Nonlinear Functional Analysis and Applications
Vol. 24, No. 3 (2019), pp. 627-637
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

A SYSTEM OF GENERALIZED NONLINEAR VARIATIONAL INCLUSIONS WITH (H, η)-MONOTONE OPERATORS

Tao Cai[1], Yan Hao[2], Minjie Fu[3] and Chahn Yong Jung[4]

[1]Department of Mathematics, Kunming University
Kunming, Yunnan 650214, China
e-mail: caitao1963@sina.com

[2]School of Mathematics and System Science, Shenyang Normal University
Shenyang, Liaoning 110034, China
e-mail: haoyan8012@163.com

[3]Department of Mathematics, Liaoning Normal University
Dalian, Liaoning 116029, China
e-mail: fuminjie1616@163.com

[4]Department of Business Administration, Gyeongsang National University
Jinju 52828, Korea
e-mail: bb5734@gnu.ac.kr

Abstract. In this paper, a new system of generalized nonlinear variational inclusions with (H, η)-monotone operators is introduced and studied in Hilbert spaces. By using the resolvent operators associated with (H, η)-monotone operators, we construct a new iterative algorithm for approximating the solution of the system of generalized nonlinear variational inclusions, prove the existence of solution of the system of generalized nonlinear variational inclusions and the convergence of the sequence generated by the algorithm. The result presented in this paper extends some known results in the literature.

[0]Received February 12, 2019. Revised June 5, 2019.
[0]2010 Mathematics Subject Classification: 47J20, 49J40.
[0]Keywords: (H, η)-monotone operator, system of generalized nonlinear variational inclusions, resolvent operator technique, iterative algorithm, convergence.
[0]Corresponding Author: C.Y. Jung(bb5734@gnu.ac.kr).

1. Introduction

It is well known that systems of variational inequalities are important generalizations of the classical variational inequality and have potential applications in mechanic, physics, optimization and control, economics and engineering sciences ([1–10]).

Utilizing the projection methods, Verma [10] investigated the existence of solutions for a system of nonlinear variational inequalities in Hilbert spaces. By means of the the resolvent operators, Nie et al. [5] discussed the approximation solvability of a system of nonlinear variational inequalities involving strongly monotone and pseudocontractive mappings. Using the resolvent operators associated with (H, η)-monotone operators, Fang et al. [2] studied a system of variational inclusions in Hilbert spaces, and obtained the existence of solutions for the system of variational inclusions. Peng [6] and Peng and Zhu [8] introduced systems of quasi-variational inequalities and generalized mixed quasi-variational inclusions with (H, η)-monotone operators, respectively, and proved the existence theorems of solutions and convergence results of iterative algorithms for the systems of quasi-variational inequalities and generalized mixed quasi-variational inclusions.

The purpose of this paper is to introduce and study a new system of generalized nonlinear variational inclusions with (H, η)-monotone operators, which includes the systems of variational inclusions in [2, 8] as special cases. By using the resolvent operator techniques associated with the (H, η)-monotone operators, we suggest an iterative algorithm for computing approximation solutions of the system of generalized nonlinear variational inclusions, prove the existence of solutions for the system of generalized nonlinear variational inclusions and discuss the convergence of the iterative sequence generated by the algorithm. The result obtained in this paper improves some results in the literature.

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space with norm and inner product denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let $CB(\mathcal{H})$ denote the families of all nonempty closed bounded subsets of $\mathcal{H}$ and $\widetilde{D}(\cdot,\cdot)$ denote the Hausdorff metric on $CB(\mathcal{H})$ defined by

$$\widetilde{D}(A_1, B_1) = \max\Big\{ \sup_{a\in A_1} d(a, B_1), \sup_{b\in B_1} d(A_1, b)\Big\}, \quad \forall A_1, B_1 \in CB(\mathcal{H}),$$

where $d(a, B_1) = d(B_1, a) = \inf_{b\in B_1} \|a - b\|$.

Definition 2.1. ([4]) Let $\eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$, $H : \mathcal{H} \to \mathcal{H}$ be two mappings and $M : \mathcal{H} \to 2^{\mathcal{H}}$ be a set-valued mapping. M is said to be

(1) *η-monotone* if

$$\langle x - y, \eta(u, v)\rangle \geq 0, \quad \forall u, v \in \mathcal{H},\, x \in Mu,\, y \in Mv,$$

(2) *(H, η)-monotone* if M is η-monotone and $(H + \lambda M)(\mathcal{H}) = \mathcal{H}$ for all $\lambda > 0$.

