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HIGHER-ORDER OPTIMALITY CONDITIONS IN NONSMOOTH CONE-CONSTRAINED MULTIOBJECTIVE PROGRAMMING

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Abstract. This paper presents higher-order necessary and sufficient optimality conditions in nonsmooth multiobjective optimizaton problems involving a cone-constraint and a set constraint in terms of the Ginchev directional derivatives of higher order.

1. Introduction

Let f and g be maps from a normed space X into other normed spaces Y and Z, respectively. Let C be a subset of X, and let Q and S be closed convex cones in Y and Z, respectively. This paper addresses higher-order necessary and sufficient conditions for efficient solutions (with respect to the cone Q) of the following multiobjective optimization problem:

$$\text{(MP)} \qquad \begin{aligned} &\min f(x), \\ &\text{subject to} \\ &\quad -g(x) \in S, \\ &\quad x \in C. \end{aligned}$$

Denote by M the feasible set of Problem (MP)

$$M = \Big\{x \in C : -g(x) \in S\Big\}.$$

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Higher order optimality conditions for Problem (MP) have been extensively studied by many authors (see, e.g., [2], [4], [5], [7]-[14]). Ginchev [4] introduces notions of lower and upper directional derivatives of higher order for extended-real-valued functions based on the Taylor expansion and establishes higher order optimality conditions for Problem (MP) without constraint in case f is an extended-real-valued function. It is shown in [4] that if solutions of the considering problem are isolated, then characteristic conditions for those are obtained. The notions of isolated minimizer and strict minimizer are extended to multiobjective programming by Jiménez [6] and Ginchev [5] to become respectively strict local Pareto minimizer of order n and strict local Pareto minimizer. Luu-Kien [9] study further minimizers of these types and establish higher-order necessary and sufficient conditions in terms of the Ginchev higher-order directional derivatives for multiobjective optimization problems with set constraints.

The purpose of this paper is to develop further the results obtained in [9] to multiobjective optimization problems involving both cone-constraint and set constraint. The remainder of the paper is organized as follows. After some preliminaries, Section 3 will be devoted to developing higher-order necessary conditions for weak efficiency in general case and for strict local Pareto minimizer in case $Y = \mathbb{R}^r$ and Q is the nonnegative orthant $\mathbb{R}^r_+$ in $\mathbb{R}^r$. Section 4 deals with higher-order sufficient conditions for strict local Pareto minimizer of order n in general case and in case $Y = \mathbb{R}^r$, $Q = \mathbb{R}^r_+$.

2. Preliminaries

Recall [4] that the nth order lower and upper directional derivatives $f_-^{(n)}(\overline{x}; v)$ and $f_+^{(n)}(\overline{x}; v)$, respectively, of an extended-real-valued function f defined on X at $\overline{x} \in X$ in a direction v are defined as

$$f_-^{(0)}(\overline{x}; v) = \liminf_{t\downarrow 0, u\to v} f(\overline{x} + tu),$$

$$f_-^{(n)}(\overline{x}; v) = \liminf_{t\downarrow 0, u\to v} \frac{n!}{t^n}\Big[f(\overline{x} + tu) - \sum_{j=0}^{n-1} \frac{t^j}{j!} f_-^{(j)}(\overline{x}; v)\Big],$$

$$f_+^{(0)}(\overline{x}; v) = \limsup_{t\downarrow 0, u\to v} f(\overline{x} + tu),$$

$$f_+^{(n)}(\overline{x}; v) = \limsup_{t\downarrow 0, u\to v} \frac{n!}{t^n}\Big[f(\overline{x} + tu) - \sum_{j=0}^{n-1} \frac{t^j}{j!} f_+^{(j)}(\overline{x}; v)\Big] \quad (n = 1, 2, \dots).$$

In case f is a mapping from X into Y, the nth order directional derivatives of f at $\overline{x}$ in the direction v are defined as follows

$$f^{(0)}(\overline{x};v) = \lim_{t\downarrow 0, u\to v} f(\overline{x}+tu),$$

$$f^{(n)}(\overline{x};v) = \lim_{t\downarrow 0, u\to v} \frac{n!}{t^n}\Big[f(\overline{x}+tu) - \sum_{j=0}^{n-1}\frac{t^j}{j!}f^{(j)}(\overline{x};v)\Big]$$

$$(n = 1, 2, \dots),$$

if these limits exists. Note that if f is Fréchet differentiable at $\overline{x}$ with the Fréchet derivatives $f'(\overline{x})$, then $f^{(1)}(\overline{x};v) = f'(\overline{x})v \quad (\forall\, v \in X)$.

