1-\rho^2}{2\lambda^2}\psi_1$ and $h = h^e + h^o = h^e + \psi_3 = h^e + \frac{1-\rho^2}{2\lambda^2}\psi_1$, we deduce that

$$(I)\left\{\begin{array}{lcl} f &=& -\lambda^2\,f_0 + \lambda^2\,b + \psi_1,\\ g &=& \frac{1+\rho^2}{2}\,f_0 + \rho\,g_0 + \frac{1-\rho^2}{2}\,b + \frac{1-\rho^2}{2\lambda^2}\psi_1,\\ h &=& \lambda\,\rho\,f_0 + \lambda\,g_0 - \lambda\,\rho\,b - \frac{\rho}{\lambda}\psi_1.\end{array}\right.$$

Moreover, since f^e, g^e and h^e are even functions, and $\psi_1 = f^o$, we get that

$$\begin{cases} \psi_1(-x) = -\psi_1(x) \\ -f_0(-x) + b(-x) = -f_0(x) + b(x) \\ \frac{1}{2} f_0(-x) + \rho\, g_0(-x) + \frac{1}{2} b(-x) = \frac{1}{2} f_0(x) + \rho\, g_0(x) + \frac{1}{2} b(x), \\ \rho\, f_0(-x) + g_0(-x) - \rho\, b(-x) = \rho\, f_0(x) + g_0(x) - \rho\, b(x), \end{cases}$$

which implies $f_0(-x) = f_0(x)$, $g_0(-x) = g_0(x)$, $b(-x) = b(x)$ and $\psi_1(-x) = -\psi_1(x)$ for all $x \in G$. Thus we obtain, by writing b instead of $b + \frac{1}{\lambda^2}\, \psi_1$ in (I), the result (5) of Theorem 4.1.

(2)

$$\begin{cases} f^e &= \lambda^2 M + a\, m + b, \\ g^e &= \beta\lambda\,(1 - \frac{1}{2}\beta\lambda\,)M + (1 - \beta\lambda\,)m - \frac{1}{2}\beta^2\, a\, m - \frac{1}{2}\beta^2\, b, \\ h^e &= \lambda\,(1 - \beta\lambda\,)M - \lambda\, m - \beta\, a\, m - \beta\, b, \end{cases}$$

where $m : G \to \mathbb{C}$ is a nonzero bounded multiplicative function, $M : G \to \mathbb{C}$ is a non bounded multiplicative function, $a : G \to \mathbb{C}$ is a nonzero additive function, $b : G \to \mathbb{C}$ is a bounded function and $\beta \in \mathbb{C}$, $\lambda \in \mathbb{C}\backslash\{0\}$ are constants. Then $\beta\, f^e + h^e = \beta\lambda^2\, M + \beta\, a\, m + \beta\, b + \lambda\,(1 - \beta\lambda\,)M - \lambda\, m - \beta\, a\, m - \beta\, b = \lambda\,(M - m)$. So that

$$M(-x) - m(-x) = M(x) - m(x) \tag{4.5}$$

for all $x \in G$. Moreover, since f^e and g^e are even functions, and

$$a(-x) + a(x) = a(-x + x) = a(e) = 0$$

for all $x \in G$, we get that

$$\lambda^2\, M(-x) - a(x)\, m(-x) + b(-x) = \lambda^2\, M(x) + a(x)\, m(x) + b(x) \tag{4.6}$$

and

$$\begin{aligned} &\beta\lambda\,(1 - \frac{1}{2}\beta\lambda\,)M(-x) + (1 - \beta\lambda\,)m(-x) + \frac{1}{2}\beta^2\, a(x)\, m(-x) - \frac{1}{2}\beta^2\, b(-x) \\ &= \beta\lambda\,(1 - \frac{1}{2}\beta\lambda\,)M(x) + (1 - \beta\lambda\,)m(x) - \frac{1}{2}\beta^2\, a(x)\, m(x) - \frac{1}{2}\beta^2\, b(x), \end{aligned} \tag{4.7}$$

for all $x \in G$. By multiplying (4.6) by $\frac{1}{2}\beta^2$ and adding the result to (4.7) we get that

$$\beta\lambda\,(M(x) - m(x)) - \beta\lambda\,(M(-x) - m(-x)) + m(x) - m(-x) = 0$$

for all $x \in G$. We deduce, by taking (4.5) into account, that $m(-x) = m(x)$ and $M(-x) = M(x)$ for all $x \in G$. When we substitute this back into (4.6) we get that

$$-a(x)\, m(x) + b(-x) = a(x)\, m(x) + b(x)$$

for all $x \in G$. Hence $a(x) = -b^o(x)\, m(-x)$ for all $x \in G$. As b and m are bounded functions we derive that the additive function a is bounded, so

$a(x) = 0$ for all $x \in G$, which contradicts the condition on a. Therefore the present case does not occur.

(3)

$$\begin{cases} f^e &= \frac{1}{2}a^2\, m + \frac{1}{2}a_1\, m + b, \\ g^e &= -\frac{1}{4}\beta^2\, a^2\, m + \beta\, a\, m - \frac{1}{4}\beta^2\, a_1\, m + m - \frac{1}{2}\beta^2\, b, \\ h^e &= -\frac{1}{2}\beta\, a^2\, m + a\, m - \frac{1}{2}\beta\, a_1\, m - \beta\, b, \end{cases}$$

where $m : G \to \mathbb{C}$ is a nonzero bounded multiplicative function, $a, a_1 : G \to \mathbb{C}$ are additive functions such that a is nonzero, $b : G \to \mathbb{C}$ is a bounded function and $\beta \in \mathbb{C}$ is a constant.

