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FIXED POINT RESULTS ON S-METRIC SPACE VIA SIMULATION FUNCTIONS

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Abstract. In this paper, we prove some common fixed point theorems for two mappings satisfying generalized contractive condition in S-metric spaces via simulation functions. Our results extend and improve several previous well-known works.

1. Introduction and preliminaries

The Banach contraction principle is the most celebrated fixed point theorem and has been generalized in various directions. Fixed point problems for contractive mappings in metric spaces with a partially order have been studied by many authors (see [1], [2], [6]). In the present paper, we introduce

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the notion of S-metric spaces and give some properties of them (see [4], [9]). In addition, we give an illustrative example to support our results.

First we recall some notions, lemmas, and examples which will be useful later.

Definition 1.1. ([9]) Let X be a nonempty set. A function $S : X^3 \to [0, \infty)$ is said to be an S-metric on X, if for each $x, y, z, a \in X$,

(1) $S(x, y, z) \geq 0$,
(2) $S(x, y, z) = 0$ if and only if $x = y = z$,
(3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an *S-metric space.*

Example 1.2. ([9]) We can easily check that the following examples are S-metric spaces.

(1) Let $X = \mathbb{R}^n$ and $||\cdot||$ a norm on X. Then $S(x, y, z) = ||y + z - 2x|| + ||y - z||$ is an S-metric on X. In general, if X is a vector space over $\mathbb{R}$ and $||\cdot||$ a norm on X, then it is easy to see that

$$S(x, y, z) = ||\alpha y + \beta z - \lambda x|| + ||y - z||,$$

where $\alpha + \beta = \lambda$ for every $\alpha, \beta \geq 1$, is an S-metric on X.

(2) Let X be a nonempty set and d_1, d_2 be two ordinary metrics on X. Then

$$S(x, y, z) = d_1(x, z) + d_2(y, z),$$

is an S-metric on X.

Lemma 1.3. ([7]) *Let (X, S) be an S-metric space. Then, we have $S(x, x, y) = S(y, y, x)$, for all $x, y \in X$.*

Definition 1.4. ([8]) Let (X, S) be an S-metric space and $A \subset X$.

(1) A sequence $\{x_n\}$ in X converges to x if $S(x_n, x_n, x) \to 0$ as $n \to \infty$, that is for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $S(x_n, x_n, x) < \varepsilon$. This case, we denote by $\lim_{n\to\infty} x_n = x$ and we say that x is the limit of $\{x_n\}$ in X.
(2) A sequence $\{x_n\}$ in X is said to be Cauchy if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \varepsilon$ for each $n, m \geq n_0$.
(3) The S-metric space (X, S) is said to be complete if every Cauchy sequence is convergent.

Definition 1.5. ([5]) Let (X, S) and $(X\,', S\,')$ be two S-metric spaces, and let $f : (X, S) \to (X\,', S\,')$ be a function. Then f is said to be continuous at a point $a \in X$ if for every sequence $\{x_n\}$ in X, $S(x_n, x_n, a) \to 0$ implies

$S'(f(x_n), f(x_n), f(a)) \to 0$. A function f is continuous on X if it is continuous at all $a \in X$.

Lemma 1.6. ([8]) *Let (X, S) be an S- metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then*

$$\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y).$$

Definition 1.7. Let (X, S) be an S-metric space. A pair $\{f, g\}$ is said to be compatible if $\lim\limits_{n\to\infty} S(fgx_n, fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n = t$ for some $t \in X$.