For an extended-real-valued f defined on X, the lower and upper Hadamard directional derivatives of f at $\overline{x}$ in the direction v is

$$\underline{d}f(\overline{x};v) = \liminf_{t\downarrow 0, u\to v} \frac{f(\overline{x}+tu) - f(\overline{x})}{t},$$

$$\overline{d}f(\overline{x};v) = \limsup_{t\downarrow 0, u\to v} \frac{f(\overline{x}+tu) - f(\overline{x})}{t}.$$

In case $\underline{d}f(\overline{x};v) = \overline{d}f(\overline{x};v)$ we donote their common value by $df(\overline{x};v)$. This is Hadamard directional derivatives of f at $\overline{x}$ in the direction v. We say that f is directionally Hadamard differentiable at $\overline{x}$ if $df(\overline{x};v)$ exists for all $v \in X$. Note that if $df(\overline{x};v)$ exists, then usual directional derivative:

$$f'(\overline{x};v) = \lim_{t\downarrow 0} \frac{f(\overline{v}+tv) - f(\overline{x})}{t}$$

also exists, and they are equal. If f is directionally Hadamard differentable at $\overline{x}$, then $df(\overline{x};.)$ is continuous on X, f is continuous at $\overline{x}$ (see [3, Theorem 3.2]), and

$$df(\overline{x};v) = f^{(1)}(\overline{x};v) = f'(\overline{x};v).$$

Turning to Problem (MP), a point $\overline{x} \in M$ is said to be a weakly local minimum (resp. local Pareto minimum) of Problem (MP) if there exists a number $\delta > 0$ such that

$$f(x) - f(\overline{x}) \notin -\mathrm{int}\, Q \quad (\forall\, x \in M \cap B(\overline{x};\delta))$$

$$(\text{resp. } f(x) - f(\overline{x}) \notin -Q \setminus \{0\} \quad \text{for all } x \in M \cap B(\overline{x};\delta)),$$

where $B(\overline{x};\delta)$ stands for the open ball of radius δ around $\overline{x}$, int$Q$ indicates the interior of Q.

Following [6], the point $\overline{x}$ is called a strict local Pareto minimum of order n (resp. strict local Pareto minimum) of Problem (MP) if there exists numbers $\delta > 0$ and $\alpha > 0$ such that

$$(f(\overline{x}) + Q) \cap B(f(\overline{x}); \alpha\|x - \overline{x}\|^n) = \emptyset \quad (\forall\, x \in M \cap B(\overline{x};\delta) \setminus \{\overline{x}\})$$

$$(\text{resp. } f(x) - f(\overline{x}) \notin -Q \quad \text{for all } x \in M \cap B(\overline{x};\delta) \setminus \{\overline{x}\}).$$

3. Higher order necessary conditions for efficiency

Recall [1] that the contingent cone to the set C at $\overline{x} \in clC$ is

$$K_C(\overline{x}) = \Big\{ v \in X: \text{ there exist sequences } t_m \downarrow 0 \text{ and } v_m \to v \text{ such that } \overline{x} + t_m v_m \in C \text{ for all } m \Big\},$$

where clC indicates the closure of C. In this section we assume that $\operatorname{int} S \neq \emptyset$.

We shall begin with a higher-order necessary condition for weakly local minima of Problem (MP) in terms of higher order directional derivatives.

Theorem 3.1. *Let* $\operatorname{int} Q \neq \emptyset$ *and* $\overline{x}$ *be a weakly local minimum of Problem (MP). Assume that* g *is directionally Hadamard differentiable at* $\overline{x}$*, and for each* $v \in K_C(\overline{x}) \cap \{u : -g'(\overline{x};u) \in \operatorname{int} S\}$*, the directional derivatives* $f^{(j)}(\overline{x};v)$ $(j = 0,1,\dots,n)$ *exist. Then the following conditions hold:*
(i) $f^{(0)}(\overline{x};v) - f(\overline{x}) \notin -\operatorname{int} Q \quad (\forall\, v \in K_C(\overline{x}) \cap \{u : -g'(\overline{x};u) \in \operatorname{int} S\})$;
(ii) *If for* $v \in K_C(\overline{x}) \cap \{u : -g'(\overline{x};u) \in \operatorname{int} S\}$, $f^{(0)}(\overline{x};v) = f(\overline{x})$, $f^{(j)}(\overline{x};v) = 0$ $(j = 1,\dots,n-1)$*, then* $f^{(n)}(\overline{x};v) \notin -\operatorname{int} Q$.

