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ON GENERALIZED HARMONIC VECTOR VARIATIONAL INEQUALITIES USING HC_*-CONDITION

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Abstract. In this paper some results on HC_*-condition are established in the harmonic invex set and are used to establish the existence theorem of the solution of the generalized harmonic variational inequalities and its dual problem using generalized harmonically monotone property of the operator.

1. Introduction

In 2014, Insan [5] has developed the structure of harmonically convex (HC) set and HC function. Using HC functions, he developed the Hermite-Hadamard type inequalities. Later Noor and Noor [7] have studied the harmonic variational inequality problems (HVIP) in the HC set related with a differentiable HC (DHC) function. Recently Mishra et al. have developed the harmonically

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invex (HI) set and have studied the harmonic variational inequality problems (HVIP) in it.

In section 2, the componentwise multiplication and division operations are defined in a separable reflexive Banach space given with the Schauder basis with the help of an operator S. In section 3, the generalized HVIP (GHVIP) and the generalized dual HVIP (GDHVIP) are defined in ordered TVS with the help of operator S. In section 4, the concept of HC_*-condition is redefined in separable reflexive Banach space. Some results of the bi-function η are established in the HI set in the presence of HC_*-condition and the existence of the solution of the problems GHVIP and GDHVIP are also discussed in the HI set in the presence of HC_*-condition.

2. Component-wise operations of the elements of separable reflexive Banach space

Suppose X is a topological vector space (TVS) in a separable reflexive Banach space $\mathbb{B}$ with the Schauder basis $\mathcal{B}$ given by

$$\mathcal{B} = \Big\{ e_i : e_i = (0, 0, \cdots, 0, 1, 0, 0, \dots), \text{where } i^{\text{th}} \text{ element is } \ 1 \Big\}$$

where the basis vectors $e_i, e_j \in \mathcal{B}$ satisfies the Kronecker's delta property for general product, i.e.,

$$e_i e_j = \delta_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

Let $\mathbb{K}$ be the scalar field, i.e., either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. A binary operation $\otimes : \mathcal{B} \times \mathcal{B} \to X$ is called a **component-wise vector product** via Kronecker's delta is defined in X by the rule

$$\begin{aligned} e_i \otimes e_j &= \begin{cases} e_i \delta_{ij}, & \text{if } i = j; \\ 0, & \text{if } i \neq j \end{cases} \\ a \otimes b &= ab \text{ for all } a, b \in \mathbb{K}. \end{aligned}$$

Again an operator S is introduced to find the **component-wise vector product** $\boxtimes$ of two elements of X as

$$x \boxtimes y = S(x \otimes y) = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} x_i y_j e_i \otimes e_j = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} x_i y_j e_i \delta_{ij} = \sum_{i=1}^{\infty} x_i y_i e_i.$$

For the nonzero elements set $X_{\circledcirc} = X \setminus \{0\} = \{x \in X : x_i \neq 0, i = 1, 2, \cdots\}$, the reciprocals of the elements set $X_{\circledcirc}$ is

$$[X_{\circledcirc}]^{-1} = \left\{ x^{-1} \in X_{\circledcirc} : x^{-1} = \sum_{i=1}^{\infty} x_i^{-1} e_i \right\}.$$

For simplicity the index notations S_{ij} and S_i^j are defined as follows

$$S_{ij} = \sum_{j=1}^{\infty} e_i \otimes e_j = \begin{cases} \sum\limits_{i=1}^{\infty} e_i, & \text{if } i = j; \\ 0, & \text{if } i \neq j \end{cases}$$

and

$$S_i^j = \sum_{i=1}^{\infty} e_i e_j = \sum_{i=1}^{\infty} \delta_{ij}.$$

The component-wise binary operations in the topological vector space X modeled in the reflexive separable Banach space $\mathbb{B}$ over a scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ are $\oplus$ and $\ominus$ which are same as tensor addition and tensor subtraction respectively. Other component-wise binary operations defined on X over $\mathbb{K}$ are defined as follows:

Definition 2.1. For $x = \sum\limits_{i=1}^{\infty} x_i e_i$ and $y = \sum\limits_{i=1}^{\infty} y_i e_i$ in X, and $\alpha \in \mathbb{K}$ the scalar field, we have

