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APPROXIMATION OF COMMON FIXED POINT OF THREE MULTI-VALUED ρ-QUASI-NONEXPANSIVE MAPPINGS IN MODULAR FUNCTION SPACES

Godwin Amechi Okeke[1] and Jong Kyu Kim[2]

[1]Department of Mathematics, School of Physical Sciences
Federal University of Technology, Owerri
P.M.B. 1526 Owerri, Imo State, Nigeria
e-mail: gaokeke1@yahoo.co.uk

[2]Department of mathematics Education Kyungnam University
Changwon, Gyeongnam 51767, Korea
e-mail: jongkyuk@kyungnam.ac.kr

Abstract. It is our purpose to continue the study of approximation of fixed point of multi-valued nonlinear mappings in a modular function space which is initiated by Khan and Abbas [14]. Some convergence results were established for three multi-valued ρ-quasi-nonexpansive mappings using a three step iterative scheme.

1. Introduction

In 2014, Khan and Abbas [14] initiated the study of approximating fixed points of multi-valued nonlinear mappings in modular function spaces. The purpose of this paper is to continue this recent trend in the study of fixed point theory of multi-valued nonlinear mappings in modular function spaces. For over a century now, the study of fixed point theory of multi-valued nonlinear mappings has attracted the interest of many well-known mathematicians and mathematical scientists (see [1], [4], [6], [9], [11], [22], [24], [25]).

The theory of modular spaces had been initiated in 1950 by Nakano [23] in connection with the theory of ordered spaces which was further generalized by

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[0]Corresponding author: J.K.Kim(jongkyuk@kyungnam.ac.kr).

Musielak and Orlicz [21]. Modular function spaces are natural generalizations of both function and sequence variants of several important, from application perspective, spaces like Musielak-Orlicz, Orlicz, Lorentz, Orlicz-Lorentz, Kothe, Lebesgue, Calderon-Lozanovskii spaces and several others. Interest in quasi-nonexpansive mappings in modular function spaces stems mainly in the richness of structure of modular function spaces, that besides, being Banach spaces (or F-spaces in a more general settings), are equipped with modular equivalents of norm or metric notions and also equipped with almost everywhere convergence and convergence in submeasure. It is known that modular type conditions are much more natural as modular type assumptions can be more easily verified than their metric or norm counterparts, particulary in applications to integral operators, approximation and fixed point results. Moreover, there are certain fixed point results that can be proved only using the apparatus of modular function spaces. Hence, fixed point theory results in modular function spaces, in this perspective, should be considered as complementary to the fixed point theory in normed and metric spaces (see [8], [12], [17], [19]).

Fixed point point theory in modular function spaces has attracted the interest of many mathematicians. Several authors have proved the very interesting fixed point results in the framework of modular function spaces, (see [3], [5], [10], [12], [13], [20]). Abbas *et al.* [2] proved the existence and uniqueness of common fixed point of certain nonlinear mappings satisfying some contractive conditions in partially ordered modular function spaces. Öztürk, Abbas and Girgin [28] established some interesting fixed point results of nonlinear mappings satisfying integral type contractive conditions in the framework of modular spaces endowed with a graph. Recently, Khan and Abbas initiated the study of approximating fixed points of multi-valued nonlinear mappings in the framework of modular function spaces [14]. Abbas and Ali [15] used a three step iterative scheme to approximate the fixed point of multi-valued ρ-quasi-nonexpansive mappings in modular function spaces. Rafiq [29] introduced the modified Noor iterative scheme, which was extensively studied by several authors, (see, e.g. Fukhar-ud-din and Khan [7], Xue and Fan [33], Okeke and Akewe [26], Okeke and Olaleru [27]).

In this paper, we introduce a modular version of the Noor iterative scheme and approximate the common fixed point of three ρ-quasi-nonexpansive mappings in the framework of modular function spaces.

2. Preliminaries

Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a δ-ring of subsets of Ω such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and

$A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$ (for instance, $\mathcal{P}$ can be the class of sets of finite measure in a σ-finite measure space). By 1_A, we denote the characteristic function of the set A in Ω. By ε, we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}_\infty$, we denote the space of all extended measurable functions, i.e., any function $f : \Omega \to [-\infty, \infty]$ such that there exists a sequence $\{g_n\} \subset \varepsilon$, $|g_n| \leq |f|$ and $g_n(\omega) \to f(\omega)$ for each $\omega \in \Omega$.