Notice that $\beta\, f^e + h^e = a\, m$ and $2g^e = \beta^2\, f^e + 2\beta\, h^e + 2m$, then m and $a\, m$ are even functions. As seen earlier we have $a(-x) = -a(x)$ for all $x \in G$. Hence $-a(x)\, m(x) = a(x)\, m(x)$ for all $x \in G$, so $a = 0$, which contradicts the condition on a. We conclude that the present possibility does not occur.

(4)

$$f^e(x + y) = f^e(x)m(y) + m(x)f^e(y) + (a(x)m(x) + b(x))(a(y)m(y) + b(y))$$

for all $x, y \in G$,

$$g^e = -\frac{1}{2}\beta^2\, f^e + (1 + \beta\, a)m + \beta\, b$$

and

$$h^e = -\beta\, f^e + a\, m + b,$$

where $m : G \to \mathbb{C}$ is a nonzero bounded multiplicative function, $a : G \to \mathbb{C}$ is a nonzero additive function, $b : G \to \mathbb{C}$ is a bounded function and $\beta \in \mathbb{C}$ is a constant.

The second and the third identities above imply

$$m = -\frac{1}{2}\beta^2\, f^e + g^e - \beta\, h^e,$$

from which we deduce that $m(-x) = m(x)$ for all $x \in G$. Moreover the third identity above implies that the function $a\, m + b$ is even. Since $a(-x) = -a(x)$ for all $x \in G$, we get that

$$-a(x)m(x) + b(-x) = a(x)m(x) + b(x)$$

for all $x \in G$. Hence $a = -b^o\, m$. As b and m are bounded functions and a is an additive function we deduce that $a = 0$, which contradicts the condition on a. We conclude that the present possibility does not occur.

(5) f^e, g^e and h^e satisfy the functional equation

$$f^e(x + y) = f^e(x)g^e(y) + g^e(x)f^e(y) + h^e(x)h^e(y) \tag{4.8}$$

for all $x, y \in G$.

If $f^o = 0$ then $f^e = f$. Moreover, taking into account that f^e and h^e are linearly independent, we derive from (3.2) that $g^o = 0$ and $h^o = 0$, hence $g^e = g$ and $h^e = h$. So the functional equation (4.8) becomes $f(x-y) = f(x)g(y) + g(x)f(y) + h(x)h(y)$ for all $x, y \in G$. This is the result (8) of Theorem 4.1.

If $f^o \neq 0$ then, according to (3.2), there exist two constants $\alpha, \beta \in \mathbb{C}$ and an even function $b \in \mathcal{B}(G)$ such that

$$g^e = \alpha\, f^e + \beta\, h^e + b. \tag{4.9}$$

By substituting (4.9) into (4.8) we get, by a similar computation to the one of Case A of the proof of [4, Lemma 3.4], that

$$\begin{aligned} f^e(x+y) = (2\alpha - \beta^2) f^e(x) f^e(y) + f^e(x) b(y) + b(x) f^e(y) \\ + [\beta\, f^e(x) + h^e(x)][\beta\, f^e(y) + h^e(y)] \end{aligned} \tag{4.10}$$

for all $x, y \in G$. We have the following subcases:

Subcase B.1.1: $2\alpha \neq \beta^2$. Proceeding exactly as in Subcase A.1 of the proof of [4, Lemma 3.4] we get that

$$\begin{cases} f^e &= -\lambda^2 f_0 + \lambda^2 b, \\ g^e &= \frac{1+\rho^2}{2} f_0 + \rho\, g_0 + \frac{1-\rho^2}{2} b, \\ h^e &= \lambda\,\rho\, f_0 + \lambda\, g_0 - \lambda\,\rho\, b. \end{cases}$$

So we go back to the possibility (1) and then obtain the result (5) of Theorem 4.1.

Subcase B.1.2: $2\alpha = \beta^2$. By similar computations to the ones in Subcase A.1 of the proof of [4, Lemma 3.4] we get that there exist a constant $\eta \in \mathbb{C}$ such that

$$H(x+y) = H(x)m(y) + m(x)H(y) + \eta\, H(x)H(y) \tag{4.11}$$

for all $x, y \in G$ and

$$b = m \tag{4.12}$$

where $\eta \in \mathbb{C}$, $H := \beta\, f^e + h^e$ and $m \in \mathcal{B}(G)$ is an even multiplicative function.

If $\eta = 0$ then H satisfies the functional equation

$$H(x+y) = H(x)m(y) + m(x)H(y)$$

for all $x, y \in G$. As f^e and h^e are linearly independent modulo $\mathcal{B}(G)$ we have $H \neq 0$, hence m is a nonzero multiplicative function on the group G. So, from the functional equation above we deduce that there exists an additive function $a : G \to \mathbb{C}$ such that $H = a\, m$. Since H is even so is a, hence $a = 0$ which contradicts the fact that $H \neq 0$.

If $\eta \neq 0$ then, by multiplying both sides of (4.11) by η and adding $m(x+y)$ to both sides of the obtained identity, we get, by a small computation, that

$$m(x+y) + \eta^2 H(x+y) = [m(x) + \eta\, H(x)][m(y) + \eta\, H(y)]$$

for all $x, y \in G$. So there exist an even multiplicative function $M : G \to \mathbb{C}$ and a constant $\lambda \in \mathbb{C}\backslash\{0\}$ such that $H = \lambda(M - m)$. By substituting this into (4.10) and taking (4.12) into account we obtain

$$\begin{aligned} f^e(x+y) &= f^e(x)m(y) + m(x)f^e(y) + \lambda^2(M(x) - m(x))(M(y) - m(y)) \\ &= f^e(x)m(y) + m(x)f^e(y) + \lambda^2 M(x+y) \\ &\quad - \lambda^2 M(x)m(y) - \lambda^2 m(x)M(y) + \lambda^2 m(x+y) \end{aligned}$$