(i) the component-wise product $\boxtimes$ is given by $x \boxtimes y = \sum\limits_{i=1}^{\infty} x_i y_i \; e_i = y \boxtimes x$,

(ii) component-wise division $\boxdot$ is defined by $x \boxdot y = x \boxtimes y^{-1} = \sum\limits_{i=1}^{\infty} x_i y_i^{-1} \; e_i$,

(iii) component-wise multiplicative identity element in X is
$1_X = (1, 1, \cdots) = \sum\limits_{i=1}^{\infty} e_i$, because $x \boxtimes 1_X = \sum\limits_{i=1}^{\infty} x_i \; e_i = x = 1_X \boxtimes x$,

(iv) component-wise multiplicative inverse element of x in X is x^{-1} because $x \boxdot x = x \boxtimes x^{-1} = \sum\limits_{i=1}^{\infty} x_i x_i^{-1} \; e_i = \sum\limits_{i=1}^{\infty} e_i = (1, 1, \cdots) = 1_X$.

By the algebraic structure of basis set $(\mathcal{B}, S) = \{e_i\}_{i=1}^{\infty}$ defined by the operator S, the structures of the points in $X_{\circledcirc} = X \setminus \{0\}$ the strictly non-zero point set of X, X^{-1} the reciprocal space of X, component-wise addition of spaces $X \oplus X$, component-wise subtraction of spaces $X \ominus X$, component-wise multiplication of spaces $X \boxtimes X$ and component-wise division of spaces $X \boxdot X_{\circledcirc}$ are defined as follows

(1) $X \oplus X = \left\{ z \in X : z = x \oplus y = \sum\limits_{i=1}^{\infty} (x_i + y_i) e_i \right\}$,

(2) $X \ominus X = \left\{ z \in X : z = x \ominus y = \sum\limits_{i=1}^{\infty} (x_i - y_i) e_i \right\}$,

(3) $X \boxtimes X = \left\{ z \in X : z = x \boxtimes y = \sum\limits_{i=1}^{\infty} x_i y_i e_i \right\}$,

(4) $X \boxdot X_{\circledcirc} = \left\{ z \in X : z = x \boxdot y = \sum\limits_{i=1}^{\infty} x_i y_i^{-1} e_i \right\}$.

3. Modeling of Generalized HVIP

Assume that $X = (X, \tau)$ is a TVS with topology τ in a separable reflexive (real or complex) Banach space $\mathbb{B}$ with the Schauder basis $\mathcal{B} \subset \tau$, $K \subset X$ and $Y = (Y, P)$ is an ordered TVS (OTVS) equipped with closed convex pointed cone P with nonempty interior, i.e., *int* $P \neq \varnothing$. Here $X = \prod_{i=1}^{\infty} X_i$ is a product space in the Banach space $\mathbb{B}$, $K = \prod_{i=1}^{\infty} K_i$ and $Y = \prod_{i=1}^{\infty} Y_i^*$. Here $K_i \subset X_i = \mathbb{K}$ given with $X_i^* = \mathbb{K}$ the dual space of X_i and $Y_i^* = \mathbb{K}$ the dual space of Y_i for each $i = 1, 2, \cdots$.

3.1. **Expression of GHVIP using S operator.** Let $T : K \subset X \to L(X, Z)$ be any map where Z is either a scalar field $\mathbb{K}$ or a vector field. The problem is to find $y \in K$ such that

$$\langle T(y), z\rangle \geq_Z 0 \ \forall \ z \in X \tag{3.1}$$

where the pairing $\langle f, z\rangle$ denotes the value of $f \in L(X, Z)$ at $z \in X$. Above problem is written by a summable series form using the operator S.