Definition 2.1. Let $\rho : M_\infty \to [0, \infty]$ be a nontrivial, convex and even function. We say that ρ is a regular convex function pseudomodular if

(1) $\rho(0) = 0$;
(2) ρ is monotone, i.e., $|f(\omega)| \leq |g(\omega)|$ for any $\omega \in \Omega$ implies $\rho(f) \leq \rho(g)$, where $f, g \in \mathcal{M}_\infty$;
(3) ρ is orthogonally subadditive, i.e., $\rho(f1_{A\cup B}) \leq \rho(f1_A) + \rho(f1_B)$ for any $A, B \in \Sigma$ such that $A \cap B \neq \emptyset$, $f \in \mathcal{M}_\infty$;
(4) ρ has Fatou property, i.e., $|f_n(\omega)| \uparrow |f(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{M}_\infty$;
(5) ρ is order continuous in ε, i.e., $g_n \in \varepsilon$ and $|g_n(\omega)| \downarrow 0$ implies $\rho(g_n) \downarrow 0$.

A set $A \in \Sigma$ is said to be ρ-null if $\rho(g1_A) = 0$ for every $g \in \varepsilon$. A property $p(\omega)$ is said to be hold ρ-almost everywhere (ρ-a.e.) if the set $\{\omega \in \Omega : p(\omega)$ does not hold$\}$ is ρ-null. As usual, we identify any pair of measurable sets whose symmetric difference is ρ-null as well as any pair of measurable functions differing only on a ρ-null set. With this in mind we define

$$\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho) = \{f \in \mathcal{M}_\infty : |f(\omega)| < \infty \ \rho\text{-}a.e.\},$$

where $f \in \mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$ is actually an equivalence class of functions equal ρ-a.e. rather than an individual function. If there is no confusion, we shall write $\mathcal{M}$ instead of $\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$.

The following definitions were given in [14].

Definition 2.2. ([14]) Let ρ be a regular function pseudomodular. We say that ρ is a regular convex function modular if $\rho(f) = 0$ implies $f = 0$ ρ-a.e.

It is known that ρ satisfies the following properties (see [18]):

(1) $\rho(0) = 0$ if and only if $f = 0$ ρ-a.e.
(2) $\rho(\alpha f) = \rho(f)$ for every scalar α with $|\alpha| = 1$ and $f \in \mathcal{M}$.
(3) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ if $\alpha + \beta = 1$, $\alpha, \beta \geq 0$ and $f, g \in \mathcal{M}$.

The function ρ is called a convex modular if, in addition, the following property is satisfied:

(3′) $\rho(\alpha f + \beta g) \leq \alpha\rho(f) + \beta\rho(g)$, for $\alpha + \beta = 1$, $\alpha, \beta \geq 0$ and $f, g \in \mathcal{M}$.

Definition 2.3. ([14]) The convex function modular ρ defines the modular function space L_ρ as

$$L_\rho = \{f \in \mathcal{M}; \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\}.$$

Generally, the modular ρ is not subadditive and therefore does not behave as a norm or a distance. However, the modular space L_ρ can be equipped with an F-norm defined by

$$\|f\|_\rho = \inf\left\{\alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \le \alpha\right\}.$$

In the case ρ is convex modular,

$$\|f\|_\rho = \inf\left\{\alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \le 1\right\}$$

defines a norm on the modular space L_ρ, and it is called the Luxemburg norm.

The following uniform convexity type properties of ρ can be found in [5].

Definition 2.4. ([5]) Let ρ be a nonzero regular convex function modular defined on Ω. For $t \in (0,1)$, $r > 0$, $\epsilon > 0$, define

$$D(r_1, \epsilon) = \{(f,g) : f, g \in L_\rho, \rho(f) \le r, \rho(g) \le r, \rho(f-g) \ge \epsilon r\}.$$

Let

$$\delta_1^t(r,\epsilon) = \inf\left\{1 - \frac{1}{r}\rho(tf + (1-t)g) : (f,g) \in D(r_1,\epsilon)\right\}, \text{ if } D(r_1,\epsilon) \neq \emptyset,$$

and $\delta_1(r,\epsilon) = 1$, if $D(r_1,\epsilon) = \emptyset$. As a conventional notation, $\delta_1 = \delta_1^{\frac{1}{2}}$.

