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TRIGONOMETRIC APPROXIMATION OF FUNCTIONS BY HAUSDORFF-MATRIX PRODUCT OPERATORS

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Abstract. In this paper, we obtain the error approximation of a function in weighted $W(L_r, \xi(t)), r \geq 1$ class by Hausdorff-Matrix $(\wedge T)$ product means of its Fourier series. Our main theorems generalize the results of Nigam ([18], [19]), Nigam and Sharma [20], Singh and Srivastava [28] and Lal [14]. Thus, these results become the particular cases of our theorems. Some important corollaries are also deduced from our main theorems.

1. Introduction

The studies of error estimation of a function f in $Lip\alpha$ class using different single means have been made by the researchers ([1], [3], [4]-[10], [13], [15]-[17], [21]-[27]) in past few decades. Nigam ([18], [19]), Nigam and Sharma [20], Singh and Srivastava [28], Albayrak *et al.* [2] and Lal [14] have studied error estimation of a function f in weighted $W(L_r, \xi(t)), r \geq 1$ class and its subclass $Lip\alpha$ using different product means in recent past.

In this paper, we obtain the error estimation of a function in weighted $W(L_r, \xi(t)), r \geq 1$ class by Hausdorff-Matrix $(\wedge T)$ product means of its Fourier series.

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Let f be a 2π-periodic function and Lebesgue integrable on $[-\pi, \pi]$. The Fourier series of f at a point x is defined by

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1.1}$$

with n^{th} partial sums $s_n(f; x)$.

In 1921, Hausdorff [12] proved the following theorem:

Theorem 1.1. *Given the sequence $(\mu_n)_{n=1}^{\infty}$, defines*

$$\Delta^p \mu_n = \sum_{i=0}^{p} \binom{p}{i} (-1)^i \mu_{n+i}.$$

Then, the matrix with elements

$$\lambda_{mn} = \begin{cases} \binom{m}{n} \Delta^{m-n} \mu_n & \textit{for } n \leq m \\ 0 & \textit{for } n > m. \end{cases} \tag{1.2}$$

is regular if and only if μ_n is the moment sequence

$$\mu_n = \int_0^1 x^n d\chi(x), \tag{1.3}$$

where χ, known as mass function, is a real, bounded variation function defined on the interval [0,1] satisfying the conditions:

$$\chi(0+) = \chi(0) = 0 \textit{ and } \chi(1) = 1. \tag{1.4}$$

A sequence μ_n that satisfies the condition (1.3) is known as a moment sequence, while a sequence that satisfies both the conditions (1.3) and (1.4), is known as a Hausdorff moment sequence. The matrix in (1.2) that satisfies both (1.3) and (1.4) is known as a Hausdorff ($\wedge$) matrix (method).

The Hausdorff means of Fourier series are defined by

$$\wedge_m(f; x) = \sum_{n=0}^{m} \lambda_{mn} s_n(f; x), m = 0, 1, 2, 3, ... \tag{1.5}$$

The Fourier series (1.1) is said to be summable to s by Hausdorff ($\wedge$) method if

$$\wedge_m(f; x) \to s \text{ as } m \to \infty.$$

An infinite series $T = [c_{mn}]; m, n = 0, 1, ...$ is called a regular matrix (method) if it transforms any convergent sequence into convergent sequence with the same limit.

In 1911, Toeplitz [30] presented the following equivalence conditions for regularity.

Theorem 1.2. *The matrix $T = [c_{mn}]$ is regular if and only if*

(i) $\lim_{m\to\infty} c_{mn} = 0, \forall n \geq 0;$

(ii) $\lim_{m\to\infty} \sum_{n=0}^{m} c_{mn} = 1;$

(iii) $\exists M > 0, \sum_{m=0}^{\infty} |c_{mn}| < M, \forall m \geq 0.$

The matrix (T) method of Fourier series is given by

$$T_m(f;x) = \sum_{n=0}^{m} c_{mn} s_n(f;x), m = 0, 1, 2, 3, ...$$

The Fourier series (1.1) is said to be summable to s by Matrix (T) method if $T_m(f;x) \to s$ as $m \to \infty$.

By superimposing Hausdorff ($\wedge$) method on Matrix (T) method, Hausdorff-Matrix ($\wedge T$) method is obtained, which is defined as

$$K_n^{\wedge T}(f;x) = \sum_{k=0}^{n} \lambda_{n,k} \sum_{\nu=0}^{k} c_{k,\nu} s_\nu(f;x).$$

If $K_n^{\wedge T}(f;x) \to s$ as $n \to \infty$, then the Fourier series (1.1) is said to be summable to s by Hausdorff-Matrix ($\wedge T$) method.

Remark 1.3. It is worthwhile to mention here that Hausdorff matrices represent a wider class of summability matrices. Cesàro $(C, 1)$ and the Euler matrix $(E, d); d > 0$ are Hausdorff matrices and their products are also Hausdorff matrices. Therefore, Hausdorff-Matrix ($\wedge T$) product means, which is considered in the present paper, is more powerful than the individual operators such as Hausdorff ($\wedge$), Matrix (T), $(C, 1)$, (E, d) means.