(1) Let $Z = \mathbb{K}$ the scalar field and $T : K \to X^*$. Let $T_i : K_i \to \mathbb{K}$ where $K_i \subset X_i$. By taking $T = \sum_{i=1}^{\infty} T_i e_i$ where the problem (3.1) can be written as $\langle T(y), z\rangle = (T \boxtimes y) \otimes z \geq 0$, i.e.,

$$0 \leq \langle T(y), z\rangle = S(T \otimes y) \otimes z \ \forall \ y \in K, \ z \in X.$$

(2) If $Z = Y$ is the vector field in the Banach space and P is the closed convex pointed cone in Y with nonempty interior. Since $T : K \to L(X, Y)$, taking $T = \sum_{i=1}^{\infty} T_i e_i$ where $T_i : K_i \subset X_i \to L(X_i, Y_i)$, the problem (3.1) can be written as

$$\begin{aligned} \langle T(y), z\rangle &= \sum_{i=1}^{\infty} \langle T_i(y_i), z_i\rangle \, e_i = (T \boxtimes y) \boxtimes z \\ &= S([S(T \otimes y)] \otimes z) \\ &\geq_P 0 \ \ \forall \ y \in K, \ z \in X. \end{aligned}$$

For our study $Z = Y \subset \mathbb{B}$. If $\xi \in Y$, then for the expression $\xi \in P$ or $\xi \geq_P 0$, we mean $\xi \geqq 0$, implying $\xi_i \geq 0$ for each i; $\xi \in -P$ or $\xi \leq_P 0$ we mean $\xi \leqq 0$, implying $\xi_i \leq 0$ for each i and $\xi \in -P \cap P$ we mean $\xi \equiv 0$, implying $\xi_i = 0$ for each i (component-wise) in Banach space Y.

For $T : K \to L(X, Y)$ and $\eta : K \times K \to X$, the generalized harmonic variational inequality problems (GHVIP) are defined as follows:

(a) The GHVIP is of finding

$$y \in K : \left\langle T(y), xy[\eta(x,y)]^{-1} \right\rangle \geqq 0 \ \forall \ x \in K, \qquad \text{(GHVIP)}$$

(b) the generalized dual HVIP is of finding

$$y \in K : \left\langle T(x), xy\left[\eta(x,y)\right]^{-1} \right\rangle \geqq 0 \ \forall \ x \in K. \qquad \text{(GDHVIP)}$$

Remark 3.1. The problem (GHVIP) coincides with HVIP developed by Noor and Noor [7] if $Y = \mathbb{R}$ and $\eta(x,y) = y - x$.

4. Condition HC_* and some results using Condition HC_*

Consider a set

$$D_h = \left\{ v \in X_{\circledcirc} : xyv^{-1} \in X, \ x_\lambda = y + \lambda v \in K \subset X, \ \forall \ x, y \in K, \ \lambda \in [0,1] \right\}$$

in X that forms the nonempty starlike subspace $K(v)$ of X from the point $y \in K$ in the direction $v \in X$ as

$$K(v) = \{ x \in K : x = y + \lambda v, v \in D_h, \lambda \in (0,1] \}.$$

Assume that $\eta(x,y) = \{\eta_i(x,y)\} \in \prod_{i=1}^{\infty} X_i = X$ where $\eta_i : K \times K \to X_i$ is continuous for each i. We recall the following definitions for our requirement.

Definition 4.1. The set $K \subset X$ is

(a) *η-invex* [4] w.r.t. the mapping $\eta : K \times K \to X$ if $K(\eta) = K$ where

$$K(\eta) = \{ x_\lambda \in K : x_\lambda = y + \lambda \eta(x,y) \ \forall \ x, y \in K, \lambda \in [0,1] \},$$

(b) harmonically *η-invex* (HI) [6] w.r.t. the mapping $\eta : K \times K \to X$ if $K_h(\eta) = K$ where

$$K_h(\eta) = \left\{ xyx_\lambda^{-1} \in K ; x, y \in K, x_\lambda \in K(\eta), 0 \leq \lambda \leq 1 \right\}.$$

In other words, K is η-HI if

$$x_\lambda \in K \text{ if and only if } xyx_\lambda^{-1} \in K \ \forall \ x, y \in K, 0 \leq \lambda \leq 1.$$

Definition 4.2. Let X be a TVS and $K \neq \varnothing$ be a η-invex subset of $X_{\circledcirc}$. The bi-function η is said to satisfy *harmonic condition* $HC_\star$ on $K_h(\eta)$ if the following conditions hold:

(a) $\eta(y,x) = -\eta(x,y)$ for all $x, y \in K$, i.e., η is anti-symmetric on K,

(b) for all $x, y \in K$, $y_\lambda = xy\left[y + \lambda\eta(x,y)\right]^{-1}$ and $0 < \lambda \leq 1$, we have

$$y_\lambda y\left[\eta\left(y_\lambda, y\right)\right]^{-1} = xy\left[(1-\lambda)\eta(x,y)\right]^{-1},$$

(c) for all $x, y \in K$, $y_\lambda = xy\left[y + \lambda\eta(x,y)\right]^{-1}$ and $\lambda > 0$, we have

$$y_\lambda x\left[\eta\left(y_\lambda, x\right)\right]^{-1} = -xy\left[\lambda\eta(x,y)\right]^{-1},$$

(d) for all $x, y \in K$, $x_\lambda = xy\,[x+\lambda\eta(y,x)]^{-1}$ and $\lambda > 0$, we have

$$x_\lambda y\,[\eta(x_\lambda, y)]^{-1} = -xy\,[\lambda\eta(y,x)]^{-1},$$

(e) for all $x, y \in K$, $x_\lambda = xy\,[x+\lambda\eta(y,x)]^{-1}$ and $\lambda > 0$, we have

$$x_\lambda x\,[\eta(x_\lambda, x)]^{-1} = xy\,[(1-\lambda)\eta(y,x)]^{-1}.$$

Remark 4.3. The harmonic conditions of η can be written as

$$z_\lambda y\,[\eta(z_\lambda, y)]^{-1} = \begin{cases} xy\,[(1-\lambda)\eta(x,y)]^{-1}, & \text{if } z_\lambda = xy\,[y+\lambda\eta(x,y)]^{-1}; \\ -xy\,[\lambda\eta(y,x)]^{-1}, & \text{if } z_\lambda = xy\,[x+\lambda\eta(y,x)]^{-1} \end{cases}$$

and

$$z_\lambda x\,[\eta(z_\lambda, x)]^{-1} = \begin{cases} -xy\,[\lambda\eta(x,y)]^{-1}, & \text{if } z_\lambda = xy\,[y+\lambda\eta(x,y)]^{-1}; \\ xy\,[(1-\lambda)\eta(y,x)]^{-1}, & \text{if } z_\lambda = xy\,[x+\lambda\eta(y,x)]^{-1}. \end{cases}$$

Example 4.4. If $\eta(x,y) = x - y$, then for all $x, y \in K \subset X$, η satisfies the condition $HC_\star$ on $K_h(\eta)$.

Example 4.5. If $p : K \to K$ is a ρ-projective linear map [3] and $p(x) = \widetilde{x} \in K$ for all $x \in K$ and 0 otherwise. The map $\eta : \text{Ran } p \times \text{Ran } p \to X$ defined by $\eta(\widetilde{x}, \widetilde{y}) = \widetilde{x} - \widetilde{y}$ satisfies $\widetilde{x}_\lambda = \widetilde{y} + \lambda\eta(\widetilde{x}, \widetilde{y}) \in X_\circledcirc$ for all $\widetilde{x}, \widetilde{y} \in \text{Ran } p$ and $\lambda \in [0,1]$. The harmonically invex set $K_h(\eta)$ is constructed by

$$K_h(\eta) = \left\{ \widetilde{z} \in K : \eta(\widetilde{x}, \widetilde{y}) \in X_\circledcirc, \text{and } \widetilde{z} = \widetilde{x}\widetilde{y}\,[\widetilde{x}_\lambda]^{-1} \in K,\ \forall\, \widetilde{x}, \widetilde{y} \in \text{Ran } p. \right\}$$

Then η satisfies $HC_\star$-condition on $K_h(\eta)$.