Definition 2.5. A nonzero regular convex function modular ρ is said to satisfy $(UC1)$ if for every $r > 0$, $\epsilon > 0$, $\delta_1(r,\epsilon) > 0$. Note that for every $r > 0$, $D_1(r,\epsilon) \neq \emptyset$ for $\epsilon > 0$ small enough. ρ is said to satisfy $(UUC1)$ if for every $s \ge 0$, $\epsilon > 0$, there exists $\eta_1(s,\epsilon) > 0$ depending only upon s and ϵ such that $\delta_1(r,\epsilon) > \eta_1(s,\epsilon) > 0$ for any $r > s$.

Definition 2.6. Let L_ρ be a modular space. The sequence $\{f_n\} \subset L_\rho$ is said to be:

(1) ρ-convergent to $f \in L_\rho$ if $\rho(f_n - f) \to 0$ as $n \to \infty$;
(2) ρ-Cauchy, if $\rho(f_n - f_m) \to 0$ as n and $m \to \infty$.

Kilmer *et al.* [16] defined ρ-distance from $f \in L_\rho$ to a set $D \subset L_\rho$ as follows:

$$\mathrm{dist}_\rho(f, D) = \inf\{\rho(f-h) : h \in D\}.$$

Definition 2.7. A subset $D \subset L_\rho$ is said to be:

(1) ρ-closed if the ρ-limit of a ρ-convergent sequence of D always belongs to D;
(2) ρ-a.e. closed if the ρ-a.e. limit of a ρ-a.e. convergent sequence of D always belongs to D;
(3) ρ-compact if every sequence in D has a ρ-convergent subsequence in D;
(4) ρ-a.e. compact if every sequence in D has a ρ-a.e. convergent subsequence in D;
(5) ρ-bounded if

$$\operatorname{diam}_\rho(D) = \sup\{\rho(f-g) : f, g \in D\} < \infty.$$

A set $D \subset L_\rho$ is called ρ-proximinal if for each $f \in L_\rho$ there exists an element $g \in D$ such that $\rho(f-g) = \operatorname{dist}_\rho(f, D)$. We shall denote the family of nonempty ρ-bounded ρ-proximinal subsets of D by $P_\rho(D)$, the family of nonempty ρ-closed ρ-bounded subsets of D by $C_\rho(D)$ and the family of ρ-compact subsets of D by $K_\rho(D)$. Let $H_\rho(.,.)$ be the ρ-Hausdorff distance on $C_\rho(L_\rho)$, that is,

$$H_\rho(A, B) = \max\left\{\sup_{f\in A} \operatorname{dist}_\rho(f, B), \sup_{g\in B} \operatorname{dist}_\rho(g, A)\right\}, \ A, B \subset C_\rho(L_\rho).$$

A multi-valued map $T : D \to C_\rho(L_\rho)$ is said to be:

(a) ρ-nonexpansive (see, e.g. Khan and Abbas [14]) if

$$H_\rho(Tf, Tg) \leq \rho(f-g), \ f, g \in D. \tag{2.1}$$

(b) ρ-quasi-nonexpansive mapping if

$$H_\rho(Tf, p) \leq \rho(f-p) \text{ for all } f \in D \text{ and } p \in F_\rho(T). \tag{2.2}$$

for all $f \in D$ and $p \in F_\rho(T)$, where $F_\rho(T)$ is the set of all fixed points of T, that is, $p \in Tp$.

A sequence $\{t_n\} \subset (0,1)$ is called bounded away from 0 if there exists $a > 0$ such that $t_n \geq a$ for every $n \in \mathbb{N}$. Similarly, $\{t_n\} \subset (0,1)$ is called bounded away from 1 if there exists $b < 1$ such that $t_n \leq b$ for every $n \in \mathbb{N}$.