Remark 1.4. Particular cases of Hausdorff-Matrix ($\wedge T$) method:

Hausdorff-Matrix ($\wedge T$) means reduces to

(i) $\wedge \left(H, \frac{1}{m+1}\right)$ or $\wedge H$ means if $c_{mn} = \frac{1}{m-n+1} \log(m+1)$.
(ii) $\wedge(C, 1)$ or $\wedge C^1$ means if $c_{mn} = \frac{1}{m+1}$.
(iii) $\wedge (N, p_m)$ or $\wedge N_p$ means if $c_{mn} = \frac{p_{m-n}}{P_m}$, where $P_m = \sum_{n=0}^{m} p_n \neq 0$.

(iv) $\wedge(N,p,q)$ or $\wedge N_{p,q}$ means if $c_{mn}=\frac{p_{m-n}q_n}{R_m}$, where $R_m=\sum_{n=0}^{m}p_nq_{m-n}$.

(v) $\wedge\left(\bar{N},p_m\right)$ or $\wedge\bar{N}_p$ means if $c_{mn}=\frac{p_n}{P_m}$.

(vi) $\wedge(E,d)$ or $\wedge E_d$ means if $c_{mn}=\frac{1}{(1+d)^m}\begin{pmatrix} m \\ n \end{pmatrix}d^{m-n}$.

(vii) Cesàro-Matrix($(C,m)T$) or C_mT means if the mass function $\chi(x)=m\int_0^x(1-t)^{m-1}dt$.

(viii) Hölder-Matrix($(H,m)T$) or H_mT means if the mass function $\chi(x)=\int_0^x\frac{1}{(m-1)}\left(\log\frac{1}{t}\right)^{m-1}dt$.

(ix) Euler-Matrix ($(E,d)T$) or E_dT means if the mass function

$$\chi(x)=\begin{cases}0, & \text{if } x\in[0,b]\\ 1, & \text{if } x\in[b,1]\end{cases},\ \text{where } b=\frac{1}{1+d}, d>0.$$

Remark 1.5. In view of above Remark 1.4, Hausdorff-Matrix ($\wedge T$) means also reduces to *(i)* C_mN_p, *(ii)* $C_mN_{p,q}$, *(iii)* $C_m\bar{N}_p$, *(iv)* H_mN_p, *(v)* $H_mN_{p,q}$, *(vi)* $H_m\bar{N}_p$, *(vii)* E_dN_p,, *(viii)* $E_dN_{p,q}$, *(ix)* $E_d\bar{N}_p$ *(x)* C_mE_d, *(xi)* E_dC_m means for $m,d>0$.

Remark 1.6. Since Cesàro means, Euler means and their product means are again Hausdorff means then our main theorems also hold for Cesáro means, Euler means and their product C_mE_d and E_dC_m means for $m,d>0$.

Remark 1.7. Our main theorems also hold for all the cases mentioned in Remarks 1.5 (case (i) to (ix)) and sub-cases mentioned in Remark 1.5 (case (i) to (ix)).

$L_\infty-$ norm of a function $f:R\to R$ is defined by

$$\|f\|_\infty=\sup_{x\in[0,2\pi]}\{|f(x)|:x\in R\}.$$

L_r- norm of a function $f\in L_r[0,2\pi]$ is defined by

$$\|f\|_r=\left(\frac{1}{2\pi}\int_0^{2\pi}|f(x)|^r dx\right)^{\frac{1}{r}},\ r\geq 1.$$

The degree of approximation of a function $f:R\to R$ by a trigonometric polynomial t_n of degree n under sup norm $\|\cdot\|_\infty$ is given by [31] and is defined as

$$\|t_n-f\|_\infty=\sup\{\,|t_n(x)-f(x)|:x\in R\}$$

and the degree of approximation $E_n(f)$ of a function $f\in L_r$ is given by

$$E_n(f)=\min_{t_n}\|t_n-f\|_r \tag{1.6}$$

This method of approximation is called trigonometric Fourier approximation (TFA).

A function $f \in Lip\alpha$ if

$$f(x+t) - f(x) = O\left(|t|^{\alpha}\right) for\ 0 < \alpha \leq 1.$$

$f \in Lip(\alpha, r)$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O\left(|t|^{\alpha}\right), 0 < \alpha \leq 1, \text{ and, } r \geq 1.$$

Given a positive increasing function $\xi(t)$, then $f \in W(L_r, \xi(t))$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r \sin^{\beta r} \frac{x}{2}\, dx\right)^{\frac{1}{r}} = O\left(\xi(t)\right), \quad \beta \geq 0, \ r \geq 1.$$

We use the following notation:

$$\phi(t) = \phi(x,t) = f(x+t) + f(x-t) - 2f(x)$$

2. Lemmas

Lemma 2.1. *For* $t \in \left[0, \frac{1}{n+1}\right]$, $|K_n(t)| = O(n+1)$.