for all $x, y \in G$. Since m is a nonzero multiplicative function on the group G we have $m(x) \neq 0$ for all $x \in G$. So, by dividing both sides of the functional equation above we get that

$$\frac{f^e(x+y) - \lambda^2 M(x+y)}{m(x+y)} + \lambda^2 = [\frac{f^e(x) - \lambda^2 M(x)}{m(x)} + \lambda^2] + [\frac{f^e(y) - \lambda^2 M(y)}{m(y)} + \lambda^2]$$

for all $x, y \in G$, hence there exists an additive function $a : G \to \mathbb{C}$ such that

$$\frac{f^e(x) - \lambda^2 M(x)}{m(x)} + \lambda^2 = a(x)$$

for all $x \in G$. Since f^e, M and m are even functions so is the additive function a, then $a(x) = 0$ for all $x \in G$. Hence $f^e = \lambda^2(M - m)$. Then $f^e = \lambda H = \lambda\beta\, f^e + \lambda\, h^e$, which contradicts the linear independence modulo $\mathcal{B}(G)$ of f^e and h^e. We conclude that the Subcase B.1.1 does not occur.

Subcase B.2: $h^o \notin \mathcal{B}(G)$. Since $\mathcal{B}(G)$ is a two-sided invariant and $(-I)$-invariant linear space of complex-valued functions on G, then we deduce, according to Lemma 3.2, that $h^e = \gamma\, f^e$ and $g^o = -\gamma\, h^o - \eta\, f^o$, where $\gamma, \eta \in \mathbb{C}$ are two constants. We split the discussion into the cases $\gamma = 0$ and $\gamma \neq 0$.

Subcase B.2.1: $\gamma = 0$. Then, from Lemma 3.1(1), (3.16) and (3.17), we deduce that $h^o = h$ and $g^o \in \mathcal{B}(G)$. So we get, from the identities (3.4) and

(3.5), that

$$\begin{aligned} f(x+y) &= f(x)g(y) + g(x)f(y) + h(x)h(y) - 2f^o(x)g^o(y) - 2g^o(x)f^o(y) \\ &\quad - 2h(x)h(y) + \psi(x,y) + \varphi_3(x,y) \\ &= [f^e(x) + f^o(x)][g^e(y) + g^o(y)] + [g^e(x) + g^o(x)][f^e(y) + f^o(y)] \\ &\quad - h(x)h(y) - 2f^o(x)g^o(y) - 2g^o(x)f^o(y) + \psi(x,y) + \varphi_3(x,y) \\ &= f^e(x)g^e(y) + g^e(x)f^e(y) - h(x)h(y) + (f^e(x)g^o(y) + g^e(x)f^o(y)) \\ &\quad + (g^o(x)f^e(y) + f^o(x)g^e(y)) - f^o(x)g^o(y) - g^o(x)f^o(y) + \psi(x,y) \\ &\quad + \varphi_3(x,y) \end{aligned}$$

for all $x, y \in G$. Hence, taking into account that $h^e = 0$, and by using (3.2) and (3.15), a small computation shows that

$$f^e(x+y) = f^e(x)g^e(y) + g^e(x)f^e(y) + k(x)k(y) + \Psi(x,y) \tag{4.13}$$

for all $x, y \in G$, where

$$k := ih \tag{4.14}$$

and

$$\Psi(x,y) := \psi(x,-y) + \varphi_1(y,x) - \varphi_1(x,y) - f^o(x+y) - f^o(x)g^o(y) - g^o(x)f^o(y) \tag{4.15}$$

for all $x, y \in G$. As the functions f^o, g^o and ψ are bounded we deduce, from (3.12), (4.13) and (4.15), that the function

$$(x,y) \mapsto f^e(x+y) - f^e(x)g^e(y) - g^e(x)f^e(y) - k(x)k(y)$$

is bounded. Hence, according to Proposition 3.4 we obtain one of the following possibilities:

(1)

$$\left\{ \begin{array}{rcl} f^e & = & -\lambda^2 f_0 + \lambda^2 b, \\ g^e & = & \frac{1+\rho^2}{2} f_0 + \rho\, g_0 + \frac{1-\rho^2}{2} b, \\ k & = & \lambda\,\rho\, f_0 + \lambda\, g_0 - \lambda\,\rho\, b, \end{array} \right.$$

where $b : G \to \mathbb{C}$ is a bounded function, $\rho \in \mathbb{C}$, $\lambda \in \mathbb{C} \setminus \{0\}$ are constants and $f_0, g_0 : G \to \mathbb{C}$ are functions satisfying the cosine functional equation

$$f_0(x+y) = f_0(x)f_0(y) - g_0(x)g_0(y), \;\; x, y \in G.$$

Since f^e and g^e are even functions, k is an odd function and $\lambda \neq 0$ we get that

$$f_0(-x) - b(-x) = f_0(x) - b(x), \tag{4.16}$$

$$f_0(-x) + 2\rho\, g_0(-x) + b(-x) = f_0(x) + 2\rho\, g_0(x) + b(x) \tag{4.17}$$

and

$$\rho\,(f_0(-x) - b(-x)) + g_0(-x) = -\rho\,(f_0(x) - b(x)) - g_0(x) \tag{4.18}$$

for all $x \in G$. The identity (4.16) implies

$$f_0^o = b^o. \tag{4.19}$$

By using this and the identity $k = \lambda\,\rho\,f_0 + \lambda\,g_0 - \lambda\,\rho\,b$, and taking into account that k is an odd function we obtain

$$k = \lambda\,g_0^o. \tag{4.20}$$

By multiplying both sides of (4.16) by ρ and subtracting (4.18) from the result we deduce that

$$g_0^e = -\rho\,(f_0 - b). \tag{4.21}$$

Moreover, we derive from (4.17) that

$$2\rho\,(g_0(x) - g_0(-x)) = -(f_0(x) - f_0(-x)) - (b(x) - b(-x))$$

for all $x \in G$, which implies, by taking (4.19) into account, that

$$\rho\,g_0^o = -b^o. \tag{4.22}$$

From (4.20), (4.22) and (4.14) we get that

$$\rho\,h = \lambda\,ib^o. \tag{4.23}$$

Since b is a bounded function on G we deduce from (4.23) that $\rho\,h$ is also a bounded function. As $h \notin \mathcal{B}(G)$ we get that $\rho = 0$. It follows that

$$(II)\begin{cases} f^e &= -\lambda^2\,f_0 + \lambda^2\,b,\\ g^e &= \frac{1}{2}f_0 + \frac{1}{2}\,b,\\ k &= \lambda\,g_0.\end{cases}$$