The following lemma will be needed in this study.

Lemma 2.8. ([3]) *Let ρ be a function modular, $\{f_n\}$ and $\{g_n\}$ be two sequences in X_ρ. Then*

$$\lim_{n\to\infty} \rho(g_n) = 0 \implies \limsup_{n\to\infty} \rho(f_n + g_n) = \limsup_{n\to\infty} \rho(f_n)$$

and

$$\lim_{n\to\infty}\rho(g_n)=0 \implies \liminf_{n\to\infty}\rho(f_n+g_n)=\liminf_{n\to\infty}\rho(f_n).$$

Lemma 2.9. ([5]) *Let ρ satisfy $(UUC1)$ and let $\{t_k\}\subset(0,1)$ be bounded away from 0 and 1. If there exists $R>0$ such that*

$$\limsup_{n\to\infty}\rho(f_n)\le R,\ \limsup_{n\to\infty}\rho(g_n)\le R$$

and

$$\lim_{n\to\infty}\rho(t_nf_n+(1-t_n)g_n)=R,$$

then $\lim_{n\to\infty}\rho(f_n-g_n)=0$.

The above lemma is an analogue of a famous Lemma due to Schu [31] in Banach spaces.

A function $f\in L_\rho$ is called a fixed point of $T:L_\rho\to P_\rho(D)$ if $f\in Tf$. The set of all fixed points of T will be denoted by $F_\rho(T)$.

Khan and Abbas [14] proved the following lemma.

Lemma 2.10. ([14]) *Let $T:D\to P_\rho(D)$ be a multi-valued mapping and*

$$P_\rho^T(f)=\{g\in Tf:\rho(f-g)=dist_\rho(f,Tf)\}.$$

Then the following statements are equivalent.

(1) *$f\in F_\rho(T)$, that is, $f\in Tf$.*
(2) *$P_\rho^T(f)=\{f\}$, that is, $f=g$ for each $g\in P_\rho^T(f)$.*
(3) *$f\in F(P_\rho^T(f))$, that is, $f\in P_\rho^T(f)$. Further $F_\rho(T)=F(P_\rho^T(f))$ where $F(P_\rho^T(f))$ is the set of fixed points of $P_\rho^T(f)$.*

Definition 2.11. A multi-valued mapping $T:D\to C_\rho(D)$ is said to satisfy condition (I) if there exists a nondecreasing function $l:[0,\infty)\to[0,\infty)$ with $l(0)=0$, $l(r)>0$ for all $r\in(0,\infty)$ such that

$$\mathrm{dist}_\rho(f,Tf)\ge l(\mathrm{dist}_\rho(f,F_\rho(T)))$$

for all $f\in D$.

It is a multi-valued version of condition (I) of Senter and Dotson [32] in the framework of modular function spaces.

The following examples were presented by Razani *et al.* [30].

Example 2.12. Let $(X,\|.\|)$ be a normed space. Then $\|\cdot\|$ is a modular on X. But the converse is not true.

Definition 2.13. Let $(X, \|.\|)$ be a normed space. For any $k \geq 1$, $\|\cdot\|^k$ is a modular on X.

3. Main results

We begin this section by introducing a three step iteration for approximating the common fixed point of three multi-valued ρ-quasi-nonexpansive mappings in modular function spaces.

Let $T_i : D \to C_\rho(L_\rho)$, $i = 1, 2, 3$ be three multi-valued ρ-quasi-nonexpansive mapping, we hereby introduce the following three-step iterative scheme.

$$\begin{cases} f_{n+1} = (1-\alpha_n)f_n + \alpha_n u_n \\ g_n = (1-\beta_n)f_n + \beta_n w_n \\ h_n = (1-\gamma_n)f_n + \gamma_n v_n \end{cases} \tag{3.1}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in $[0, 1]$ satisfying certain conditions, $u_n \in P_\rho^{T_1}(g_n)$, $w_n \in P_\rho^{T_2}(h_n)$, $v_n \in P_\rho^{T_3}(f_n)$,

$$P_\rho^{T_i}(f) = \{g \in T_i f : \rho(f-g) = \operatorname{dist}_\rho(f, T_i f)\}, \ i = 1, 2, 3. \tag{3.2}$$

The three-step iterative scheme (3.1) could be seen as the modular version of the modified Noor iterative scheme introduced by Rafiq [29] and extensively studied by several authors (see, e.g. [26], [27], [33]).