Lemma 1.8. *Let (X, S) be an $S-$metric space. If there exists two sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} S(x_n, x_n, y_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} x_n = t$ for some $t \in X$, then $\lim_{n\to\infty} y_n = t$.*

Proof. By the triangle inequality in $S-$metric space, we have

$$S(y_n, y_n, t) \leq 2S(y_n, y_n, x_n) + S(t, t, x_n).$$

Now, by taking the upper limit as $n \to \infty$ in above inequality we get

$$\limsup_{n\to\infty} S(y_n, y_n, t) \leq 2 \limsup_{n\to\infty} S(y_n, y_n, x_n) + \limsup_{n\to\infty} S(t, t, x_n) = 0.$$

Hence $\lim_{n\to\infty} y_n = t$. □

Definition 1.9. A function $\xi : [0, \infty[\times[0, \infty[\longrightarrow \mathbb{R}$ is said to be a simulation function if it satisfies the following conditions:

(ξ_1) $\xi(t, s) \leq s - t$, for all $t, s \geq 0$,

(ξ_2) if $\{t_n\}$, $\{s_n\}$ are sequences in $]0, \infty[$ such that

$$0 < \lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n < \infty,$$

then

$$\limsup_{n\to\infty} \xi(t_n, s_n) < 0.$$

Following are some examples of simulation functions [3].

Example 1.10. Let $\xi : [0, \infty[\times[0, \infty[\longrightarrow \mathbb{R}$, be defined by

(i) $\xi(t, s) = \lambda s - t$ for all $t, s \in [0, \infty[$, where $\lambda \in [0, 1[$.

(ii) $\xi(t, s) = \psi(s) - \varphi(t)$ for all $t, s \in [0, \infty[$, where $\varphi, \psi : [0, \infty[\longrightarrow [0\infty[$ are two continuous functions such that $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \leq \varphi(t)$ for all $t > 0$.

(iii) $\xi(t,s) = s\frac{f(t,s)}{g(t,s)}t$ for all $t,s \in [0,\infty[$, where $f,g : [0,\infty[\times[0,\infty[\longrightarrow]0,\infty[$ are two continuous functions with respect to each variable such that $f(t,s) > g(t,s)$ for all $t,s > 0$.

(iv) $\xi(t,s) = s - \varphi(s) - t$ for all $t,s \in [0,\infty[$, where $\varphi : [0,\infty[\longrightarrow [0,\infty[$ is a lower semi-continuous function such that $\varphi(t) = 0$ if and only if $t = 0$.

(v) $\xi(t,s) = s\varphi(s) - t$ for all $t,s \in [0,\infty[$, where $\varphi : [0,\infty[\longrightarrow [0,\infty[$ is such that $\lim\limits_{t\to r^+} \varphi(t) < 1$ for all $r > 0$.

Each of the function considered in (i)-(v) is a simulation function.

2. Fixed points via simulation functions in S-metric

The following lemmas, are needed to establish the main result.

Lemma 2.1. *Let (X,S) be an S-metric space and let $f,g : X \longrightarrow X$ be two mappings. Suppose that there exists a simulation function ξ such that*

$$\xi(S(fx,fy,fz), S(gx,gy,gz)) \geq 0, \forall\, x,y,z \in X. \tag{2.1}$$

Let $f(X) \subseteq g(X)$. Then there exists $\{y_n\}$ a sequence in X such that

$$\lim_{n\to\infty} S(y_{n-1}, y_{n-1}, y_n) = 0.$$

Proof. Let $x_0 \in X$. Since $f(X) \subseteq g(X)$, for every $n \in \mathbb{N}$ we have $y_n = f(x_n) = g(x_{n+1})$. If there exists $n_0 \in \mathbb{N}$ such that $y_{n_0} = y_{n_0+1}$, then it follows from (2.1) and (ξ_1) that for all $n \in \mathbb{N}$, we have

$$\begin{aligned} 0 &\leq \xi(S(fx_{n_0+1}, fx_{n_0+1}, fx_{n_0+2}), S(gx_{n_0+1}, gx_{n_0+1}, gx_{n_0+2})) \\ &= \xi(S(y_{n_0+1}, y_{n_0+1}, y_{n_0+2}), S(y_{n_0}, y_{n_0}, y_{n_0+1})) \\ &\leq S(y_{n_0}, y_{n_0}, y_{n_0+1}) - S(y_{n_0+1}, y_{n_0+1}, y_{n_0+2}). \end{aligned}$$