Proposition 4.6. *Suppose that $K \subset X_\circledcirc$ where $\eta : K \times K \to X_\circledcirc$ satisfies the $HC_\star$-condition on $K_h(\eta)$. Let $x, y \in K$ and $\lambda \in (0,1)$. Then*

(1) *For $\overline{y} = xy\,[y+\lambda\eta(x,y)]^{-1}$, we have*
  (*a*) $y\overline{y}\,[\eta(y,\overline{y})]^{-1} = -xy\,[(1-\lambda)\eta(x,y)]^{-1}$,
  (*b*) $x\overline{y}\,[\eta(x,\overline{y})]^{-1} = xy\,[\lambda\eta(x,y)]^{-1}$;

(2) *for $\overline{y} = xy\,[x+\lambda\eta(y,x)]^{-1}$ and $\lambda \in (0,1)$, we have*
  (*a*) $y\overline{y}\,[\eta(y,\overline{y})]^{-1} = xy\,[\lambda\eta(y,x)]^{-1}$,
  (*b*) $x\overline{y}\,[\eta(x,\overline{y})]^{-1} = -xy\,[(1-\lambda)\eta(y,x)]^{-1}$.

Proof. Given $\eta(y,x) = -\eta(x,y)$ for $x, y \in K$ and $\lambda \in (0,1)$.

(1) Let $\overline{y} = y_\lambda = xy\,[y+\lambda\eta(x,y)]^{-1}$.

(*a*) Using condition (*b*) of $HC_\star$, then we have

$$y\overline{y}\,[\eta(y,\overline{y})]^{-1} = x_\lambda y\,[\eta(y, y_\lambda)]^{-1} = -x_\lambda y\,[\eta(y_\lambda, y)]^{-1} = -xy\,[(1-\lambda)\eta(x,y)]^{-1}.$$

(*b*) Using condition (*c*) of $HC_\star$, then we have

$$x\overline{y}\,[\eta(x,\overline{y})]^{-1} = x_\lambda x\,[\eta(x, y_\lambda)]^{-1} = -x_\lambda x\,[\eta(y_\lambda, x)]^{-1} = xy\,[\lambda\eta(x,y)]^{-1}.$$

(2) Let $\overline{y} = x_\lambda = xy\,[x + \lambda\eta(y,x)]^{-1}$.

(a) Using condition (d) of $HC_\star$, then we have

$$y\overline{y}\,[\eta\,(y,\overline{y})]^{-1} = x_\lambda y\,[\eta\,(y,x_\lambda)]^{-1} = -x_\lambda y\,[\eta\,(x_\lambda,y)]^{-1} = xy\,[\lambda\eta(y,x)]^{-1}\,,$$

(b) Using condition (e) of $HC_\star$, then we have

$$x\overline{y}\,[\eta\,(x,\overline{y})]^{-1} = x_\lambda x\,[\eta\,(x,x_\lambda)]^{-1} = -x_\lambda x\,[\eta\,(x_\lambda,x)]^{-1} = -xy\,[(1-\lambda)\eta(y,x)]^{-1}\,,$$

which is the required result. □

Proposition 4.7. *Let $\eta : K \times K \to X$ satisfies condition $HC_\star$ on η-HI set $K_h(\eta)$, and $x, y \in K$.*

(a) *If $\overline{y} = xy\,[y + \lambda\eta(x,y)]^{-1} \in K_h(\eta)$, then*

$$\lambda x\overline{y}\,[\eta(x,\overline{y})]^{-1} + (1-\lambda)y\overline{y}\,[\eta(y,\overline{y})]^{-1} = 0.$$

(b) *If $\overline{y} = xy\,[x + \lambda\eta(y,x)]^{-1} \in K_h(\eta)$, then*

$$(1-\lambda)x\overline{y}\,[\eta(\overline{y},x)]^{-1} + \lambda y\overline{y}\,[\eta(\overline{y},y)]^{-1} = 0.$$