The following lemma which could be seen as an extension of Lemma 2 of Khan and Abbas [14] will be needed for the proof of main theorems.

Lemma 3.1. *Let $T_i : D \to P_\rho(D)$ be three multi-valued mappings and*

$$P_\rho^{T_i}(f) = \{g \in T_i f : \rho(f-g) = dist_\rho(f, T_i f)\}, \ i = 1, 2, 3. \tag{3.3}$$

Then the following statements are equivalent:

(1) *$f \in \bigcap_{i=1}^3 F_\rho(T_i)$, that is, $f \in \bigcap_{i=1}^3 T_i f$,*
(2) *$\bigcap_{i=1}^3 P_\rho^{T_i}(f) = \{f\}$, that is, $f = g$ for each $g \in \bigcap_{i=1}^3 P_\rho^{T_i}(f)$.*
(3) *$f \in \bigcap_{i=1}^3 F(P_\rho^{T_i}(f))$, that is, $f \in \bigcap_{i=1}^3 P_\rho^{T_i}(f)$. And also, we have $\bigcap_{i=1}^3 F_\rho(T_i) = \bigcap_{i=1}^3 F(P_\rho^{T_i}(f))$ where $F(P_\rho^{T_i}(f))$ is the set of fixed points of $P_\rho^{T_i}(f)$, $i = 1, 2, 3$.*

Proof. We adopt the similar method of the proof of Khan and Abbas [14].
(1) $\implies$ (2). Let $f \in \bigcap_{i=1}^3 F_\rho(T_i)$. Then $f \in \bigcap_{i=1}^3 T_i f$, this means that $dist_\rho(f, T_i f) = 0$. Hence, for any $g \in P_\rho^{T_i}(f)$, $i = 1, 2, 3$, $\rho(f-g) = dist_\rho(f, T_i f)$ $= 0$ implies that $\rho(f-g) = 0$. This means that $f = g$. Hence, $P_\rho^{T_i}(f) = \{f\}$, $i = 1, 2, 3$.
(2) $\implies$ (3). It is easy to prove this implication.

(3) $\implies$ (1). Recall $f \in \bigcap_{i=1}^{3} F(P_\rho^{T_i}(f))$ and by definition of $P_\rho^{T_i}(f)$ as in (3.3), we obtain $dist_\rho(f, T_i f) = \rho(f-f) = 0$. Thus, $f \in \bigcap_{i=1}^{3} T_i f$ by ρ-closedness of $T_i f$, $(i = 1, 2, 3)$. □

Next, we establish the following results. We shall adopt the method of proof of Abbas and Ali [15].

Theorem 3.2. *Let ρ satisfy $(UUC1)$ and D a nonempty $\rho-$closed, $\rho-$bounded and convex subset of L_ρ. Let $T_i : D \to P_\rho(D)$, $(i = 1, 2, 3)$ be three multi-valued mappings such that $P_\rho^{T_i}$, $(i = 1, 2, 3)$ are $\rho-$quasi-nonexpansive mappings. Suppose that*

$$\bigcap_{i=1}^{3} F_\rho(T_i) = F_\rho(T_1) \cap F_\rho(T_2) \cap F_\rho(T_3) \neq \emptyset.$$

Let $\{f_n\} \subset D$ be defined by three step iterative process (3.1), where $u_n \in P_\rho^{T_1}(g_n)$, $w_n \in P_\rho^{T_2}(h_n)$, $v_n \in P_\rho^{T_3}(f_n)$, $0 < \alpha_n, \beta_n, \gamma_n < 1$. Then, for all $p \in F_\rho(T_1) \cap F_\rho(T_2) \cap F_\rho(T_3)$, there exists the limit