Since $S(y_{n_0}, y_{n_0}, y_{n_0+1}) = 0$, the above inequality shows that

$$S(y_{n_0+1}, y_{n_0+1}, y_{n_0+2}) = 0,$$

therefore $y_{n_0+1} = y_{n_0+2}$. Thus, $y_{n_0} = y_{n_0+1} = y_{n_0+2} = \cdots$, which implies that

$$\lim_{n\to\infty} S(y_{n-1}, y_{n-1}, y_n) = 0.$$

Now, suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. Then, it follows from (2.1) and (ξ_1) that for all $n \in \mathbb{N}$, we have

$$\begin{aligned} 0 &\leq \xi(S(fx_n, fx_n, fx_{n+1}), S(gx_n, gx_n, gx_{n+1})) \\ &= \xi(S(y_n, y_n, y_{n+1}), S(y_{n-1}, y_{n-1}, y_n)) \\ &\leq S(y_{n-1}, y_{n-1}, y_n) - S(y_n, y_n, y_{n+1}). \end{aligned}$$

The above inequality shows that

$$S(y_n, y_n, y_{n+1}) \leq S(y_{n-1}, y_{n-1}, y_n), \ \ \forall\, n \in \mathbb{N},$$

which implies that $\{S(y_{n-1}, y_{n-1}, y_n)\}$ is a decreasing sequence of positive real numbers. So there is some $r \geq 0$ such that $\lim_{n\to\infty} S(y_{n-1}, y_{n-1}, y_n) = r$. Suppose that $r > 0$. It follows from the condition (ξ_2), with $t_n = S(y_n, y_n, y_{n+1})$ and $s_n = S(y_{n-1}, y_{n-1}, y_n)$, that

$$0 \leq \limsup_{n\longrightarrow\infty} \xi(S(y_n, y_n, y_{n+1}), S(y_{n-1}, y_{n-1}, y_n) \ < 0,$$

which is a contradiction. Then we conclude that $r = 0$, which ends the proof. □

Lemma 2.2. *Let (X, S) be an $S-$metric space and let $f, g : X \longrightarrow X$ be two mappings. Suppose that there exists a simulation function ξ such that* (2.1) *holds. Let $f(X) \subseteq g(X)$. Then there exists $\{y_n\}$ be a sequence in X such that*

$$S(y_m, y_m, y_n) \leq S(y_{m-1}, y_{m-1}, y_{n-1}).$$

Proof. By similar argument of Lemma 2.1, for every $n \in \mathbb{N}$ we have $y_n = f(x_n) = g(x_{n+1})$. Hence, it follows from (2.1) and (ξ_1) that for all $m, n \in \mathbb{N}$, we have

$$\begin{aligned} 0 &\leq \xi(S(fx_m, fx_m, fx_n), S(gx_m, gx_m, gx_n)) \\ &= \xi(S(y_m, y_m, y_n), S(y_{m-1}, y_{m-1}, y_{n-1})) \\ &\leq S(y_{m-1}, y_{m-1}, y_{n-1}) - S(y_m, y_m, y_n). \end{aligned}$$

The above inequality shows that

$$S(y_m, y_m, y_n) \leq S(y_{m-1}, y_{m-1}, y_{n-1}), \ \forall\ m, n \in \mathbb{N}.$$

□

Lemma 2.3. *Let (X, S) be an S-metric space and let $f, g : X \longrightarrow X$ be two mappings. Suppose that there exists a simulation function ξ such that* (2.1) *holds. Let $f(X) \subseteq g(X)$. Then there exists a bounded sequence $\{y_n\}$ in X.*

Proof. By similar argument of Lemma 2.1, for every $n \in \mathbb{N}$ we have $y_n = f(x_n) = g(x_{n+1})$. If there exists $n_0 \in \mathbb{N}$ such that $y_{n_0} = y_{n_{0+1}}$, we set

$$M = \max\{S(y_i, y_i, y_j) : i, j \leq n_0\},$$

then in this case for all $i, j = 0, 1, 2, \cdots$ we have $S(y_i, y_i, y_j) \leq M$. Let us assume that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$ and $\{y_n\}$ is not a bounded sequence. Then, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $n_1 = 1$ and for each $k \in \mathbb{N}$, n_{k+1} is the minimum integer such that $S(y_{n_k+1}, y_{n_k+1}, y_{n_k}) > 1$ and