Let $\psi_1 := g^o$ and $\psi_2 := f^o$. By using that $h^e = 0$, (3.2), the first and the second identities in (II) we obtain

$$\begin{aligned}\varphi_1(x,y) &= (-\lambda^2\,f_0(x) + \lambda^2\,b(x))\psi_1(y) + (\frac{1}{2}\,f_o(x) + \frac{1}{2}\,b(x))\psi_2(y)\\ &= f_0(x)[-\lambda^2\,\psi_1(y) + \frac{1}{2}\psi_2(y)] + b(x)[\lambda^2\,\psi_1(y) + \frac{1}{2}\psi_2(y)],\end{aligned}$$

for all $x, y \in G$. So, taking (3.12) into account and the that the functions ψ, b, ψ_1 and ψ_2 are bounded, we deduce from the identity above that the function

$$x \mapsto f_0(x)[-\lambda^2\,\psi_1(y) + \frac{1}{2}\psi_2(y)]$$

belongs to $\mathcal{B}(G)$ for all $y \in G$. Since

$$f^e = -\lambda^2\,f_0 + \lambda^2\,b,$$

$f^e \notin \mathcal{B}(G)$ and $b \in \mathcal{B}(G)$ we deduce that $f_0 \notin \mathcal{B}(G)$. Hence

$$-\lambda^2\,\psi_1(y) + \frac{1}{2}\psi_2(y) = 0$$

for all $y \in G$, which implies that

$$\psi_2 = 2\lambda^2 \psi_1.$$

Since

$$f = f^e + f^o = f^e + \psi_2 = f^e + 2\lambda^2 \psi_1, \; g^e + g^o = g^e + \psi_1$$

we deduce, taking (4.14) and (II) into account, that

$$(III) \begin{cases} f &= -\lambda^2 f_0 + \lambda^2 b + 2\lambda^2 \psi_1, \\ g &= \frac{1}{2} f_0 + \frac{1}{2} b + \psi_1, \\ h &= -\lambda i \, g_0. \end{cases}$$

On the other hand, we get from the identities (4.22), (4.19), (4.21) and $\psi_1 = g^o$, that

$$b(-x) = b(x), \; f_0(-x) = f_0(x), \; g_0(-x) = -g_0(x) \text{ and } \psi_1(-x) = -\psi_1(x)$$

for all $x \in G$, and $\psi_1 \in \mathcal{B}(G)$. So we obtain, by writing b and λ instead of $b + 2\psi_1$ and $-\lambda i$ respectively in (III), the result (6) of Theorem 4.1.

(2)

$$\begin{cases} f^e &= \lambda^2 M + am + b, \\ g^e &= \beta\lambda(1 - \frac{1}{2}\beta\lambda)M + (1 - \beta\lambda)m - \frac{1}{2}\beta^2 a \, m - \frac{1}{2}\beta^2 b, \\ k &= \lambda(1 - \beta\lambda)M - \lambda \, m - \beta \, a \, m - \beta \, b, \end{cases}$$

where $m : G \to \mathbb{C}$ is a nonzero bounded multiplicative function, $M : G \to \mathbb{C}$ is a non bounded multiplicative function, $a : G \to \mathbb{C}$ is a nonzero additive function, $b : G \to \mathbb{C}$ is a bounded function and $\beta \in \mathbb{C}$, $\lambda \in \mathbb{C} \setminus \{0\}$ are constants.

We have $\beta \, k = -\frac{1}{2}\beta^2 f^e + g^e - m$, which implies, taking into account that k is an odd function, that $\beta \, k = -m^o$. Hence $\beta \, k \in \mathcal{B}(G)$. As $k \notin \mathcal{B}(G)$ we get that $\beta = 0$. Then $g^e = m$ and $k = \lambda(M - m)$. Since $\lambda \neq 0$ we get that $m(-x) = m(x)$ and $M(-x) - m(-x) = -M(x) + m(x)$ for all $x \in G$. So that $2m(x) = M(-x) + M(x)$ for all $x \in G$. Since m and M are multiplicative functions we deduce, according to [21, Corollary 3.19], that $m = M$, which contradicts the conditions $m \in \mathcal{B}(G)$ and $M \notin \mathcal{B}(G)$. Thus the present possibility does not occur.

(3)

$$\begin{cases} f^e &= \frac{1}{2} a^2 \, m + \frac{1}{2} a_1 \, m + b, \\ g^e &= -\frac{1}{4}\beta^2 \, a^2 \, m + \beta \, a \, m - \frac{1}{4}\beta^2 \, a_1 \, m + m - \frac{1}{2}\beta^2 \, b, \\ k &= -\frac{1}{2}\beta \, a^2 \, m + a \, m - \frac{1}{2}\beta \, a_1 \, m - \beta \, b, \end{cases}$$

where $m : G \to \mathbb{C}$ is a nonzero bounded multiplicative function, $a, a_1 : G \to \mathbb{C}$ are additive functions such that a is nonzero, $b : G \to \mathbb{C}$ is a bounded function and $\beta \in \mathbb{C}$ is a constant.