Definition 2.2. Let H, $g : \mathcal{H} \to \mathcal{H}$, $\eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be three mappings. g is said to be

(1) *η-monotone* if

$$\langle gu - gv, \eta(u, v)\rangle \geq 0, \quad \forall u, v \in \mathcal{H};$$

(2) *strictly η-monotone* if g is η-monotone and

$$\langle gu - gv, \eta(u, v)\rangle = 0 \iff u = v;$$

(3) *strongly monotone* if there exists a constant $r > 0$ such that

$$\langle gu - gv, u - v\rangle \geq r\|u - v\|^2, \quad \forall u, v \in \mathcal{H};$$

(4) *strongly monotone with respect to H* if there exists a constant $\gamma > 0$ such that

$$\langle gu - gv, Hu - Hv\rangle \geq r\|u - v\|^2, \quad \forall u, v \in \mathcal{H};$$

(5) *strongly η-monotone* if there exists a constant $r > 0$ such that

$$\langle gu - gv, \eta(u, v)\rangle \geq r\|u - v\|^2, \quad \forall u, v \in \mathcal{H};$$

(6) *Lipschitz continuous* if there exists a constant $s > 0$ such that

$$\|gu - gv\| \leq \|u - v\|, \quad \forall u, v \in \mathcal{H}.$$

Definition 2.3. Let $\eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a mapping. η is said to be

(1) *monotone* if

$$\langle \eta(u, v), u - v\rangle \geq 0, \quad \forall u, v \in \mathcal{H};$$

(2) *Lipschitz continuous* if there exists a constant $\tau > 0$ such that

$$\|\eta(u, v)\| \leq \tau\|u - v\|, \quad \forall u, v \in \mathcal{H}.$$

Definition 2.4. Let $M : \mathcal{H} \to CB(\mathcal{H})$ and $N : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be mappings.

(1) M is said to be *$\widetilde{D}$-Lipschitz continuous* if there exists a constant $\xi > 0$ such that

$$\widetilde{D}(Mu, Mv) \leq \xi\|u - v\|, \quad \forall u, v \in \mathcal{H};$$

(2) N is said to be *Lipschitz continuous* in the first argument if there exists a constant $\xi > 0$ such that

$$\|N(u,x) - N(v,x)\| \leq \xi\|u - v\|, \quad \forall u, v, x \in \mathcal{H};$$

(3) N is said to be *mixed Lipschitz continuous* if there exist two constants $\xi > 0$ and $\zeta > 0$ such that

$$\|N(u,v) - N(x,y)\| \leq \xi\|u - x\| + \zeta\|v - y\|, \quad \forall u, v, x, y \in \mathcal{H}.$$

Similarly we can define the Lipschitz continuity of N in the second argument.

Definition 2.5. Let $F : \mathcal{H} \to \mathcal{H}$, and $N : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be mappings. N is said to be *F-strongly monotone in the first argument* if there exists a constant $\beta > 0$ such that

$$\langle N(u,x) - N(v,x), Fu - Fv\rangle \geq \beta\|u - v\|^2, \quad \forall u, v, x \in \mathcal{H}.$$

Definition 2.6. ([4]) Let $\eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a mapping, $H : \mathcal{H} \to \mathcal{H}$ be a strictly η-monotone mapping and $M : \mathcal{H} \to 2^{\mathcal{H}}$ be an (H, η)-monotone mapping. Then the resolvent operator $R^{H,\eta}_{M,\rho} : \mathcal{H} \to \mathcal{H}$ is defined by

$$R^{H,\eta}_{M,\rho}(x) = (H + \rho M)^{-1}(x), \quad \forall x \in \mathcal{H}.$$

Lemma 2.7. ([4]) *Let $\eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a Lipschitz continuous mapping with constant $\tau > 0$, $H : \mathcal{H} \to \mathcal{H}$ be a strongly η-monotone mapping with constant $\gamma > 0$ and $M : \mathcal{H} \to 2^{\mathcal{H}}$ be an (H, η)-monotone mapping. Then the resolvent operator $R^{H,\eta}_{M,\rho} : \mathcal{H} \to \mathcal{H}$ is Lipschitz continuous with constant $\frac{\tau}{\gamma}$, that is,*

$$\left\|R^{H,\eta}_{M,\rho}(x) - R^{H,\eta}_{M,\rho}(y)\right\| \leq \frac{\tau}{\gamma}\|x - y\|, \quad \forall x, y \in \mathcal{H}.$$