Proof. The minimality of $\overline{x}$ implies that there exists a neighborhood U of $\overline{x}$ satisfying

$$f(x) - f(\overline{x}) \in -(Y \setminus \operatorname{int} Q) \quad (\forall\, x \in M \cap U). \tag{3.1}$$

Now for $v \in K_C(\overline{x}) \cap \{u : -g'(\overline{x};u) \in \operatorname{int} S\}$, there exists sequences $t_m \downarrow 0$ and $v_m \to v$ such that $\overline{x} + t_m v_m \in M$ $(\forall\, m)$, where M is the feasible set of Problem (MP). So there is a natural number N_1 such that for every $m \geqslant N_1$,

$$\overline{x} + t_m v_m \in C \cap U. \tag{3.2}$$

Since g is directionally Hadamard differentiable at $\overline{x}$, $dg(\overline{x};.)$ is continuous on X (see [3, Theorem 3.2]), and $dg(\overline{x};v_m) = g'(\overline{x},v_m)$. Hence, there is a natural number N_2 $(\geqslant N_1)$ such that for every $m \geqslant N_2$,

$$-g'(\overline{x};v_m) \in \operatorname{int} S.$$

Consequently, there is a natural number N_3 $(\geqslant N_2)$ such that for every $m \geqslant N_3$,

$$\begin{aligned} g(\overline{x} + t_m v_m) &= g(\overline{x}) + t_m \Big[g'(\overline{x};v_m) + \frac{o(t_m)}{t_m} \Big] \\ &\in g(\overline{x}) - S \\ &\subset -S - S \subset -S, \end{aligned}$$

where $o(t_m)/t_m \to 0$ as $m \to +\infty$. This together with (3.2) yields that for every $m \geqslant N_3$,

$$\overline{x} + t_m v_m \in M \cap U.$$

Making use of (3.1) it results that

$$f(\overline{x} + t_m v_m) - f(\overline{x}) \in -(Y \setminus \operatorname{int} Q), \tag{3.3}$$

which leads to the following

$$\lim_{m \to +\infty} f(\overline{x} + t_m v_m) - f(\overline{x}) = f^{(0)}(\overline{x}; v) - f(\overline{x}) \in -(Y \setminus \operatorname{int} Q),$$

which gives (i).

We now observe that if for $v \in K_C(\overline{x}) \cap \{u : -g'(\overline{x}; u) \in \operatorname{int} S\}$, the directional derivatives $f^{(j)}(\overline{x}; v)$ $(j = 0, 1, \ldots, n)$ exist and $f^{(0)}(\overline{x}; v) = f(\overline{x})$, $f^{(1)}(\overline{x}; v) = \cdots = f^{(n-1)}(\overline{x}; v) = 0$, then

$$\begin{aligned} f^{(n)}(\overline{x}; v) &= \lim_{m \to +\infty} \frac{n!}{t_m^n} \Big[f(\overline{x} + t_m v_m) - \sum_{j=0}^{n-1} \frac{t_m^j}{j!} f^{(j)}(\overline{x}; v) \Big] \\ &= \lim_{m \to +\infty} \frac{n!}{t_m^n} \big[f(\overline{x} + t_m v_m) - f(\overline{x}) \big], \end{aligned}$$

which along with (3.3) yields that

$$f^{(n)}(\overline{x}; v) \notin - \operatorname{int} Q,$$

as was to be shown. □

Let us consider the case $Y = \mathbb{R}^r$, $Q = \mathbb{R}^r_+$, $f = (f_1, \ldots, f_r)$. Denote by $f_{i,+}^{(j)}$ $(j = 0, 1, \ldots, n)$ the jth order upper Ginchev directional derivatives of f_i at $\overline{x} \in X$ in a direction $v \in X$, that is

$$\begin{aligned} f_{i,+}^{(0)}(\overline{x}; v) &= \limsup_{t \downarrow 0, u \to v} f_i(\overline{x} + tu), \\ f_{i,+}^{(j)}(\overline{x}; v) &= \limsup_{t \downarrow 0, u \to v} \frac{j!}{t^j} \Big[f_i(\overline{x} + t_m v_m) - \sum_{k=0}^{j-1} \frac{t^k}{k!} f_{i,+}^{(k)}(\overline{x}; v) \Big] \\ & \qquad (j = 1, \ldots, n). \end{aligned}$$

A higher-order necessary condition for strict local Pareto minima of Problem (MP) in terms of upper directional derivatives of higher order can be stated as follows.