Proof. Letting $\overline{y} = xy\,[y + \lambda\eta(x,y)]^{-1} \in K_h(\eta) \subset K$ for $\lambda \in (0,1)$ and using Proposition 4.6 we have

$$x\overline{y}\,[\eta(x,\overline{y})]^{-1} = -xy\,[\lambda\eta(x,y)]^{-1} \ \text{ and } \ y\overline{y}\,[\eta(y,\overline{y})]^{-1} = -xy\,[(1-\lambda)\eta(x,y)]^{-1}$$

for all $x, y \in K_h(\eta)$. Thus

$$\begin{aligned}&\lambda\left(x\overline{y}\,[\eta(x,\overline{y})]^{-1}\right) + (1-\lambda)\left(y\overline{y}\,[\eta(y,\overline{y})]^{-1}\right)\\&\quad= \lambda\left(xy\,[\lambda\eta(x,y)]^{-1}\right) + (1-\lambda)\left(xy\,[-(1-\lambda)\eta(x,y)]^{-1}\right)\\&\quad= xy\,[\eta(x,y)]^{-1} - xy\,[\eta(x,y)]^{-1} = 0\end{aligned}$$

for all $x, y \in K$. This proves (a).

Since $\eta(x,\overline{y}) = -\eta(\overline{y},x)$, letting $\overline{y} = xy\,[x + \lambda\eta(y,x)]^{-1} \in K_h(\eta)$ for $\lambda \in (0,1)$ and using Proposition 4.6, we have

$$x\overline{y}\,[\eta(\overline{y},x)]^{-1} = xy\,[(1-\lambda)\eta(y,x)]^{-1} \ \text{ and } \ y\overline{y}\,[\eta(\overline{y},y)]^{-1} = -\ xy\,[\lambda\eta(y,x)]^{-1}$$

for all $x, y \in K_h(\eta)$. Thus

$$\begin{aligned}&(1-\lambda)x\overline{y}\,[\eta(\overline{y},x)]^{-1} + \lambda y\overline{y}\,[\eta(\overline{y},y)]^{-1}\\&\quad= (1-\lambda)\left(xy\,[(1-\lambda)\eta(y,x)]^{-1}\right) + \lambda\left(-\ xy\,[\lambda\eta(y,x)]^{-1}\right)\\&\quad= xy\,[\eta(y,x)]^{-1} - xy\,[\eta(y,x)]^{-1} = 0\end{aligned}$$

for all $x, y \in K$ which proves (b). Hence the proof is completed. □

Proposition 4.8. *Let $\eta \in X_{\circledcirc}$ satisfies the $HC_{\star}$-condition on η-HI set $K_h(\eta)$ and $T : K \to X^*$. If for $x, y \in K$ and $\lambda \in (0,1)$, $\overline{y} = xy\left[y + \lambda\eta(x,y)\right]^{-1} \in K_h(\eta)$, then we have*

(a) $\lambda \left\langle T(\overline{y}), x\overline{y}\left[\eta(x,\overline{y})\right]^{-1}\right\rangle + (1-\lambda)\left\langle T(\overline{y}), y\overline{y}\left[\eta(y,\overline{y})\right]^{-1}\right\rangle \equiv 0,$

(b) $\lambda \left\langle T(u), x\overline{y}\left[\eta(x,\overline{y})\right]^{-1}\right\rangle + (1-\lambda)\left\langle T(u), y\overline{y}\left[\eta(y,\overline{y})\right]^{-1}\right\rangle \equiv 0,$
for any $u \in K_h(\eta)$,

(c) $\lambda \left\langle T(x), x\overline{y}\left[\eta(\overline{y},x)\right]^{-1}\right\rangle + (1-\lambda)\left\langle T(y), y\overline{y}\left[\eta(\overline{y},y)\right]^{-1}\right\rangle$
$\equiv -\left\langle T(x) - T(y), xy\left[\eta(x,y)\right]^{-1}\right\rangle.$

Proof. Since all the conditions of Proposition 4.7 are satisfied, we have

$$\lambda x\overline{y}\left[\eta(x,\overline{y})\right]^{-1} + (1-\lambda)y\overline{y}\left[\eta(y,\overline{y})\right]^{-1} = 0$$

for all $x, y \in K$, for any $\overline{y} = xy\left[y + \lambda\eta(x,y)\right]^{-1} \in K_h(\eta)$ and $t \in [0,1]$. Thus