3. A system of generalized mixed quasi-variational inclusions and iterative algorithm

In this section, we will introduce a new system of generalized nonlinear variational inclusions with (H, η)-monotone mappings and construct a new iterative algorithm for solving the system of generalized nonlinear variational inclusions in Hilbert spaces. In what follows, unless other specified, we always assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two real Hilbert spaces, $F, P : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1$, $G, Q : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_2$, $f_i, g_i, r_i, H_i : \mathcal{H}_i \to \mathcal{H}_i$, $\eta_i : \mathcal{H}_i \times \mathcal{H}_i \to \mathcal{H}_i$ are mappings for $i \in \{1, 2\}$ and $A_1, C_1 : \mathcal{H}_1 \to CB(\mathcal{H}_1)$, $B_1, D_1 : \mathcal{H}_2 \to CB(\mathcal{H}_2)$ are four set-valued mappings. Let $M : \mathcal{H}_1 \times \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $N : \mathcal{H}_2 \times \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ be two mappings, $M(\cdot, x)$ be an (H_1, η_1)-monotone mapping, $N(\cdot, y)$ be an

(H_2, η_2)-monotone mapping for all $x \in \mathcal{H}_1$, $y \in \mathcal{H}_2$. We consider the following problem of finding (x, y, u, v, w, z) with $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$, $u \in A_1 x$, $v \in B_1 y$, $w \in C_1 x$, $z \in D_1 y$ such that

$$\begin{cases} 0 \in F(x,y) + P(u,v) + M((f_1 - g_1)x, r_1 x), \\ 0 \in G(x,y) + Q(w,z) + N((f_2 - g_2)y, r_2 y). \end{cases} \tag{3.1}$$

The problem (3.1) is called *a system of generalized nonlinear variational inclusions.*

Special cases

(i) If $M((f_1 - g_1)x, r_1 x) = M(g_1 x)$ and $N((f_2 - g_2)y, r_2 y) = N(g_2 y)$ for any $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$, then the problem (3.1) reduces to finding (x, y, u, v, w, z) with $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$, $u \in A_1 x$, $v \in B_1 y$, $w \in C_1 x$, $z \in D_1 y$ such that

$$\begin{cases} 0 \in F(x,y) + P(u,v) + M(g_1 x), \\ 0 \in G(x,y) + Q(w,z) + N(g_2 y), \end{cases} \tag{3.2}$$

which was introduced and studied by Peng and Zhu [8].

(ii) If $P = Q = 0$, $M((f_1 - g_1)x, r_1 x) = M(x)$ and $N((f_2 - g_2)y, r_2 y) = N(y)$ for any $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$, then the problem (3.1) is equivalent to finding $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$ such that

$$\begin{cases} 0 \in F(x,y) + M(x), \\ 0 \in G(x,y) + N(y), \end{cases} \tag{3.3}$$

which was introduced by Fang et al. [2].

Lemma 3.1. *Let ρ_i be a positive constant, $\eta_i : \mathcal{H}_i \times \mathcal{H}_i \to \mathcal{H}_i$ be a mapping and $H_i : \mathcal{H}_i \to \mathcal{H}_i$ be a strictly η_i-monotone mapping for each $i \in \{1, 2\}$. Let $M : \mathcal{H}_1 \times \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ be (H_1, η_1)-monotone and $N : \mathcal{H}_2 \times \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ be (H_2, η_2)-monotone. Then (x, y, u, v, w, z) with $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$, $u \in A_1 x$, $v \in B_1 y$, $w \in C_1 x$, $z \in D_1 y$ is a solution of the problem* (3.1) *if and only if*

$$\begin{cases} (f_1 - g_1)x = R^{H_1,\eta_1}_{M(\cdot, r_1 x),\rho_1}(H_1(f_1 - g_1)x - \rho_1 F(x,y) - \rho_1 P(u,v)), \\ (f_2 - g_2)y = R^{H_2,\eta_2}_{M(\cdot, r_2 y),\rho_2}(H_2(f_2 - g_2)y - \rho_2 G(x,y) - \rho_2 Q(w,z)), \end{cases} \tag{3.4}$$

where

$$R^{H_1,\eta_1}_{M(\cdot, r_1 x),\rho_1} = (H_1 + \rho_1 M(\cdot, r_1 x))^{-1}, \quad R^{H_2,\eta_2}_{M(\cdot, r_2 y),\rho_2} = (H_2 + \rho_2 N(\cdot, r_2 y))^{-1}.$$

Proof. The fact directly follows from Definition 2.6. □

Based on Lemma 3.1 and Nadler's lemma, we suggest the following iterative algorithm for the problem (3.1).