Theorem 3.2. *Let $\overline{x}$ be a strict local Pareto minimum of Problem (MP). Assume that g is directionally Hadamard differentiable at $\overline{x}$. Then for every $v \in K_C(\overline{x}) \cap \{u : -g'(\overline{x}; u) \in \operatorname{int} S\}$, there is $i \in \{1, \ldots, r\}$ such that*

(a) $f_{i,+}^{(0)}(\overline{x}; v) \geqslant f_i(\overline{x})$,

(b) *If* $f^{(0)}_{i,+}(\overline{x};v) = f_i(\overline{x})$, $f^{(j)}_{i,+}(\overline{x};v) \leqslant 0$ $(j = 1,\dots,n-1)$, *then*

$$f^{(n)}_{i,+}(\overline{x};v) \geqslant 0.$$

Proof. We first invoke Theorem 3.7 [6] to deduce that there exist a neighborhood U of $\overline{x}$ and sets V_i $(i = 1,\dots,s; s \leqslant r)$ such that $\{V_j, j = 1,\dots,s\}$ is a covering of $(M \cap U) \setminus \{\overline{x}\}$ and verifying

$$f_j(x) > f_j(\overline{x}) \quad \text{for all } x \in \ M_j \setminus \{\overline{x}\}, \tag{3.4}$$

where M is the feasible set of (MP), $M_j = (M \cap U \cap V_j) \cup \{\overline{x}\}$. Hence,

$$M \cap U = \bigcup_{j=1}^{s} M_j.$$

Setting $W_j = V_j \cup ((C \setminus M) \cap U)$, we can see that $\{W_j, j = 1,\dots,s\}$ is a covering of $(C \cap U) \setminus \{\overline{x}\}$, and

$$C \cap U = \bigcup_{j=1}^{s} C_j, \tag{3.5}$$

where $C_j = (C \cap U \cap W_j) \cup \{\overline{x}\}$ $(j = 1,\dots,s)$. Putting $D = \{x : -g(x) \in S\}$, we deduce that for each $j = 1,\dots,s$,

$$\begin{aligned} C_j \cap D &= ((C \cap U \cap W_j) \cup \{\overline{x}\}) \cap D \\ &= \big[M \cap U \cap (V_j \cup ((C \setminus M) \cap U)\big] \cup \{\overline{x}\} \\ &= (M \cap U \cap V_j) \cup \{\overline{x}\} = M_j, \end{aligned}$$

which along with (3.4) yields that

$$f_j(x) > f_j(\overline{x}) \quad \text{for all } x \in C_j \cap D \setminus \{\overline{x}\}\ (j = 1,\dots,s). \tag{3.6}$$

On the other hand, making use of a result due to Aubin-Frankowska [1, Table 4.1], it follows from (3.5) that

$$K_C(\overline{x}) = K_{C\cap U}(\overline{x}) = \bigcup_{j=1}^{s} K_{C_j}(\overline{x}).$$

Note also that the existence of $dg(\overline{x};.)$ implies the existence of $g'(\overline{x};.)$ and they are equal. Taking $v \in K_C(\overline{x}) \cap \{u : -g'(\overline{x};u) \in \operatorname{int} S\}$, there is $i \in \{1,\dots,s\}$ such that $v \in K_{C_i}(\overline{x}) \cap \{u : -g'(\overline{x};u) \in \operatorname{int} S\}$, and so there exist sequences $t_m \downarrow 0$ and $v_m \to v$ such that

$$\overline{x} + t_m v_m \in C_i \setminus \{\overline{x}\}. \tag{3.7}$$

Since g is directionally Hadamard differentiable at $\overline{x}$, $g'(\overline{x};.)$ is continuous on X (see [3, Theorem 3.2]). Hence, for sufficiently large m, $g'(\overline{x};v_m)+\dfrac{o(t_m)}{t_m}\in -S$, as $-g'(\overline{x};v)\in \operatorname{int} S$. Consequently,

$$\begin{aligned} g(\overline{x}+t_mv_m) &= g(\overline{x})+t_m\Big[g'(\overline{x};v_m)+\frac{o(t_m)}{t_m}\Big] \\ &\in -S-S\subset -S, \end{aligned}$$

which together with (3.7) yields that

$$\overline{x}+t_mv_m\in M_i\setminus\{\overline{x}\}.$$

In view of (3.6), we get that for sufficiently large m,

$$f_i(\overline{x}+t_mv_m)>f_i(\overline{x}), \tag{3.8}$$

which implies that

$$f^{(0)}_{i,+}(\overline{x};v)\geqslant \limsup_{m\to+\infty} f_i(\overline{x}+t_mv_m)\geqslant f_i(\overline{x}).$$

We thus arrive at (i).