$$\lim_{n\to\infty} \rho(f_n - p)$$

such that

$$\lim_{n\to\infty} dist_\rho(f_n, P_\rho^{T_i}(f_n)) = 0, \ (i = 1, 2, 3).$$

Proof. From Lemma 3.1, we know that for $p \in F_\rho(T_1) \cap F_\rho(T_2) \cap F_\rho(T_3)$, $P_\rho^{T_i}(p) = \{p\}$ and $\bigcap_{i=1}^{3} F_\rho(T_i) = \bigcap_{i=1}^{3} F(P_\rho^{T_i})$, $i = 1, 2, 3$. Since

$$\begin{aligned} \rho(f_{n+1} - p) &= \rho[(1-\alpha_n) f_n + \alpha_n u_n - p] \\ &= \rho[(1-\alpha_n)(f_n - p) + \alpha_n (u_n - p)]. \end{aligned} \tag{3.4}$$

By the convexity of ρ, we have

$$\begin{aligned} \rho(f_{n+1} - p) &\le (1-\alpha_n)\rho(f_n - p) + \alpha_n \rho(u_n - p) \\ &\le (1-\alpha_n)\rho(f_n - p) + \alpha_n H_\rho(P_\rho^{T_1}(g_n), P_\rho^{T_1}(p)) \\ &\le (1-\alpha_n)\rho(f_n - p) + \alpha_n \rho(g_n - p). \end{aligned} \tag{3.5}$$

Next, since

$$\rho(g_n - p) = \rho[(1-\beta_n) f_n + \beta_n w_n - p],$$

from the convexity of ρ, we obtain

$$\begin{aligned} \rho(g_n - p) &= \rho[(1-\beta_n) f_n + \beta_n w_n - p] \\ &\le (1-\beta_n)\rho(f_n - p) + \beta_n \rho(w_n - p) \\ &\le (1-\beta_n)\rho(f_n - p) + \beta_n H_\rho(P_\rho^{T_2}(h_n) - P_\rho^{T_2}(p)) \\ &\le (1-\beta_n)\rho(f_n - p) + \beta_n \rho(h_n - p). \end{aligned} \tag{3.6}$$

And also, since

$$\rho(h_n - p) = \rho[(1-\gamma_n)f_n + \gamma_n v_n - p],$$

from the convexity of ρ, we obtain

$$\begin{aligned}\rho(h_n - p) &\le (1-\gamma_n)\rho(f_n - p) + \gamma_n \rho(v_n - p)\\ &\le (1-\gamma_n)\rho(f_n - p) + \gamma_n H_\rho(P_\rho^{T_3}(f_n) - P_\rho^{T_3}(p))\\ &\le (1-\gamma_n)\rho(f_n - p) + \gamma_n \rho(f_n - p)\\ &= \rho(f_n - p).\end{aligned} \tag{3.7}$$

Using (3.6) and (3.7), we have

$$\begin{aligned}\rho(g_n - p) &\le (1-\beta_n)\rho(f_n - p) + \beta_n \rho(f_n - p)\\ &= \rho(f_n - p).\end{aligned} \tag{3.8}$$

Therefore,

$$\begin{aligned}\rho(f_{n+1} - p) &\le (1-\alpha_n)\rho(f_n - p) + \alpha_n \rho(f_n - p)\\ &= \rho(f_n - p).\end{aligned} \tag{3.9}$$

It has shown that $\lim_{n\to\infty} \rho(f_n - p)$ exist for each $p \in F_\rho(T_1) \cap F_\rho(T_2) \cap F_\rho(T_3)$.

Suppose that

$$\lim_{n\to\infty} \rho(f_n - p) = L. \tag{3.10}$$

Then we have to show that $\lim_{n\to\infty} dist_\rho(f_n, P_\rho^{T_i}(f_n)) = 0$, $i = 1, 2, 3$.