Algorithm 3.2. *For any given $x_0 \in \mathcal{H}_1$ and $y_0 \in \mathcal{H}_2$, compute the sequences $\{x_n\}_{n\geq 0}$, $\{y_n\}_{n\geq 0}$, $\{u_n\}_{n\geq 0}$, $\{v_n\}_{n\geq 0}$, $\{w_n\}_{n\geq 0}$ and $\{z_n\}_{n\geq 0}$ by iterative schemes*

$$\begin{aligned} x_{n+1} &= x_n - (f_1 - g_1)x_n + R^{H_1,\eta_1}_{M(\cdot,r_1x_n),\rho_1}(H_1(f_1-g_1)x_n \\ &\quad - \rho_1 F(x_n,y_n) - \rho_1 P(u_n,v_n)), \\ y_{n+1} &= y_n - (f_2 - g_2)y_n + R^{H_2,\eta_2}_{N(\cdot,r_2y_n),\rho_2}(H_2(f_2-g_2)y_n \\ &\quad - \rho_2 G(x_n,y_n) - \rho_2 Q(w_n,z_n)), \end{aligned} \tag{3.5}$$

$$\begin{aligned} u_n \in A_1x_n, &\quad \|u_{n+1}-u_n\| \leq \left(1+\frac{1}{n+1}\right)\widetilde{D}(A_1x_{n+1},A_1x_n), \\ v_n \in B_1y_n, &\quad \|v_{n+1}-v_n\| \leq \left(1+\frac{1}{n+1}\right)\widetilde{D}(B_1y_{n+1},B_1y_n), \\ w_n \in C_1x_n, &\quad \|w_{n+1}-w_n\| \leq \left(1+\frac{1}{n+1}\right)\widetilde{D}(C_1x_{n+1},C_1x_n), \\ z_n \in D_1y_n, &\quad \|z_{n+1}-z_n\| \leq \left(1+\frac{1}{n+1}\right)\widetilde{D}(D_1y_{n+1},D_1y_n) \end{aligned} \tag{3.6}$$

for all $n \geq 0$, where ρ_1 and ρ_2 are positive constants.

4. Existence of solutions of the problem (3.1) and convergence of Algorithm 3.2

In this section, we will prove the existence of solutions for the problem (3.1) and the convergence of the iterative sequences generated by Algorithm 3.2.

Theorem 4.1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces, $A_1, C_1 : \mathcal{H}_1 \to CB(\mathcal{H}_1)$, $B_1, D_1 : \mathcal{H}_2 \to CB(\mathcal{H}_2)$ be $\widetilde{D}$-Lipschitz continuous with constants $l_{A_1}, l_{B_1}, l_{C_1}, l_{D_1}$, respectively. Let $\eta_i : \mathcal{H}_i \times \mathcal{H}_i \to \mathcal{H}_i$ be Lipschitz continuous with constant τ_i, $H_i : \mathcal{H}_i \to \mathcal{H}_i$ be strongly η_i-monotone and Lipschitz continuous with constants γ_i and h_i, respectively, $r_i : \mathcal{H}_i \to \mathcal{H}_i$ be Lipschitz continuous with constant δ_i, $f_i, g_i : \mathcal{H}_i \to \mathcal{H}_i$ be mappings such that $f_i - g_i$ be Lipschitz continuous and strongly monotone with constants l_i and δ_i, respectively, for $i \in \{1, 2\}$. Let $F : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1$ be $H_1(f_1 - g_1)$-strongly monotone in the first argument with constant μ_1, Lipschitz continuous in the first and second arguments with constants l_{F_1} and l_{F_2}, respectively. Let $P : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1$ be mixed Lipschitz continuous with constants l_{P_1} and l_{P_2}, $G : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_2$ be $H_2(f_2 - g_2)$-strongly monotone in the second argument with constant μ_2, Lipschitz continuous in the first and second arguments with constants l_{G_1} and l_{G_2}, respectively, $Q : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_2$ be mixed Lipschitz continuous with constants l_{Q_1} and l_{Q_2}. Let $M : \mathcal{H}_1 \times \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $N : \mathcal{H}_2 \times \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ satisfy that $M(\cdot, x)$ is an (H_1, η_1)-monotone mapping for each $x \in \mathcal{H}_1$, $N(\cdot, y)$ is an*