$$S(y_m, y_m, y_{n_k}) \leq 1, \ \ for\ n_k \leq m \leq n_{k+1} - 1.$$

By the triangle inequality, we obtain

$$\begin{aligned} 1 &< S(y_{n_{k+1}}, y_{n_{k+1}}, y_{n_k}) \\ &\leq 2S(y_{n_{k+1}}, y_{n_{k+1}}, y_{n_{k+1}-1}) + S(y_{n_{k+1}-1}, y_{n_{k+1}-1}, y_{n_k}) \\ &\leq 2S(y_{n_{k+1}}, y_{n_{k+1}}, y_{n_{k+1}-1}) + 1. \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using Lemma 2.1, we get

$$1 \leq \liminf_{k\to\infty} S(y_{n_{k+1}}, y_{n_{k+1}}, y_{n_k}) \leq \limsup_{k\to\infty} S(y_{n_{k+1}}, y_{n_{k+1}}, y_{n_k}) \leq 1.$$

Hence, we have

$$\lim_{k\to\infty} S(y_{n_{k+1}}, y_{n_{k+1}}, y_{n_k}) = 1. \tag{2.2}$$

Again, from Lemma 2.2, we have

$$\begin{aligned} S(y_{n_{k+1}}, y_{n_{k+1}}, y_{n_k}) &\leq S(y_{n_{k+1}-1}, y_{n_{k+1}-1}, y_{n_k-1}) \\ &= S(y_{n_k-1}, y_{n_k-1}, y_{n_{k+1}-1}) \\ &\leq 2S(y_{n_k-1}, y_{n_k-1}, y_{n_k}) + S(y_{n_{k+1}-1}, y_{n_{k+1}-1}, y_{n_k}) \\ &\leq 2S(y_{n_k-1}, y_{n_k-1}, y_{n_k}) + 1 \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (2.2), we deduce that there exist

$$\lim_{k\to\infty} S(y_{n_{k+1}-1}, y_{n_{k+1}-1}, y_{n_k-1}) = 1.$$

Then by condition (ξ_2), with

$$t_k = S(y_{n_{k+1}}, y_{n_{k+1}}, y_{n_k})$$

and

$$s_k = S(y_{n_{k+1}-1}, y_{n_{k+1}-1}, y_{n_k-1}),$$

we obtain

$$0 \leq \limsup_{k\to\infty} \xi(S(y_{n_{k+1}}, y_{n_{k+1}}, y_{n_k}), S(y_{n_{k+1}-1}, y_{n_{k+1}-1}, y_{n_k-1})) < 0,$$

which is a contradiction. This completes the proof. □

Lemma 2.4. *Let (X, S) be an S-metric space and let $f, g : X \to X$ be two mappings. Suppose that there exists a simulation function ξ such that* (2.1) *holds. Let $f(X) \subseteq g(X)$. Then there exists a Cauchy sequence $\{y_n\}$ in X.*

Proof. By similar argument of Lemma 2.1, for every $n \in \mathbb{N}$ we have $y_n = f(x_n) = g(x_{n+1})$. If there exists $n_0 \in \mathbb{N}$ such that $y_{n_0} = y_{n_{0+1}}$, then $\{y_n\}$ is a Cauchy sequence. Let us assume that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$ and let

$$C_n = \sup\{S(y_i, y_i, y_j) : i, j \geq n\},\ n \in \mathbb{N}.$$

From Lemma 2.3, we know that $C_n < \infty$ for every $n \in \mathbb{N}$. Since $\{C_n\}$ is a positive decreasing sequence, there is some $C \geq 0$ such that

$$\lim_{n\to\infty} C_n = C. \tag{2.3}$$

Let us suppose that $C > 0$. By the definition of C_n, for every $k \in \mathbb{N}$, there exists $n_k, m_k \in \mathbb{N}$ such that $m_k > n_k \geq k$ and

$$C_k - \frac{1}{k} < S(y_{m_k}, y_{m_k}, y_{n_k}) \leq C_k.$$

Letting $k \to \infty$ in the above inequality, we get

$$\lim_{k\to\infty} S(y_{m_k}, y_{m_k}, y_{n_k}) = C. \tag{2.4}$$

Again, from Lemma 2.2 and the definition of C_n, we deduce

$$S(y_{m_k}, y_{m_k}, y_{n_k}) \leq S(y_{m_k-1}, y_{m_k-1}, y_{n_k-1}) \leq C_{k-1}.$$