Notice that $\beta\, k = -\frac{1}{2}\beta^2\, f^e + g^e - m$. As in the possibility above we get that $\beta = 0$. Hence we obtain

$$(IV) \begin{cases} f^e &= \frac{1}{2}a^2\, m + \frac{1}{2}a_1\, m + b, \\ g^e &= m, \\ k &= a\, m. \end{cases}$$

From the second identity of (IV) we deduce that $m(-x) = m(x)$ for all $x \in G$. As $f^e(-x) = f^e(x)$, $a(-x) = -a(x)$ and $a_1(-x) = -a_1(x)$ for all $x \in G$, we deduce from the first identity of (IV) that

$$\frac{1}{2}a^2(x)m(x) - \frac{1}{2}a_1(x)m(x) + b(-x) = \frac{1}{2}a^2(x)m(x) + \frac{1}{2}a_1(x)m(x) + b(x)$$

for all $x \in G$. So

$$a_1(x)m(x) = b(x) - b(-x)$$

for all $x \in G$, from which we get, taking into account that $m(-x) = m(x)$ for all $x \in G$ and m is a nonzero multiplicative function on the group G, that $a_1 = -2mb^o$. As $m, b \in \mathcal{B}(G)$ and a_1 is an additive function we deduce that $a_1 = 0$ and $b(-x) = b(x)$ for all $x \in G$. Hence the first identity of (IV) becomes $f^e = \frac{1}{2}a^2\, m + b$. So, taking into account that $g^e = m$ and $h^e = 0$, the identity (3.2) becomes

$$\begin{aligned} \varphi_1(x,y) &= [\frac{1}{2}a^2(x)m(x) + b(x)]g^o(y) + m(x)f^o(y) \\ &= \frac{1}{2}a^2(x)m(x)g^o(y) + b(x)g^o(y) + m(x)f^o(y), \end{aligned}$$

for all $x, y \in G$. As the functions m, b, g^o and f^o are bounded and m is a nonzero multiplicative function on the group G, we deduce from the identity above that the function

$$x \mapsto a^2(x)g^o(y)$$

belongs to $\mathcal{B}(G)$ for all $y \in G$. Since a^2 is a non bounded function, because of the fact that a is a nonzero additive function on G, we deduce that $g^o = 0$. We infer from (IV), taking (4.14) into account, and using that $f = f^e + f^o$ and $g = g^e + g^o$, that

$$\begin{cases} f &= \frac{1}{2}a^2\, m + b + f^o, \\ g &= m, \\ h &= -ia\, m. \end{cases}$$

By writing b instead of $b + f^o$ in the identities above we obtain the result (7) of Theorem 4.1.

(4) f^e satisfies the functional equation

$$f^e(x+y) = f^e(x)m(y) + m(x)f^e(y) + (a(x)m(x) + b(x))(a(y)m(y) + b(y)) \tag{4.24}$$

for all $x, y \in G$,

$$g^e = -\frac{1}{2}\beta^2 f^e + (1 + \beta\, a)m + \beta\, b$$

and

$$k = -\beta\, f^e + a\, m + b,$$

where $m : G \to \mathbb{C}$ is a nonzero bounded multiplicative function, $a : G \to \mathbb{C}$ is a nonzero additive function, $b : G \to \mathbb{C}$ is a bounded function and $\beta \in \mathbb{C}$ is a constant.

A simple computation shows that $\beta\, k = -\frac{1}{2}\beta^2 f^e + g^e - m$. Thus, as in the possibility (2), we have $\beta = 0$. Hence

$$g^e = m \tag{4.25}$$

and

$$k = a\, m + b. \tag{4.26}$$

From (4.24) and (4.26) we deduce that f^e and k satisfy the functional equation

$$f^e(x+y) = f^e(x)m(y) + m(x)f^e(y) + k(x)k(y).$$

As a is a nonzero additive function, m is a nonzero multiplicative bounded function and b is bounded we derive from (4.26) that $k \neq 0$. Moreover $k(-x) = -k(x)$ for all $x \in G$, and from (4.25) we get that $m(-x) = m(x)$ for all $x \in G$. Hence, according to Proposition 3.3, f^e and k are of the form

$$f^e = \frac{1}{2}A^2 m \tag{4.27}$$

and

$$k = A\, m, \tag{4.28}$$

where $A : G \to \mathbb{C}$ is a nonzero additive function. It follows, from (4.26), (4.28) and that $m(-x) = m(x)$ for all $x \in G$, that $A - a = b\, m$. Hence, $A - a$ is a bounded additive function. Therefore $A = a$ and $b = 0$. We deduce, taking (4.27) and (4.28) into account, that

$$f^e = \frac{1}{2}a^2 m. \tag{4.29}$$

and

$$k = a\, m. \tag{4.30}$$

Moreover, since the functions m and ψ are bounded, we deduce by using (3.2), (3.12) and (4.25), that the function $x \to f^e(x)g^o(y)$ belongs to $\mathcal{B}(G)$ for all $y \in G$. As seen earlier, we have $f^e \notin \mathcal{B}(G)$. Hence

$$g^o = 0. \tag{4.31}$$

Thus, by using (4.14), (4.25), (4.29), (4.30) and (4.31), and taking into account that $f^o \in \mathcal{B}(G)$, we conclude, by writing b instead of f^o, that

$$\begin{cases} f &= \frac{1}{2}a^2\, m + b, \\ g &= m, \\ h &= -ia\, m. \end{cases}$$

The result occurs in (7) of Theorem 4.1.

(5) f^e, g^e and k satisfy the functional equation

$$f^e(x+y) = f^e(x)g^e(y) + g^e(x)f^e(y) + k(x)k(y) \tag{4.32}$$

for all $x, y \in G$.