Next, we take $v\in K_C(\overline{x})\cap\{u:-g'(\overline{x};u)\in \operatorname{int} S\}$ satisfying

$$f^{(0)}_{i,+}(\overline{x};v)=f_i(\overline{x}),\quad f^{(j)}_{i,+}(\overline{x};v)\leqslant 0\quad (j=1,\dots,n-1). \tag{3.9}$$

Taking account of (3.8) and (3.9), we get

$$\begin{aligned} f^{(n)}_{i,+}(\overline{x};v) &\geqslant \limsup_{m\to+\infty}\frac{n!}{t_m^n}\Big[f_i(\overline{x}+t_mv_m)-f_i(\overline{x})-\sum_{j=1}^{n-1}\frac{t_m^j}{j!}f^{(j)}_{i,+}(\overline{x};v)\Big] \\ &\geqslant 0, \end{aligned}$$

which completes the proof. □

Remark 3.3. *Theorems 3.1 and 3.2 obtained here are generalizations of Theorems 3.1 and 5.1 [9], respectively.*

Theorem 3.2 is illustrated by the following example.

Example 3.4. *Let* $X=Y=\mathbb{R}^2$, $Q=S=R^2_+$, $\overline{x}=(0,0)$, *and*

$$C=\{x=(x_1,x_2):x_1^2+x_2^2\leqslant 5,x_1\geqslant 0,x_2\leqslant 0\}.$$

Let f and g be defined as

$$\begin{aligned} f(x) &= (f_1(x), f_2(x)),\\ f_1(x) &= |x_1|^k + |x_2|^k,\\ f_2(x) &= -|x_2|,\\ g(x) &= (x_1^2 - |x_1|, x_2^2 + 2x_2), \end{aligned}$$

where k is a positive integer number. Then $M = [0,1] \times [-2,0]$, $K_C(\overline{x}) = \mathbb{R}_+ \times \mathbb{R}_-$, and $\overline{x} = (0,0)$ is a strict local Pareto minimum of f under the constraints $-g(x) \in S$ and $x \in C$, where $\mathbb{R}_- = -\mathbb{R}_+$.

For $u = (u_1, u_2) \in \mathbb{R}^2$, $g'(\overline{x}; u) = (-|u_1|, 2u_2)$, and hence,

$$\{u \in \mathbb{R}^2 : -g'(\overline{x}; u) \in \mathbb{R}^2_+\} = \mathbb{R} \times \mathbb{R}_-.$$

Therefore, $K_C(\overline{x}) \cap \{u : -g'(\overline{x}; u) \in \mathbb{R}^2_+\} = \mathbb{R}_+ \times \mathbb{R}_-$. Then, for $v = (v_1, v_2) \in K_C(\overline{x}) \cap \{u : -g'(\overline{x}; u) \in \mathbb{R}^2_+\}$,

$$f^{(0)}_{1,+}(0; v) = f_1(0), \quad f^{(j)}_{1,+}(0; v) = 0 \quad (j = 1, \ldots, k-1),$$

$$f^{(k)}_{1,+}(0; v) = |v_1|^k + |v_2|^k \geqslant 0.$$

4. Higher-order sufficient conditions for efficiency

We set $S_{g(\overline{x})} = \overline{cone(S + g(\overline{x}))}$, where $\overline{cone(S + g(\overline{x}))}$ denotes the closure of the cone generated by $S + g(\overline{x})$. Let $\dim X < +\infty$.

We are now in a position to formulate a higher-order sufficient condition for strict local Pareto minima of order n of Problem (MP) in terms of higher order directional derivatives in finite dimensions.