(H_2,η_2)-monotone mapping for each $y \in \mathcal{H}_2$ and there exist positive constants ξ_1 and ξ_2 satisfying

$$\big\|R^{H_1,\eta_1}_{M(\cdot,r_1x),\rho_1}(z) - R^{H_1,\eta_1}_{M(\cdot,r_1y),\rho_1}(z)\big\| \le \xi_1\|r_1x - r_1y\|, \quad \forall x,y,z \in \mathcal{H}_1, \tag{4.1}$$

$$\big\|R^{H_2,\eta_2}_{N(\cdot,r_2x),\rho_2}(z) - R^{H_2,\eta_2}_{N(\cdot,r_2y),\rho_2}(z)\big\| \le \xi_2\|r_2x - r_2y\|, \quad \forall x,y,z \in \mathcal{H}_2. \tag{4.2}$$

Let

$$\begin{aligned}
A &= l_{F_1}^2 - (l_{P_1}l_{A_1} + l_{F_2} + l_{P_2}l_{B_1})^2;\\
B &= \mu_1 - \frac{\gamma_1}{\tau_1}\Big(1 - \sqrt{1 - 2\sigma_1 + l_1^2} - \xi_1\delta_1\Big)(l_{P_1}l_{A_1} + l_{F_2} + l_{P_2}l_{B_1});\\
C &= h_1^2l_1^2 - \frac{\gamma_1^2}{\tau_1^2}\Big(1 - \sqrt{1 - 2\sigma_1 + l_1^2} - \xi_1\delta_1\Big)^2;\\
A' &= l_{G_2}^2 - (l_{Q_1}l_{C_1} + l_{G_1} + l_{Q_2}l_{D_1})^2;\\
B' &= \mu_2 - \frac{\gamma_2}{\tau_2}\Big(1 - \sqrt{1 - 2\sigma_2 + l_2^2} - \xi_2\delta_2\Big)(l_{Q_1}l_{C_1} + l_{G_1} + l_{Q_2}l_{D_1});\\
C' &= h_2^2l_2^2 - \frac{\gamma_2^2}{\tau_2^2}\Big(1 - \sqrt{1 - 2\sigma_2 + l_2^2} - \xi_2\delta_2\Big)^2.
\end{aligned}$$

If there exist constants ρ_1 and ρ_2 satisfying

$$\begin{aligned}
0 < \rho_1 < \frac{\gamma_1\big(1 - \sqrt{1 - 2\sigma_1 + l_1^2} - \xi_1\delta_1\big)}{\tau_1(l_{P_1}l_{A_1} + l_{F_2} + l_{P_2}l_{B_1})},\\
0 < \rho_2 < \frac{\gamma_2\big(1 - \sqrt{1 - 2\sigma_2 + l_2^2} - \xi_2\delta_2\big)}{\tau_2(l_{Q_1}l_{C_1} + l_{G_1} + l_{Q_2}l_{D_1})}
\end{aligned} \tag{4.3}$$

and one of

$$A > 0, \quad B^2 > AC, \quad \left|\rho_1 - \frac{B}{A}\right| < \frac{\sqrt{B^2 - AC}}{A}; \tag{4.4}$$

$$A = 0, \quad 2B\rho_1 > C; \tag{4.5}$$

$$A < 0, \quad B^2 > AC, \quad \left|\rho_1 - \frac{B}{A}\right| > -\frac{\sqrt{B^2 - AC}}{A} \tag{4.6}$$

and one of

$$A' > 0, \quad {B'}^2 > A'C', \quad \left|\rho_2 - \frac{B'}{A'}\right| < \frac{\sqrt{{B'}^2 - A'C'}}{A'}; \tag{4.7}$$

$$A' = 0, \quad 2B'\rho_2 > C'; \tag{4.8}$$

$$A' < 0, \quad {B'}^2 > A'C', \quad \left|\rho_2 - \frac{B'}{A'}\right| > -\frac{\sqrt{{B'}^2 - A'C'}}{A'}, \tag{4.9}$$

then the problem (3.1) *admits a solution* (x, y, u, v, w, z) *with* $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$, $u \in A_1x$, $v \in B_1y$, $w \in C_1x$, $z \in D_1y$ *and the sequences* $\{x_n\}_{n\geq 0}$, $\{y_n\}_{n\geq 0}$, $\{u_n\}_{n\geq 0}$, $\{v_n\}_{n\geq 0}$, $\{w_n\}_{n\geq 0}$, $\{z_n\}_{n\geq 0}$ *generated by Algorithm* 3.2 *converge to* x, y, u, v, w, z, *respectively.*