If $f^o = 0$ then $f^e = f$. Moreover we derive from (3.17) that $g^e = g$. So, by using (4.14), the functional equation (4.32) becomes

$$f(x+y) = f(x)g(y) + g(x)f(y) - h(x)h(y)$$

for all $x, y \in G$. As $h = h^o$ we derive that f, g and h satisfy the functional equation

$$f(x-y) = f(x)g(y) + g(x)f(y) + h(x)h(y)$$

for all $x, y \in G$. This is the result (8) of Theorem 4.1.

If $f^o \neq 0$ then, according to (3.2), there exist a constant $\eta \in \mathbb{C}$ and an even function $\varphi \in \mathcal{B}(G)$ such that

$$g^e = \eta\, f^e + \varphi.$$

Substituting this into (4.32) we obtain

$$f^e(x+y) = 2\eta\, f^e(x)f^e(y) + f^e(x)\varphi(y) + \varphi(x)f^e(y) + k(x)k(y) \tag{4.33}$$

for all $x, y \in G$.

If $\eta = 0$, then the functional equation (4.33) can be written

$$f^e(x+y) = f^e(x)\varphi(y) + \varphi(x)f^e(y) + k(x)k(y) \tag{4.34}$$

for all $x, y \in G$.

Notice that $\varphi \neq 0$. Indeed, if $\varphi = 0$ then we get, by putting $y = e$ in (4.34) and taking (4.14) into account, that

$$f^e(x) + h(x)h(e) = 0$$

for all $x \in G$. Since $h = h^o$ we have $h(e) = 0$. Hence $f^e(x) = 0$ for all $x \in G$, and then $f = f^o$, which implies $f \in \mathcal{B}(G)$ which contradicts that f and h are linearly independent modulo $\mathcal{B}(G)$. Moreover we derive from (4.34), according to [4, Lemma 3.2], that φ is a multiplicative function because f^e and k are linearly independent modulo $\mathcal{B}(G)$ and $\varphi \in \mathcal{B}(G)$. Let $m := \varphi$. Then the functional equation (4.34) becomes

$$f^e(x+y) = f^e(x)m(y) + m(x)f^e(y) + k(x)k(y)$$

for all $x, y \in G$. Since f^e is an even function, m a nonzero multiplicative function on the group G such that

$$m(-x) = \varphi(-x) = \varphi(x) = m(x)$$

for all $x \in G$, and k an odd function we deduce, according to Proposition 3.3, that $f^e = \frac{1}{2}a^2 m$ and $k = a\,m$ where $a : G \to \mathbb{C}$ is a nonzero additive function. So, taking (4.14), (3.17) and (3.18) into account, and using that $f^o \in \mathcal{B}(G)$, $\gamma = \eta = 0$ and $\varphi = m$, we derive, by setting $b = f^o$, that

$$\begin{cases} f &= \frac{1}{2}a^2\, m + b, \\ g &= m, \\ h &= -ia\, m. \end{cases}$$

This is the result (7) of Theorem 4.1.

If $\eta \neq 0$, let $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\lambda^2 = \frac{1}{2\eta}$. The functional equation (4.33) can be written, by multiplying both sides by $\frac{1}{\lambda^2}$ and adding $\varphi(x+y)$ to the obtained functional equation, as follows

$$\begin{aligned} \frac{1}{\lambda^2}\, f^e(x+y) + \varphi(x+y) = {} & [\frac{1}{\lambda^2}\, f^e(x) + \varphi(x)][\frac{1}{\lambda^2}\, f^e(y) + \varphi(y)] \\ & + \frac{1}{\lambda^2}\, k(x)k(y) + \varphi(x+y) - \varphi(x)\varphi(y) \end{aligned}$$

for all $x, y \in G$. As $\varphi \in \mathcal{B}(G)$ we get that the function

$$x \mapsto \frac{1}{\lambda^2}\, f^e(x+y) + \varphi(x+y) - [\frac{1}{\lambda^2}\, f^e(x) + \varphi(x)][\frac{1}{\lambda^2}\, f^e(y) + \varphi(y)] - \frac{1}{\lambda^2}\, k(x)k(y)$$

belongs to the two-sided invariant linear space $\mathcal{B}(G)$ for all $y \in G$. Since the functions f^e and h are linearly independent modulo $\mathcal{B}(G)$ so are $\frac{1}{\lambda^2}\, f^e + \varphi$ and $\frac{1}{\lambda^2}\, k$. Hence, according to [22, Lemma 3.1] and taking (4.14) into account, the functional equation

$$\frac{1}{\lambda^2}\, f^e(x+y) + \varphi(x+y) = [\frac{1}{\lambda^2}\, f^e(x) + \varphi(x)][\frac{1}{\lambda^2}\, f^e(y) + \varphi(y)] - \frac{1}{\lambda^2} h(x)h(y)$$

for all $x, y \in G$, is valid, from which we deduce that

$$(V)\begin{cases} f^e &= \lambda^2 f_0 - \lambda^2\varphi, \\ h &= \lambda g_0, \end{cases}$$

where

$$f_0 := \frac{1}{\lambda^2} f^e + \varphi$$

and $g_0 := \frac{1}{\lambda}h$ satisfy the functional equation

$$f_0(x+y) = f_0(x)f_0(y) - g_0(x)g_0(y)$$

for all $x, y \in G$.

Moreover, since φ is an even function and $h^e = 0$ we get easily that

$$f_0(-x) = f_0(x)$$

and

$$g_0(-x) = -g_0(x)$$

for all $x \in G$.

On the other hand, by taking into account that $f = f^e + f^o$ and $g = g^e + g^o$, and by using (3.17), (3.18) and (V), we derive by an elementary computation that

$$\begin{cases} f &= \lambda^2 f_0 - \lambda^2 b, \\ g &= \frac{1}{2} f_0 + \frac{1}{2} b \\ h &= \lambda g_0, \end{cases}$$

where $b := \varphi - \frac{1}{\lambda^2} f^o$ is a bounded function. The result occurs in (6) of Theorem 4.1.