Theorem 4.1. *Let $\overline{x}$ be a feasible point of (MP) and let $dg(\overline{x}; v)$ exist for all $v \in K_C(\overline{x}) \setminus \{0\}$. Assume that there is a positive integer number n such that for every $v \in K_C(\overline{x}) \cap \{u : -g(\overline{x}; u) \in S_{g(\overline{x})}\} \setminus \{0\}$, the directional derivatives $f^{(j)}(\overline{x}, v)$ $(j = 0, 1, \ldots, n)$ exist, and one of the following conditions (A_k) $(k = 1, \ldots, n)$ holds:*

(A_k) $\quad f^{(0)}(\overline{x}; v) = f(\overline{x}), \quad f^{(j)}(\overline{x}; v) = 0 \quad (j = 1, \ldots, k-1), f^{(k)}(\overline{x}; v) \notin -Q.$

Then $\overline{x}$ is a strict local Pareto mimimum of order n for (MP).

Proof. Contrary to the conclusion, suppose that condition (A_k) holds, but $\overline{x}$ is not a strict local Pareto of order n for (MP). We invoke Proposition 3.4 [6] to deduce that there exist $x_m \in M$, $x_m \neq \overline{x}$, $x_m \to \overline{x}$ and $b_m \in Q$ such that

$$\lim_{m \to +\infty} \frac{f(x_m) - f(\overline{x}) + b_m}{\|x_m - \overline{x}\|^n} = 0, \tag{4.1}$$

where M is the feasible set of Problem (MP). Putting $v_m = \dfrac{x_m - \overline{x}}{\|x_m - \overline{x}\|^n}$ and $t_m = \|x_m - \overline{x}\|$, we get that $t_m \downarrow 0$ and $x_m = \overline{x} + t_m v_m \in M \subset C$. Since $\dim X < +\infty$, there is a subsequence of $\{v_m\}$ converging to v_0 with $\|v_0\| = 1$. Without loss of generality, we can assume that $v_m \to v_0$, and so $v_0 \in K_C(\overline{x}) \setminus \{0\}$.

Moreover, since $dg(\overline{x}; v_0)$ exists, it results that

$$\lim_{m\to+\infty} \frac{g(\overline{x} + t_m v_m) - g(\overline{x})}{t_m} = dg(\overline{x}; v_0).$$

Observing that $g(\overline{x} + t_m v_m) \in -S$, we deduce that

$$\frac{g(\overline{x} + t_m v_m) - g(\overline{x})}{t_m} \in -S_{g(\overline{x})} \quad \text{for all } m,$$

whence,

$$g'(\overline{x}; v_0) = dg(\overline{x}; v_0) \in -S_{g(\overline{x})}.$$

We can thus contend that

$$v_0 \in K_C(\overline{x}) \cap \big\{u : g'(\overline{x}; u) \in -S_{g(\overline{x})}\big\} \setminus \{0\}.$$

On the other hand, by (4.1) it results that for each $k \in \overline{\{1, n\}}$,

$$\lim_{m\to+\infty} \frac{f(\overline{x} + t_m v_m) - f(\overline{x}) + b_m}{t_m^k} = 0. \tag{4.2}$$

If for $k \in \overline{\{1, n\}}$, $f^{(0)}(\overline{x}; v_0) = f(\overline{x})$ and $f^{(j)}(\overline{x}; v_0) = 0$ $(j = 1, \ldots, k-1)$, then

$$f^{(k)}(\overline{x}; v_0) = \lim_{m\to+\infty} \frac{k!}{t_m^k} \big[f(\overline{x} + t_m v_m) - f(\overline{x})\big].$$

Hence, the existence of $f^{(k)}(\overline{x}; v_0)$ implies the existence of the following limit:

$$\lim_{m\to+\infty} \frac{f(\overline{x} + t_m v_m) - f(\overline{x})}{t_m^k},$$

which together with (4.2) yields the existence of the limit $\lim\limits_{m\to+\infty} \dfrac{b_m}{t_m^k}$. We can also see that $\lim\limits_{m\to+\infty} \dfrac{b_m}{t_m^k} \in Q$, as $b_m \in Q$ and Q is closed. Consequently, by (4.2) it follows readily that

$$f^{(k)}(\overline{x}; v_0) \in -Q.$$

But this conflicts with condition (A_k). $\square$

Let us consider the case $Y = \mathbb{R}^r$, $Q = \mathbb{R}^r_+$, $f = (f_1, \ldots, f_r)$ and $\dim X < +\infty$. Denote by $f^{(j)}_{i,-}$ $(j = 0, 1, \ldots, n)$ the jth order lower Ginchev directional derivatives of f_i at $\overline{x}$ in the direction v, that is

$$f^{(0)}_{i,-}(\overline{x}; v) = \liminf_{t\downarrow 0, u\to v} f_i(\overline{x} + tu),$$

$$f^{(j)}_{i,-}(\overline{x}; v) = \liminf_{t\downarrow 0, u\to v} \frac{j!}{t^j}\Big[f_i(\overline{x} + tu) - \sum_{k=0}^{j-1} \frac{t^k}{k!} f^{(k)}_{i,-}(\overline{x}; v)\Big] \qquad (j = 1, \ldots, n).$$