Proof. Let $a_n = H_1(f_1 - g_1)x_n - \rho_1 F(x_n, y_n) - \rho_1 P(u_n, v_n)$. By (3.5) and Lemma 2.7, we have

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\leq \|x_n - x_{n-1} - ((f_1 - g_1)x_n - (f_1 - g_1)x_{n-1})\| \\
&\quad + \left\|R^{H_1,\eta_1}_{M(\cdot,r_1x_n),\rho_1}(a_n) - R^{H_1,\eta_1}_{M(\cdot,r_1x_{n-1}),\rho_1}(a_n)\right\| \\
&\quad + \left\|R^{H_1,\eta_1}_{M(\cdot,r_1x_{n-1}),\rho_1}(a_n) - R^{H_1,\eta_1}_{M(\cdot,r_1x_{n-1}),\rho_1}(a_{n-1})\right\|, \quad \forall n \geq 1.
\end{aligned} \tag{4.10}$$

Since $f_1 - g_1$ is strongly monotone and Lipschitz continuous with constants σ_1 and l_1, respectively, we have

$$\begin{aligned}
&\|x_n - x_{n-1} - ((f_1 - g_1)x_n - (f_1 - g_1)x_{n-1})\|^2 \\
&= \|x_n - x_{n-1}\|^2 - 2\langle x_n - x_{n-1}, (f_1 - g_1)x_n - (f_1 - g_1)x_{n-1}\rangle \\
&\quad + \|(f_1 - g_1)x_n - (f_1 - g_1)x_{n-1}\|^2 \\
&\leq (1 - 2\sigma_1 + l_1^2)\|x_n - x_{n-1}\|^2, \quad \forall n \geq 1.
\end{aligned} \tag{4.11}$$

It follows from (4.1) and the Lipschitz continuity of r_1 that

$$\begin{aligned}
&\left\|R^{H_1,\eta_1}_{M(\cdot,r_1x_n),\rho_1}(a_n) - R^{H_1,\eta_1}_{M(\cdot,r_1x_{n-1}),\rho_1}(a_n)\right\| \\
&\leq \xi_1\|r_1x_n - r_1x_{n-1}\| \leq \xi_1\delta_1\|x_n - x_{n-1}\|, \quad \forall n \geq 1.
\end{aligned} \tag{4.12}$$

By Lemma 2.7, we deduce that

$$\begin{aligned}
&\left\|R^{H_1,\eta_1}_{M(\cdot,r_1x_{n-1}),\rho_1}(a_n) - R^{H_1,\eta_1}_{M(\cdot,r_1x_{n-1}),\rho_1}(a_{n-1})\right\| \\
&\leq \frac{\tau_1}{\gamma_1}\|a_n - a_{n-1}\|, \quad \forall n \geq 1.
\end{aligned} \tag{4.13}$$

Since F is Lipschitz continuous in the second argument and P is mixed Lipschitz continuous, it follows from (3.6) that

$$\begin{aligned}
&\|a_n - a_{n-1}\| \\
&\leq \|H_1(f_1 - g_1)x_n - H_1(f_1 - g_1)x_{n-1} - \rho_1(F(x_n, y_n) - F(x_{n-1}, y_n))\| \\
&\quad + \rho_1\|F(x_{n-1}, y_n) - F(x_{n-1}, y_{n-1})\| \\
&\quad + \rho_1\|P(u_n, v_n) - P(u_{n-1}, v_{n-1})\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \|H_1(f_1-g_1)x_n - H_1(f_1-g_1)x_{n-1} - \rho_1(F(x_n,y_n)-F(x_{n-1},y_n))\| \\
&\quad + \rho_1 l_{F_2}\|y_n-y_{n-1}\| + \rho_1 l_{P_1}\left(1+\frac{1}{n}\right)\widetilde{D}(A_1x_n, A_1x_{n-1}) \\
&\quad + \rho_1 l_{P_2}\left(1+\frac{1}{n}\right)\widetilde{D}(B_1y_n, B_1y_{n-1}), \quad \forall n\geq 1.
\end{aligned}
\tag{4.14}
$$