Subcase B.2.2: $\gamma \neq 0$. Let $x, y \in G$ be arbitrary. By substituting (3.16) and (3.17) in (3.2) we obtain by an elementary computation

$$\varphi_1(x,y) = [-\eta f^e(x) + g^e(x)]f^o(y). \tag{4.35}$$

On the other hand, since $f = f^e + f^o$ and $g = g^e + g^o$ the identity (3.5) can be written

$$\begin{aligned} \psi(x,y) = {} & f^e(x-y) - f^e(x)g^e(y) - g^e(x)f^e(y) \\ & - g^e(x)f^o(y) - f^e(x)g^o(y) \\ & - f^e(y)g^o(x) - f^o(x)g^e(y) - f^o(x)g^o(y) \\ & - g^o(x)f^o(y) - h(x)h(y) + f^o(x-y). \end{aligned}$$

By using (3.17) we obtain

$$\begin{aligned}\psi(x,y) &= f^e(x-y) - f^e(x)g^e(y) - g^e(x)f^e(y) - h(x)h(y) - g^e(x)f^o(y)\\ &\quad - f^e(x)[-\gamma\, h^o(y) - \eta\, f^o(y)] - f^e(y)[-\gamma\, h^o(x) - \eta\, f^o(x)] - f^o(x)g^e(y)\\ &\quad - f^o(x)[-\gamma\, h^o(y) - \eta\, f^o(y)] - f^o(y)[-\gamma\, h^o(x) - \eta\, f^o(x)] + f^o(x-y)\\ &= f^e(x-y) - f^e(x)g^e(y) - g^e(x)f^e(y) - h(x)h(y)\\ &\quad + \gamma\, f^e(x)h^o(y) + \gamma\, f^e(y)h^o(x) + \gamma\, f^o(x)h^o(y) + \gamma\, h^o(x)f^o(y)\\ &\quad + 2\eta\, f^o(x)f^o(y) - [-\eta\, f^e(x) + g^e(x)]f^o(y)\\ &\quad - [-\eta\, f^e(y) + g^e(y)]f^o(x) + f^o(x-y),\end{aligned}$$

from which we infer, by using that $h = h^e + h^o$, and taking (3.16) and (4.35) into account, that

$$\begin{aligned}\psi(x,y) &= f^e(x-y) - f^e(x)g^e(y) - g^e(x)f^e(y) - [h^e(x) + h^o(x)][h^e(y) + h^o(y)]\\ &\quad + h^e(x)h^o(y) + h^e(y)h^o(x) + \gamma\, f^o(x)h^o(y) + \gamma\, h^o(x)f^o(y)\\ &\quad - \varphi_1(x,y) - \varphi_1(y,x) + 2\eta\, f^o(x)f^o(y) + f^o(x-y)\\ &= f^e(x-y) - f^e(x)g^e(y) - g^e(x)f^e(y) - h^e(x)h^e(y)\\ &\quad - h^o(x)h^o(y) + \gamma\, f^o(x)h^o(y) + \gamma\, h^o(x)f^o(y)\\ &\quad - \varphi_1(x,y) - \varphi_1(y,x) + 2\eta\, f^o(x)f^o(y) + f^o(x-y)\\ &= f^e(x-y) - f^e(x)g^e(y) - g^e(x)f^e(y) - \gamma^2 f^e(x)f^e(y)\\ &\quad - h^o(x)h^o(y) + \gamma\, f^o(x)h^o(y) + \gamma\, h^o(x)f^o(y)\\ &\quad - \varphi_1(x,y) - \varphi_1(y,x) + 2\eta\, f^o(x)f^o(y) + f^o(x-y).\end{aligned}$$

So that

$$\begin{aligned}&f^e(x-y) - f^e(x)[g^e(y) + \frac{1}{2}\gamma^2 f^e(y)] - [g^e(x) + \frac{1}{2}\gamma^2 f^e(x)]f^e(y)\\ &- [h^o(x) - \gamma\, f^o(x)][h^o(y) - \gamma\, f^o(y)]\\ &= \psi(x,y) + \varphi_1(x,y) + \varphi_1(y,x) - (\gamma^2 + 2\eta)f^o(x)f^o(y) - f^o(x-y)\end{aligned} \tag{4.36}$$

for all $x, y \in G$. Let

$$F_0 := f^e,\ G_0 := g^e + \frac{1}{2}\gamma^2 f^e,\ H_0 := h^o - \gamma\, f^o. \tag{4.37}$$

Since $f = f^e + f^o$, $g = g^e + g^o$ and $h = h^e + h^o$, we get by setting $\delta = -\gamma$ and $\varphi = f^o$, and taking (3.16), (3.17) and (4.37) into account, that

$$(VI)\begin{cases} f &= F_0 + \varphi,\\ g &= -\frac{1}{2}\delta^2 F_0 + G_0 + \delta\, H_0 - (\eta + \delta^2)\varphi,\\ h &= -\delta\, F_0 + H_0 - \delta\, \varphi.\end{cases}$$