First, we show that $\lim_{n\to\infty} dist_\rho(f_n, P_\rho^{T_3}(f_n)) = 0$.
Since $dist_\rho(f_n, P_\rho^{T_3}(f_n)) \le \rho(f_n - v_n)$, it suffices to show that

$$\lim_{n\to\infty} \rho(f_n - v_n) = 0. \tag{3.11}$$

Since

$$\rho(v_n - p) \le H_\rho(P_\rho^{T_3}(f_n), P_\rho^{T_3}(p)) \le \rho(f_n - p), \tag{3.12}$$

it implies that

$$\limsup_{n\to\infty} \rho(v_n - p) \le \limsup_{n\to\infty} \rho(f_n - p). \tag{3.13}$$

Hence, from (3.10), we obtain

$$\limsup_{n\to\infty} \rho(v_n - p) \le L. \tag{3.14}$$

From (3.7), we have

$$\limsup_{n\to\infty} \rho(h_n - p) \le \limsup_{n\to\infty} \rho(f_n - p). \tag{3.15}$$

Hence, we have

$$\limsup_{n\to\infty} \rho(h_n - p) \le L. \tag{3.16}$$

Similarly, using (3.8) and (3.10), we have

$$\limsup_{n\to\infty} \rho(g_n - p) \leq L. \tag{3.17}$$

From the inequality

$$\rho(w_n - p) \leq H_\rho(P_\rho^{T_2}(h_n), P_\rho^{T_2}(p)) \leq \rho(h_n - p) \leq \rho(f_n - p), \tag{3.18}$$

it implies that

$$\limsup_{n\to\infty} \rho(w_n - p) \leq \limsup_{n\to\infty} \rho(f_n - p). \tag{3.19}$$

Hence, we have

$$\limsup_{n\to\infty} \rho(w_n - p) \leq L. \tag{3.20}$$

Similarly, we can get

$$\limsup_{n\to\infty} \rho(u_n - p) \leq L. \tag{3.21}$$

But

$$\begin{aligned} \lim_{n\to\infty} \rho(f_{n+1} - p) &= \lim_{n\to\infty} \rho[(1-\alpha_n)f_n + \alpha_n u_n - p] \\ &= \lim_{n\to\infty} \rho[(1-\alpha_n)(f_n - p) + \alpha_n(u_n - p)] \\ &= L. \end{aligned} \tag{3.22}$$

Using (3.20), (3.21), (3.22) and Lemma 2.9, we obtain

$$\lim_{n\to\infty} \rho(u_n - w_n) = 0. \tag{3.23}$$

From the inequality

$$\rho(v_n - p) \leq H_\rho(P_\rho^{T_3}(f_n), P_\rho^{T_3}(p)) \leq \rho(f_n - p), \tag{3.24}$$

this yields,

$$\rho(v_n - p) \leq \limsup_{n\to\infty} \rho(f_n - p). \tag{3.25}$$

Hence, we have

$$\limsup_{n\to\infty} \rho(v_n - p) \leq L. \tag{3.26}$$

Using (3.20), (3.21), (3.26) and Lemma 2.9, we obtain

$$\lim_{n\to\infty} \rho(v_n - w_n) = 0. \tag{3.27}$$

From the inequality

$$\rho(u_n - p) \leq H_\rho(P_\rho^{T_1}(g_n), P_\rho^{T_1}(p)) \leq \rho(g_n - p) \leq L, \tag{3.28}$$

we have,

$$\limsup_{n\to\infty} \rho(u_n - p) \leq L. \tag{3.29}$$

Using (3.25), (3.28) and Lemma 2.9, we have

$$\lim_{n\to\infty} \rho(f_n - v_n) = 0. \tag{3.30}$$

Therefore, we have

$$\lim_{n\to\infty} dist_\rho(f_n, P_\rho^{T_i}(f_n)) = 0, \ (i = 1, 2, 3). \tag{3.31}$$

This completes the proof. □

Next, we establish a convergence result for the approximation of the common fixed points of three multi-valued ρ-quasi-nonexpansive mappings in modular function spaces, using the three-step iterative scheme defined in (3.1).

Theorem 3.3. *Let ρ satisfy $(UUC1)$ and D be a nonempty ρ-compact, ρ-bounded and convex subset of L_ρ. Let $T_i : D \to P_\rho(D)$, $(i = 1, 2, 3)$ be three multi-valued mappings such that $P_\rho^{T_i}$, $(i = 1, 2, 3)$ are three ρ-quasi-nonexpansive mappings with*

$$\bigcap_{i=1}^{3} F_\rho(T_i) = F_\rho(T_1) \cap F_\rho(T_2) \cap F_\rho(T_3) \neq \emptyset.$$

Let $\{f_n\}$ be a sequence defined in Theorem 3.2. Then $\{f_n\}$ is ρ-convergent to a common fixed point of T_i, $i = 1, 2, 3$.