Note that H_1 and $f_1 - g_1$ are Lipschitz continuous with constants h_1 and l_1, respectively, A_1 and B_1 are $\widetilde{D}$-Lipschitz continuous with constants l_{A_1} and l_{B_1}, and F is $H_1(f_1-g_1)$-strongly monotone in the first argument. It follows that

$$
\begin{aligned}
&\|H_1(f_1-g_1)x_n - H_1(f_1-g_1)x_{n-1} - \rho_1(F(x_n,y_n)-F(x_{n-1},y_n))\|^2 \\
&= \|H_1(f_1-g_1)x_n - H_1(f_1-g_1)x_{n-1}\|^2 \\
&\quad - 2\rho_1\langle H_1(f_1-g_1)x_n - H_1(f_1-g_1)x_{n-1}, F(x_n,y_n)-F(x_{n-1},y_n)\rangle \\
&\quad + \rho_1^2\|F(x_n,y_n)-F(x_{n-1},y_n)\|^2 \\
&\leq (h_1^2l_1^2 - 2\rho_1\mu_1 + \rho_1^2 l_{F_1}^2)\|x_n-x_{n-1}\|^2, \quad \forall n\geq 1,
\end{aligned}
\tag{4.15}
$$

By (4.10)-(4.15), we know that

$$
\|x_{n+1}-x_n\| \leq b_n \max\{\|x_n-x_{n-1}\|, \|y_n-y_{n-1}\|\}, \quad \forall n\geq 1, \tag{4.16}
$$

where

$$
\begin{aligned}
b_n &= \sqrt{1-2\sigma_1+l_1^2} + \xi_1\delta_1 + \frac{\tau_1}{\gamma_1}\Bigg[\sqrt{h_1^2l_1^2 - 2\rho_1\mu_1 + \rho_1^2 l_{F_1}^2} \\
&\qquad + \rho_1\left(l_{P_1}l_{A_1}\left(1+\frac{1}{n}\right) + l_{F_2} + l_{P_2}l_{B_1}\left(1+\frac{1}{n}\right)\right)\Bigg] \\
\to b &= \sqrt{1-2\sigma_1+l_1^2} + \xi_1\delta_1 + \frac{\tau_1}{\gamma_1}\Bigg[\sqrt{h_1^2l_1^2 - 2\rho_1\mu_1 + \rho_1^2 l_{F_1}^2} \\
&\qquad + \rho_1\big(l_{P_1}l_{A_1} + l_{F_2} + l_{P_2}l_{B_1}\big)\Bigg] \quad \text{as } n\to\infty.
\end{aligned}
\tag{4.17}
$$

Similarly, we have

$$
\|y_{n+1}-y_n\| \leq c_n \max\{\|x_n-x_{n-1}\|, \|y_n-y_{n-1}\|\}, \quad \forall n\geq 1, \tag{4.18}
$$

where

$$
\begin{aligned}
c_n &= \sqrt{1-2\sigma_2+l_2^2} + \xi_2\delta_2 + \frac{\tau_2}{\gamma_2}\Bigg[\sqrt{h_2^2l_2^2 - 2\rho_2\mu_2 + \rho_2^2 l_{G_2}^2} \\
&\qquad + \rho_2\left(l_{Q_2}l_{D_1}\left(1+\frac{1}{n}\right) + l_{G_1} + l_{Q_1}l_{C_1}\left(1+\frac{1}{n}\right)\right)\Bigg]
\end{aligned}
$$

$$\begin{aligned} \to c = \sqrt{1-2\sigma_2+l_2^2} + \xi_2\delta_2 + \frac{\tau_2}{\gamma_2}\Big[\sqrt{h_2^2 l_2^2 - 2\rho_2\mu_2 + \rho_2^2 l_{G_2}^2} \\ + \rho_2\big(l_{Q_1}l_{C_1} + l_{G_1} + l_{Q_2}l_{D_1}\big)\Big] \quad \text{as } n\to\infty. \end{aligned} \tag{4.19}$$

Put $\theta = \max\{b, c\}$ and $\theta_n = \max\{b_n, c_n\}$ for any $n \geq 0$. Obviously, (4.17) and (4.19) ensure that $\lim_{n\to\infty}\theta_n = \theta$. It follows from (4.3) and one of (4.4)-(4.6) and one of (4.7)-(4.9) that $0 < \theta < 1$. Put $\omega = \frac{1+\theta}{2}$. In view of (4.16) and (4.18) we infer that there exists $n_0 \geq 1$ satisfying

$$\begin{aligned} &\max\{\|x_{n+1}-x_n\|, \|y_{n+1}-y_n\|\} \\ &\leq \omega\max\{\|x_n - x_{n-1}\|, \|y_n - y_{n-1}\|\}, \quad \forall n \geq n_0, \end{aligned} \tag{4.20}$$