Letting $k \to \infty$ in the above inequality, using (2.3) and (2.4), we get

$$\begin{aligned} C &\leq \liminf_{k\to\infty} S(y_{m_{k-1}}, y_{m_{k-1}}, y_{n_{k-1}}) \\ &\leq \limsup_{k\to\infty} S(y_{m_{k-1}}, y_{m_{k-1}}, y_{n_{k-1}}) \\ &\leq C. \end{aligned} \tag{2.5}$$

Now, by the condition (ξ_2), with

$$t_k = S(y_{m_k}, y_{m_k}, y_{n_k})$$

and

$$s_k = S(y_{m_{k-1}}, y_{m_{k-1}}, y_{n_{k-1}}),$$

we get

$$0 \leq \limsup_{k\to\infty} \xi(S(y_{m_k}, y_{m_k}, y_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k-1})) < 0,$$

which is a contradiction. Thus we have $C = 0$, that is,

$$\lim_{n\to\infty} C_n = 0.$$

This proves that $\{y_n\}$ is a Cauchy sequence. □

Now, we present our first main result.

Theorem 2.5. *Let (X, S) be a complete S-metric space and let $f, g : X \longrightarrow X$ be two mappings with $f(X) \subseteq g(X)$ and the pair $\{f, g\}$ be compatible. Suppose that there exists a simulation function ξ such that* (2.1) *holds, that is,*

$$\xi(S(fx, fy, fz), S(gx, gy, gz)) \geq 0, \ \forall\ x, y, z \in X.$$

If g is continuous, then f and g have a coincidence point, that is, there exists $y \in X$ such that $f(y) = g(y)$. Moreover, if g is continuous and it is one to one, then f and g have unique fixed point.

Proof. Let $x_0 \in X$. Since $f(X) \subseteq g(X)$, for every $n \in \mathbb{N}$ we have $y_n = f(x_n) = g(x_{n+1})$. Now, by Lemma 2.4, the sequence $\{y_n\}$ is Cauchy. From the completeness of (X, S) there exists some $y \in X$ such that $\lim_{n\to\infty} y_n = y$. That is,

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n). \tag{2.6}$$

We claim that y is a coincidence point of f, g. Since, g is continuous, we have

$$\lim_{n\to\infty} gf(x_n) = \lim_{n\to\infty} gg(x_n) = g(y).$$

Also, since $\{f, g\}$ is compatible, we have $\lim_{n\to\infty} S(fg(x_n), fg(x_n), gf(x_n)) = 0$. Hence, by Lemma 1.8 we deduce

$$\lim_{n\to\infty} fg(x_n) = g(y).$$

From (2.1) we have,

$$\begin{aligned} 0 &\leq \xi(S(fy, fy, fgx_n), S(gy, gy, ggx_n) \\ &\leq S(gy, gy, ggx_n) - S(fy, fy, fgx_n). \end{aligned}$$

Letting $n \longrightarrow \infty$ in the above inequality, we get

$$\begin{aligned} 0 &\leq \liminf_{n\to\infty} S(gy, gy, ggx_n) - \limsup_{n\to\infty} S(fy, fy, fgx_n) \\ &= -\limsup_{n\to\infty} S(fy, fy, fgx_n) \\ &\leq 0. \end{aligned}$$