If $\varphi = 0$ the result (9) of Theorem 4.1 is obviously satisfied. In the following we assume that $\varphi \neq 0$. By using (4.35), the first identity and the second one in (4.36), and replacing f^o by φ, we get, by a small computation, that

$$\varphi_1(x, y) = -[(\eta + \frac{1}{2}\delta^2)F_0(x) - G_0(x)]\varphi(y)$$

for all $x, y \in G$. Since f^o and ψ are bounded functions, we deduce, taking (3.12) and the identity above into account, that

$$(\eta + \frac{1}{2}\delta^2)F_0 - G_0 \in \mathcal{B}(G), \tag{4.38}$$

and, from (4.36) and (4.37), we derive that the function

$$(x, y) \mapsto F_0(x - y) - F_0(x)G_0(y) - G_0(x)F_0(y) - H_0(x)H_0(y)$$

is bounded. Since f and h are linearly independent modulo $\mathcal{B}(G)$, we deduce easily, by using the first and the third identities in (4.36), that H_0 and F_0 are because $f^o \in \mathcal{B}(G)$ and $h^o \notin \mathcal{B}(G)$. Moreover we have $H_0^o = H_0$ and $H_0^o \notin \mathcal{B}(G)$, hence we go back to Subcase B.2.1. As F_0 and G_0 are even functions we derive that we have the following subcases:

Subcase B.2.2.1: F_0, G_0, and H_0 are of the form (6) with the same constraints. Then

$$F_0 = \lambda^2 f_0 - \lambda^2 b,\ G_0 = \frac{1}{2}f_o + \frac{1}{2}b,\ H_0 = \lambda\, g_0,$$

where $\lambda \in \mathbb{C}\backslash\{0\}$ is a constant and $b, f_0, g_0 : G \to \mathbb{C}$ are functions satisfying the same constraints indicated in (6) of Theorem 4.1, unless to take $b(-x) = b(x)$ for all $x \in G$, then a small computation shows, by using (4.38) and the formulas of F_0 and G_0, that

$$[\frac{1}{2} - \lambda^2(\eta + \frac{1}{2}\delta^2)]f_0 \in \mathcal{B}(G).$$

As F_0 and H_0 are linearly independent modulo $\mathcal{B}(G)$ and $b \in \mathcal{B}(G)$, we get $f_0 \notin \mathcal{B}(G)$. So that

$$\frac{1}{2} - \lambda^2(\eta + \frac{1}{2}\delta^2) = 0$$

and then

$$\eta = \frac{1}{2\lambda^2} - \frac{1}{2}\delta^2.$$

By substituting this back into (VI) we obtain the result (9) of Theorem 4.1 with the constraint (i).

Subcase B.2.2.2: F_0, G_0, and H_0 are of the form (7) with the same constraints. Then we get, taking into account that $F_0(-x) = F_0(x)$ and $b(-x) = -b(x)$ for all $x \in G$, that $b = 0$. So that

$$F_0 = \frac{1}{2}a^2\, m,\; G_0 = m,\; H_0 = -ia\, m,$$

where $m : G \to \mathbb{C}$ is a nonzero bounded multiplicative function, $a : G \to \mathbb{C}$ is a nonzero additive function such that $m(-x) = m(x)$ for all $x \in G$. By using (4.38) and the formulas of F_0 and G_0 we get, by an elementary computation, that $(\eta + \frac{1}{2}\delta^2)a^2 \in \mathcal{B}(G)$. Since a is a nonzero additive function we get that $a^2 \notin \mathcal{B}(G)$. Hence $\eta = -\frac{1}{2}\delta^2$. By substituting this back into (VI) we obtain the result (9) of Theorem 4.1 with the constraint (ii).

Subcase B.2.2.3: F_0, G_0, and H_0 satisfy the functional equation in the result (8) of Theorem 4.1, i.e.,

$$F_0(x - y) = F_0(x)G_0(y) + G_0(x)F_0(y) + H_0(x)H_0(y)$$

for all $x, y \in G$. Since F_0 and G_0 are even functions and H_0, replacing y by $-y$ yields the functional equation

$$F_0(x + y) = F_0(x)G_0(y) + G_0(x)F_0(y) + (iH_0(x))(iH_0(y)).$$

From (4.38) we derive that there exist a constant $\alpha \in \mathbb{C}$ and a function $b_0 \in \mathcal{B}(G)$ such that $G_0 = \frac{\alpha}{2}F_0 + b_0$. So that the last functional equation becomes

$$F_0(x + y) = \alpha\, F_0(x)F_0(y) + F_0(x)b_0(y) + b_0(x)F_0(y) + (iH_0(x))(iH_0(y)),$$

for all $x, y \in G$. Hence, by applying a similar idea used to solve (4.33) (see Subcase B.2.1(5)) we prove that:

If $\alpha = 0$, then $F_0 = \frac{1}{2}a^2\, m$, $G_0 = m$ and $H_0 = -ia\, m$, where $m : G \to \mathbb{C}$ is a nonzero bounded multiplicative function such that $m(-x) = m(x)$ for all $x \in G$, so we go back to Subcase B.2.2.2 and obtain the result (9) of Theorem 4.1 with the constraint (ii).

If $\alpha \neq 0$, then

$$F_0 = \lambda^2 f_0 - \lambda^2 b_0,\; G_0 = \frac{1}{2}\, f_0 + \frac{1}{2}\, b \text{ and } H_0 = \lambda\, g_0,$$

where $b : G \to \mathbb{C}$ is a bounded function, $\lambda \in \mathbb{C} \setminus \{0\}$ is a constant and $f_0, g_0 : G \to \mathbb{C}$ are functions satisfying the cosine functional equation

$$f_0(x + y) = f_0(x)f_0(y) - g_0(x)g_0(y)$$

for all $x, y \in G$, such that

$$f_0(-x) = f_0(x),\; g_0(-x) = -g_0(x)$$

and $b(-x) = -b(x)$ for all $x \in G$, so we go back to Subcase B.2.2.1 and obtain the result (9) of Theorem 4.1 with the constraint (i).

Conversely if f, g and h are of the forms (1)-(9) in Theorem 4.1 we check by elementary computations that the function

$$(x, y) \mapsto f(x-y) - f(x)g(y) - g(x)f(y) - h(x)h(y)$$

is bounded. This completes the proof. □

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