$$\begin{aligned}
&\lambda \left\langle T(\overline{y}), x\overline{y}\left[\eta(x,\overline{y})\right]^{-1}\right\rangle + (1-\lambda)\left\langle T(\overline{y}), y\overline{y}\left[\eta(y,\overline{y})\right]^{-1}\right\rangle \\
&\quad \equiv \left\langle T(\overline{y}), \lambda x\overline{y}\left[\eta(\overline{y},x)\right]^{-1} + (1-\lambda)y\overline{y}\left[\eta(\overline{y},y)\right]^{-1}\right\rangle \\
&\quad \equiv \langle T(\overline{y}), 0\rangle \equiv 0
\end{aligned}$$

for all $x, y \in K$, for any $\overline{y} = xy\left[y + t\eta(x,y)\right]^{-1} \in K_h(\eta)$ and $t \in [0,1]$ since $T(x) \in L(X,Y)$. This proves (a).

Similarly for any $u \in K_h(\eta)$, we have

$$\begin{aligned}
&\lambda \left\langle T(u), x\overline{y}\left[\eta(x,\overline{y})\right]^{-1}\right\rangle + (1-\lambda)\left\langle T(u), y\overline{y}\left[\eta(y,\overline{y})\right]^{-1}\right\rangle \\
&\quad \equiv \left\langle T(u), \lambda x\overline{y}\left[\eta(\overline{y},x)\right]^{-1} + (1-\lambda)y\overline{y}\left[\eta(\overline{y},y)\right]^{-1}\right\rangle \\
&\quad \equiv \langle T(u), 0\rangle = 0
\end{aligned}$$

for all $x, y \in K$, for any $\overline{y} = xy\left[y + t\eta(x,y)\right]^{-1} \in K_h(\eta)$ and $t \in [0,1]$. This proves (b).

Finally taking $\overline{y} = xy\left[y + \lambda\eta(x,y)\right]^{-1} \in K_h(\eta)$ for $\lambda \in (0,1)$ and using Proposition 4.6 we have

$$x\overline{y}\left[\eta(\overline{y},x)\right]^{-1} = -xy\left[\lambda\eta(x,y)\right]^{-1}$$

and

$$y\overline{y}\left[\eta(\overline{y},y)\right]^{-1} = xy\left[(1-\lambda)\eta(x,y)\right]^{-1}.$$

Thus

$$\begin{aligned}
&\lambda \left\langle T(x), x\overline{y}\,[\eta(\overline{y},x)]^{-1}\right\rangle + (1-\lambda)\left\langle T(y), y\overline{y}\,[\eta(\overline{y},y)]^{-1}\right\rangle \\
&\quad \equiv \lambda \left\langle T(x), x\overline{y}\,[\eta(x,\overline{y})]^{-1}\right\rangle + (1-\lambda)\left\langle T(y), y\overline{y}\,[\eta(y,\overline{y})]^{-1}\right\rangle \\
&\quad \equiv \lambda \left\langle T(x), -\, x\overline{y}\,[\lambda\eta(x,y)]^{-1}\right\rangle + (1-\lambda)\left\langle T(y), y\overline{y}\,[(1-\lambda)\eta(x,y)]^{-1}\right\rangle \\
&\quad \equiv -\left\langle T(x), xy\,[\eta(x,y)]^{-1}\right\rangle + \left\langle T(y), xy\,[\eta(x,y)]^{-1}\right\rangle \\
&\quad \equiv -\left\langle T(x)-T(y), xy\,[\eta(x,y)]^{-1}\right\rangle
\end{aligned}$$

for all $x, y \in K_h(\eta)$, for any $\overline{y} = xy\,[y+\lambda\eta(x,y)]^{-1} \in K_h(\eta)$ and $\lambda \in (0,1)$ which proves (c). The proof is completed. □

Definition 4.9. The mapping $T : K \to L(X,Y)$ is said to be

(i) harmonically η-monotone on K if

$$\left\langle T(x), xy\,[\eta(x,y)]^{-1}\right\rangle + \left\langle T(y), yy\,[\eta(y,x)]^{-1}\right\rangle \geqq 0 \;\forall\; x, y \in K,$$

(ii) generalized harmonically η-monotone on K if

$$\left\langle T(x)-T(y), xy\,[\eta(x,y)]^{-1}\right\rangle \geqq 0 \;\forall\; x, y \in K.$$

For strictly harmonically η-monotonicity case, equality hold in the above equation for $x = y$ only.