Thus,

$$\limsup_{n\to\infty} S(fy, fy, fgx_n) = 0.$$

That is

$$\lim_{n\to\infty} fg(x_n) = f(y),$$

therefore, $f(y) = g(y)$. Now, let there exists $u \in X$ such that $f(u) = g(u)$ then we show that $f(u) = f(y) = g(u) = g(y)$. Now, the (ξ_2) inequality implies

$$0 \leq \xi(S(fy, fy, fu), S(gy, gy, gu)) \leq S(gy, gy, gu) - S(fy, fy, fu) = 0,$$

hence $S(fy, fy, fu) = S(gy, gy, gu)$, by the condition (ξ_2), with

$$t_k = S(fy, fy, fu)$$

and

$$s_k = S(gy, gy, gu),$$

we get

$$0 \leq \limsup_{k \longrightarrow \infty} \xi(S(fy, fy, fu), S(gy, gy, gu) < 0,$$

which is a contradiction. Thus we have $fy = fu = gy = gu$.

Now, let the map g is continuous and it is one to one. If y, u are two coincidence points of f and g, in this case we show that $y = u$. Because, by above argument we have $f(y) = g(y) = f(u) = g(u)$, since g is one to one it follows that $y = u$. Now, since $g(y) = f(y)$ and the pair (f, g) is compatible we have $fg(y) = gf(y)$. Therefore, $gf(y) = fg(y) = ff(y)$. That is $f(y)$ is coincidence point for f and g. Therefore, $f(y) = y$ hence $f(y) = g(y) = y$. That is f and g have unique fixed point $y \in X$. □

We show the unifying power of simulation functions by applying Theorem 2.5 to deduce different kinds of contractive conditions in the existing literature.

Now we get the special cases of Theorem 2.5 as follows:

Corollary 2.6. *Let (X, S) be an complete S-metric space and let $f, g : X \longrightarrow X$ be two mappings with $f(X) \subseteq g(X)$ and the pair $\{f, g\}$ is compatible. Suppose that there exists $\lambda \in]0, 1[$ such that*

$$S(fx, fy, fz) \leq \lambda S(gx, gy, gz) \ \forall \ x, y, z \in X.$$

If g is continuous, then f and g have a coincidence point. Moreover, if g is continuous and it is one to one, then f and g have unique fixed point.

Proof. The result follows from Theorem 2.5, by taking as simulation function

$$\xi(t, s) = \lambda s - t,$$

for all $t, s \geq 0$. □

Corollary 2.7. *Let (X, S) be a complete S-metric space and let $f : X \longrightarrow X$ be a mapping. Suppose that there exists $\lambda \in]0, 1[$ such that*

$$S(fx, fy, fz) \leq \lambda S(x, y, z) \ \forall \ x, y, z \in X.$$

Then f has unique fixed point.

Proof. The result follows from Corollary 2.6, by taking $g = I$ identity map. □

Now we give an example to support our result.

Example 2.8. Let $X = [0, 1]$ be endowed with S-metric $S(x, y, z) = |x - z| + |y - z|$. Define f and g on X by

$$f(x) = (\tfrac{x}{2})^4, \quad g(x) = (\tfrac{x}{2})^2.$$

Then, we know that $f(X) \subseteq g(X)$ and the pair $\{f, g\}$ is compatible. Taking as simulation function

$$\xi(t,s) = \frac{1}{2}s - t,$$

for all $t, s \geq 0$. Also for each $x, y \in X$ we have

$$\begin{aligned} S(fx,fy,fz) &= |fx - fz| + |fy - fz| \\ &= |(\frac{x}{2})^4 - (\frac{z}{2})^4| + |(\frac{y}{2})^4 - (\frac{z}{2})^4| \\ &= |(\frac{x}{2})^2 - (\frac{z}{2})^2||(\frac{x}{2})^2 + (\frac{z}{2})^2| + |(\frac{y}{2})^2 - (\frac{z}{2})^2||(\frac{y}{2})^2 + (\frac{z}{2})^2| \\ &\leq \frac{1}{2}|gx - gz| + \frac{1}{2}|gy - gz| \\ &= \frac{1}{2}S(gx, gy, gz). \end{aligned}$$

Thus f and g satisfy the conditions given in Corollary 2.6 and 0 is the unique common fixed point of f and g .

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