Proof. Using the $\rho-$compactness of D, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\lim_{n\to\infty}(f_{n_k} - q) = 0$ for some $q \in D$. To establish that q is a common fixed point of T_i, $(i = 1, 2, 3)$. Let g be an arbitrary point in $P_\rho^{T_1}(q) \cap P_\rho^{T_2}(q) \cap P_\rho^{T_3}(q)$ and $f \in P_\rho^{T_1}(f_{n_k}) \cap P_\rho^{T_2}(f_{n_k}) \cap P_\rho^{T_3}(f_{n_k})$. Following the proof of Theorem 2 of Abbas and Ali [15], we obtain $\rho(q - g) = 0$. Using Theorem 3.2, we have

$$q \in \bigcap_{i=1}^{3} F(P_\rho^{T_i}) = \bigcap_{i=1}^{3} F_\rho(T_i). \tag{3.32}$$

This implies that $\{f_n\}$ is $\rho-$convergent to the common fixed point of T_1, T_2 and T_3. □

Theorem 3.4. *Let ρ satisfy $(UUC1)$ and D be a nonempty ρ-closed, $\rho-$bounded and convex subset of L_ρ. Let $T_i : D \to P_\rho(D)$, $(i = 1, 2, 3)$ be three multi-valued mappings with*

$$\bigcap_{i=1}^{3} F_\rho(T_i) = F_\rho(T_1) \cap F_\rho(T_2) \cap F_\rho(T_3) \neq \emptyset$$

satisfying condition (I) *such that* $P_\rho^{T_i}$, $(i = 1, 2, 3)$ *are* $\rho-$*quasi-nonexpansive mappings. Let* $\{f_n\}$ *be a sequence defined in Theorem* 3.2. *Then* $\{f_n\}$ *is* $\rho-$*convergent to a common fixed point of* T_1, T_2 *and* T_3.

Proof. Using Theorem 3.2, $\lim_{n\to\infty} \rho(f_n - p)$ exists for all $p \in \bigcap_{i=1}^{3} F(P_\rho^{T_i}) = \bigcap_{i=1}^{3} F_\rho(T_i)$. If $\lim_{n\to\infty} \rho(f_n - p) = 0$, there is nothing to prove. Assume that $\lim_{n\to\infty} \rho(f_n - p) = L > 0$. Using Theorem 3.2, we have

$$\rho(f_{n+1} - p) \leq \rho(f_n - p). \tag{3.33}$$

Hence

$$dist_\rho(f_{n+1}, F_\rho(T_i)) \leq dist_\rho(f_n, F_\rho(T_i)),\ (i = 1, 2, 3). \tag{3.34}$$

So, this implies that $\lim_{n\to\infty} dist_\rho(f_n, F_\rho(T_i))$ exists.

Next, we show that $\lim_{n\to\infty} dist_\rho(f_n, F_\rho(T_i)) = 0$, $(i = 1, 2, 3)$. By using condition (I) and Theorem 3.2, we obtain

$$\lim_{n\to\infty} l(dist_\rho(f_n, F_\rho(T_i))) \leq \lim_{n\to\infty} dist_\rho(f_n, Tf_n) = 0. \tag{3.35}$$

This means that

$$\lim_{n\to\infty} l(dist_\rho(f_n, F_\rho(T_i)) = 0. \tag{3.36}$$

Using the fact that l is nondecreasing function and $l(0) = 0$, it follows that

$$\lim_{n\to\infty} dist_\rho(f_n, F_\rho(T_i)) = 0,\ i = 1, 2, 3.$$

Following the proof of Theorem 3 of Abbas and Ali [15], we see that $\{f_n\}$ is a $\rho-$Cauchy sequence in a $\rho-$closed subset D of L_ρ. Hence, it converges in D. Let $\lim_{n\to\infty} f_n = q$, we see that $q \in \cap_{i=1}^{3} F_\rho(T_i)$ follows from Theorem 3.2. This completes the proof. □

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