In the following theorem, the result is shown to approach the existence theorems of dual harmonic variational inequality problems in the presence of generalized harmonically η-monotone property of T.

Theorem 4.10. *Let $K \subset X_{\circledcirc}$ harmonically η-invex set $K_h(\eta)$. Then $\overline{y} \in K_h(\eta)$ solves both the harmonic problem* (GHVIP) *and* (GDHVIP) *on $K_h(\eta)$, if η satisfies HC_*-conditions, T is generalized harmonically η-monotone on K and for some $\lambda > 0$,*

$$\begin{aligned}
\left\langle T(x_\lambda), x_\lambda\overline{y}\,[\eta(x_\lambda,\overline{y})]^{-1}\right\rangle \quad \leq \quad & (1-\lambda)\left\langle T(\overline{y}), x\overline{y}\,[\eta(x,\overline{y})]^{-1}\right\rangle \\
& +\lambda\left\langle T(x), x\overline{y}\,[\eta(x,\overline{y})]^{-1}\right\rangle
\end{aligned}$$

for all $x \in K$, $x_\lambda = x\overline{y}\,[\overline{y}+\lambda\eta(x,\overline{y})]^{-1}$, $\overline{y} = xy\,[y+\lambda\eta(x,y)]^{-1} \in K_h(\eta)$.

Proof. For each $\overline{y} \in K_h(\eta)$, T is generalized harmonically η-monotone on K, i.e.,

$$\left\langle T(x)-T(y), xy\,[\eta(x,y)]^{-1}\right\rangle \geqq 0$$

for all $x, y \in K$. By (c) of Theorem 4.8, we have

$$\begin{aligned}&\lambda \left\langle T(x), x\overline{y}\left[\eta(x,\overline{y})\right]^{-1}\right\rangle + (1-\lambda)\left\langle T(y), y\overline{y}\left[\eta(y,\overline{y})\right]^{-1}\right\rangle \\ \equiv\ & \left\langle T(x)-T(y), xy\left[\eta(x,y)\right]^{-1}\right\rangle \geqq 0\end{aligned}$$

for all $x, y \in K$ and at $\overline{y} \in K_h(\eta) \subset K$. Hence

$$\lambda \left\langle T(x), x\overline{y}\left[\eta(x,\overline{y})\right]^{-1}\right\rangle + (1-\lambda)\left\langle T(y), y\overline{y}\left[\eta(y,\overline{y})\right]^{-1}\right\rangle \geqq 0$$

for all $x, y \in K$. By taking limit as $\lambda \to 1$ we obtain

$$\left\langle T(x), x\overline{y}\left[\eta(x,\overline{y})\right]^{-1}\right\rangle \geqq 0,$$

that is, $\overline{y} \in K_h(\eta)$ solves the harmonic problem (GDHVIP) on $K_h(\eta)$.

Replacing x by $x_\lambda = x\overline{y}\left[\overline{y} + \lambda\eta(x,\overline{y})\right]^{-1}$ for some $\lambda \in (0,1)$, we have

$$\begin{aligned}0 &\leqq \left\langle T(x_\lambda), x_\lambda\overline{y}\left[\eta(x_\lambda,\overline{y})\right]^{-1}\right\rangle \\ &\leq (1-\lambda)\left\langle T(\overline{y}), x\overline{y}\left[\eta(x,\overline{y})\right]^{-1}\right\rangle + \lambda\left\langle T(x), x\overline{y}\left[\eta(x,\overline{y})\right]^{-1}\right\rangle\end{aligned}$$

for all $x \in K$. Taking limit as $\lambda \to 0$, we get

$$0 \leqq \left\langle T(\overline{y}), x\overline{y}\left[\eta(x,\overline{y})\right]^{-1}\right\rangle$$

for all $x \in K$ and $\overline{y} \in K_h(\eta)$. Hence $\overline{y} \in K_h(\eta)$ solves the harmonic problem (GHVIP) on $K_h(\eta)$. This completes the proof. □

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