which implies that $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ are Cauchy sequences. In light of the Lipschitz continuity of A_1 and (3.6), we know that

$$\|u_{n+1}-u_n\| \leq \left(1+\frac{1}{n+1}\right) l_{A_1}\|x_{n+1}-x_n\|,$$

which together with (4.20) gives that $\{u_n\}_{n\geq 0}$ is a Cauchy sequence. Similarly, we conclude that $\{v_n\}_{n\geq 0}$, $\{w_n\}_{n\geq 0}$ and $\{z_n\}_{n\geq 0}$ are Cauchy sequences. Therefore, there exist $x, u, w \in \mathcal{H}_1$ and $y, v, z \in \mathcal{H}_2$ such that $x_n \to x$, $y_n \to y$, $u_n \to u$, $v_n \to v$, $w_n \to w$, $z_n \to z$ as $n \to \infty$.

Notice that

$$\begin{aligned} d(u, A_1x) &\leq \|u-u_n\| + d(u_n, A_1x) \\ &\leq \|u-u_n\| + \widetilde{D}(A_1x, A_1x_n) \\ &\leq \|u-u_n\| + l_{A_1}\|x_n - x\| \\ &\to 0 \quad \text{as } n\to\infty. \end{aligned}$$

Since A_1x is closed, we get that $u \in A_1x$. Similarly, we have $v \in B_1y$, $w \in C_1x$, $z \in D_1y$. By Algorithm 3.2 and the Lipschitz continuity of $f_1 - g_1$, $f_2 - g_2$, H_1, H_2, F, G, P, Q, $R^{H_1,\eta_1}_{M(\cdot,r_1x),\rho_1}$, $R^{H_2,\eta_2}_{M(\cdot,r_2y),\rho_2}$, we conclude that

$$(f_1-g_1)x = R^{H_1,\eta_1}_{M(\cdot,r_1x),\rho_1}(H_1(f_1-g_1)x - \rho_1F(x,y) - \rho_1P(u,v))$$

and

$$(f_2-g_2)y = R^{H_2,\eta_2}_{N(\cdot,r_2y),\rho_2}(H_2(f_2-g_2)y - \rho_2G(x,y) - \rho_2Q(w,z)).$$

It follows from Lemma 3.1 that (x, y, u, v, w, z) is a solution of the problem (3.1). This completes the proof. □

Remark 4.2. Theorem 4.1 is a generalization of Theorem 4.1 in [8].

References

[1] Y.J. Cho, Y.P. Pang and N.J. Huang, *Algorithms for systems of nonlinear variational inequalities*, J. Korean Math. Soc., **41** (2004), 489–499.

[2] Y.P. Fang, N.J. Huang and H.B. Thompson, *A new system of variational inclusions with* (H,η)*-monotone operators in Hilbert spaces*, Comput. Math. Appl., **49** (2005), 365–374.

[3] J.K. Kim and D.S. Kim, *A new system of generalized nonlinear mixed quasi-variational inequalities in Hilbert spaces*, J. Convex Anal., **11** (2004), 235–243.

[4] H.Y. Lan, J.H. Kim and Y.J. Cho, *On a new system of generalized nonlinear A-monotone mulitivalued variational inclusions*, J. Math. Anal. Appl., **327** (2007), 481–493.

[5] H. Nie, Z. Liu, K.H. Kim and S.M. Kang, *A new system of nonlinear variational inequalities involving strongly monotone and pseudocontrative mappings*, Adv. Nonlinear var. Inequal., **6** (2003), 91–99.

[6] J.W. Peng, *System of generalized set-valued quasi-variational-like inequalities*, Bull. Austral. Math. Soc., **68** (2003), 501–515.

[7] J.W. Peng and D.L. Zhu, *Existence of solutions and convergence of iterative algorithms for a system of generalized nonlinear mixed quasi-variational inclusions*, Comput. Math. Appl., **53** (2007), 693–705.

[8] J.W. Peng and D.L. Zhu, *A new system of generalized mixed quasi-variational inclusions with* (H,η)*-monotone operators*, J. Math. Anal. Appl., **327** (2007), 175–187.

[9] R.U. Verma, *Iterative algorithms and a new system of nonlinear quasivaritional inequalities*, Adv. Nonlinear Var. Inequal., **4** (2001), 117–124.

[10] R.U. Verma, *Projection methods, algorithms, and a new system of nonlinear variational inequalities*, Comput. Math. Appl., **41** (2001), 1025–1031.