In what follows we give a higher-order sufficient condition for strict local Pareto minima of order n for Problem (MP) in terms of lower directional derivatives of higher order.

Theorem 4.2. *Let $\overline{x}$ be a feasible point of Problem (MP). Assume that g is directionally Hadamard differentiable at $\overline{x}$, and there are $i_0 \in \overline{\{1, n\}}$ and a positive integer number n such that for every $v \in K_C(\overline{x}) \cap \{u : -g'(\overline{x}; u) \in S_{g(\overline{x})}\} \setminus \{0\}$, one of the following conditions (B_k) $(k = 1, \ldots, n)$ holds:*

$$(B_k) \quad f^{(0)}_{i_0,-}(\overline{x}; v) \geqslant f_{i_0}(\overline{x}), \quad f^{(j)}_{i_0,-}(\overline{x}, v) \geqslant 0 \quad (j = 1, \ldots, k-1),$$
$$f^{(k)}_{i_0,-}(\overline{x}; v) > 0.$$

Then $\overline{x}$ is a strict local Pareto minimum of order n for Problem (MP).

Proof. Let us consider the following problem:

$$(\mathrm{P}_{i_0}) \qquad \min\Big\{f_{i_0}(x) : -g(x) \in S, x \in C\Big\},$$

where g, S, C are as in Problem (MP). Note that the feasible set of this problem is also M. We shall begin with showing that $\overline{x}$ is a strict local minimum of order n for (P_{i_0}).

Suppose, for a contradiction, that condition (B_k) holds, but $\overline{x}$ is not a strict local minimum of order n for (P_{i_0}). Then for any integer number $m \geqslant 1$, there would exists $x_m \in M$, $x_m \neq \overline{x}$, $x_m \to \overline{x}$ such that

$$f_{i_0}(x_m) \leqslant f_{i_0}(\overline{x}) + \frac{1}{m}\|x_m - \overline{x}\|^n. \tag{4.3}$$

We set $t_m = \|x_m - \overline{x}\|$, $v_m = (x_m - \overline{x})/t_m$, and get that $t_m \downarrow 0$ and $x_m = \overline{x} + t_m v_m \in M \subset C$. Since C is finite dimensional, without loss of generality, we can assume that $v_m \to v_0$ with $\|v_0\| = 1$, and so $v_0 \in K_C(\overline{x}) \setminus \{0\}$.

Since g is directionally Hadamard differentiable at $\overline{x}$, it holds that $dg(\overline{x}; v_m) = g'(\overline{x}; v_m)$. Hence,

$$g(\overline{x} + t_m v_m) = g(\overline{x}) + t_m\Big[g'(\overline{x}; v_m) + \frac{o(t_m)}{t_m}\Big],$$

which leads to the following

$$g'(\overline{x}; v_m) + \frac{o(t_m)}{t_m} \in -S_{g(\overline{x})}, \tag{4.4}$$

as $g(\overline{x} + t_m v_m) \in -S$. Moreover, since g is directionally Hadamard differentiable at $\overline{x}$, it results that $g'(\overline{x}; .)$ is continuous on X (see [3, Theorem 3.2]). By letting $m \to +\infty$, it follows from (13) that

$$g'(\overline{x}; v_0) \in -S_{g(\overline{x})}.$$

Consequently,

$$v_0 \in K_C(\overline{x}) \cap \Big\{ u : -g'(\overline{x}; u) \in S_{g(\overline{x})} \Big\} \setminus \{0\}.$$

In view of (4.3), we get

$$f_{i_0}(\overline{x} + t_m v_m) \leqslant f_{i_0}(\overline{x}) + \frac{t_m^k}{m} \|v_m\|^n \quad (k = 1, \ldots, n). \tag{4.5}$$

Now if v_0 satisfies the following conditions:

$$f^{(0)}_{i_0,-}(\overline{x}; v_0) \geqslant f_{i_0}(\overline{x}), \quad f^{(j)}_{i_0,-}(\overline{x}; v_0) \geqslant 0 \quad (j = 1, \ldots, k-1),$$

then these along with (4.5) yields that

$$\begin{aligned} f^{k)}_{i_0,-}(\overline{x}; v_0) &\leqslant \liminf_{m \to +\infty} \frac{k!}{t_m^k} \Big[f_{i_0}(\overline{x} + t_m v_m) - \sum_{j=0}^{k-1} \frac{t_m^j}{j!} f^{(j)}_{i_0,-}(\overline{x}; v_0) \Big] \\ &\leqslant \liminf_{m \to +\infty} \frac{k!}{t_m^k} \big[f_{i_0}(\overline{x} + t_m v_m) - f_{i_0}(\overline{x}) \big] \\ &\leqslant \liminf_{m \to +\infty} \frac{k!}{t_m^k} \cdot \frac{t_m^k}{m} \cdot \|v_m\|^n = 0, \end{aligned}$$

which is in contradiction to condition (B_k). Hence, $\overline{x}$ is a strict local minimum of order n of Problem (P_{i_0}).

Let us show that $\overline{x}$ is a strict local Pareto minimum of order n of Problem (MP). If it were false, then according to Proposition 3.4 [6], there would exist a sequence $x_m \in M$, $x_m \neq \overline{x}$ and $b_m = (b_{m_1}, \ldots, b_{m_r}) \in \mathbb{R}^r_+$ such that $x_m \to \overline{x}$, and

$$\lim_{m \to +\infty} \frac{f(x_m) - f(\overline{x}) + b_m}{\|x_m - \overline{x}\|^n} = 0,$$

which implies that

$$\lim_{m \to +\infty} \frac{f_{i_0}(x_m) - f_{i_0}(\overline{x}) + b_{m_{i_0}}}{\|x_m - \overline{x}\|^n} = 0.$$

Taking account of Proposition 3.4 [6] once more, we claim that $\overline{x}$ is not a strict local minimum of order n of (P_{i_0}), which is in contradiction to the above proof. $\square$

Remark 4.3. *Theorems 4.1 and 4.2 obtained here are generalizations of Theorems 4.1 and 5.2 [9], respectively.*

We close this paper with an example which will be illustrated Theorem 4.2.

Example 4.4. *Let* $X = Y = \mathbb{R}^2$, $Q = S = \mathbb{R}^2_+$, $\overline{x} = (0,0)$, $C = [-1,1] \times [-1,0]$. *Let* f *and* g *be given by*

$$\begin{aligned} f(x) &= (f_1(x), f_2(x)), \\ f_1(x) &= \sum_{i=0}^{p} \alpha_i \|x\|^{k+i} \quad (x = (x_1, x_2) \in \mathbb{R}^2), \\ f_2(x) &= -|\sin x|, \\ g(x) &= (x_1^2 - x_1, x_2^2 - |x_2|), \end{aligned}$$

where k *and* p *are positive integer numbers,* $\|x\| = \big(|x_1|^2 + |x_2|^2\big)^{1/2}$. *Then* $\overline{x}$ *is a feasible point of the following problem:*

$$\min \Big\{ f(x) : -g(x) \in \mathbb{R}^2_+, x \in C \Big\}. \tag{4.6}$$

We can see that $K_C(\overline{x}) = \mathbb{R} \times \mathbb{R}_-$, *and*

$$\Big\{ u \in \mathbb{R}^2 : -g'(\overline{x}; u) \in \mathbb{R}^2_+ \Big\} = \mathbb{R}_+ \times \mathbb{R}.$$

For $v \in K_C(\overline{x}) \cap \{u : -g'(\overline{x}; u) \in \mathbb{R}^2_+\} = \mathbb{R}_+ \times \mathbb{R}_-$, *we have*

$$\begin{aligned} & f_{1,-}^{(0)}(0; v) = f_1(0), \quad f_{1,-}^{(j)}(0, v) = 0 \quad (j = 1, \ldots, k-1), \\ & f_{1,-}^{(k)}(0; v) = \alpha_0 \|v\|^k > 0. \end{aligned}$$

By Theorem 4.2, the point $\overline{x} = (0,0)$ *is a strict local Pareto minimum of order* k *(with respect to the cone* $Q = \mathbb{R}^2_+$*) of Problem (4.6).*

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