Proof. For $t \in \left[0, \frac{1}{n+1}\right]$, $\sin nt \leq nt$, $\sin(t/2) \geq t/\pi$,

$$|K_n(t)| = \frac{1}{2\pi}\left|\left[\left\{\int_0^1 \sum_{k=0}^{n} \binom{n}{k} u^k (1-u)^{n-k} d\gamma(u)\right\}\left\{\sum_{\nu=0}^{k} c_{k,\nu} \frac{\sin\left(\nu+\frac{1}{2}\right)t}{\sin(t/2)}\right\}\right]\right| \tag{2.1}$$

First, we solve the following:

$$\begin{aligned}\left|\sum_{\nu=0}^{k} c_{k,\nu} \frac{\sin\left(\nu+\frac{1}{2}\right)t}{\sin(t/2)}\right| &\leq \sum_{\nu=0}^{k} c_{k,\nu} \frac{\left|\sin\left(\nu+\frac{1}{2}\right)t\right|}{|\sin(t/2)|} \\ &\leq \sum_{\nu=0}^{k} c_{k,\nu} \frac{\left(\nu+\frac{1}{2}\right)t}{(t/\pi)} \\ &= \frac{\pi}{2}\left\{\sum_{\nu=0}^{k} c_{k,\nu}(2\nu+1)\right\}\end{aligned}$$

$$
\begin{aligned}
&=\frac{\pi}{2}\left\{\sum_{\nu=0}^{k}\ c_{k,\nu}+2\sum_{\nu=0}^{k}\nu\ c_{k,\nu}\right\}\\
&=\frac{\pi}{2}\left\{1+2(c_{k,1}+2c_{k,2}+3c_{k,1}+....kc_{k,k})\right\}\\
&\leq\frac{\pi}{2}\left\{1+2(kc_{k,1}+kc_{k,2}+kc_{k,3}+....kc_{k,k})\right\}\\
&\leq\frac{\pi}{2}\left\{1+2k(c_{k,1}+c_{k,2}+c_{k,3}+....c_{k,k})\right\}\\
&\leq\frac{\pi}{2}\left\{1+2k(c_{k,0}+c_{k,1}+c_{k,2}+....c_{k,k})-2kc_{k,0}\right\}\\
&\leq\frac{\pi}{2}\left\{1+2k(1-c_{k,0}\right\}\\
&\leq\frac{\pi}{2}(1+2k)\\
&=O(k+1). \qquad (2.2)
\end{aligned}
$$

Using (2.2) in (2.1), we have

$$
\begin{aligned}
|K_n(t)| &=\frac{1}{2\pi}\left|\left[\left\{\int_0^1\sum_{k=0}^{n}\binom{n}{k}u^k(1-u)^{n-k}(k+1)d\gamma(u)\right\}\right]\right|\\
&=\frac{1}{2\pi}\left|\left[\left\{\int_0^1 g(u)d\gamma(u)\right\}\right]\right|, \qquad (2.3)
\end{aligned}
$$

where $g(u)=\sum_{k=0}^{n}\binom{n}{k}u^k(1-u)^{n-k}(k+1)$.

Now solving,

$$
\begin{aligned}
g(u)&=\sum_{k=0}^{n}\binom{n}{k}u^k(1-u)^{n-k}(k+1)\\
&=(1-u)^n\sum_{k=0}^{n}\binom{n}{k}\left\{\frac{u}{1-u}\right\}^k(k+1)\\
&=(1-u)^n\left[\sum_{k=0}^{n}\binom{n}{k}k\left\{\frac{u}{1-u}\right\}^k+\sum_{k=0}^{n}\binom{n}{k}\left\{\frac{u}{1-u}\right\}^k\right]\\
&=(1-u)^n\left[\sum_{k=0}^{n}\binom{n}{k}k\ p^k+\sum_{k=0}^{n}\binom{n}{k}p^k\right],\text{ where } p=\frac{u}{1-u}\\
&=(1-u)^n\left[\sum_{k=0}^{n}\binom{n}{k}k\ p^k+(1+p)^n\right]. \qquad (2.4)
\end{aligned}
$$

And also, we have

$$\sum_{k=0}^{n}\binom{n}{k}k\ p^k = 0\binom{n}{0}p^0 + 1\binom{n}{1}p^1 + 2\binom{n}{2}p^2 + ... + n\binom{n}{n}p^n$$
$$= p\left[\binom{n}{1} + 2\binom{n}{2}p + ... + n\binom{n}{n}p^{n-1}\right]. \tag{2.5}$$

We know that

$$(1+p)^n = \left[\binom{n}{1} + \binom{n}{2}p + ... + \binom{n}{n}p^n\right]. \tag{2.6}$$

Differentiating (2.6) with respect to p on both sides,

$$n(1+p)^{n-1} = \left[\binom{n}{1} + ... + n\binom{n}{n}p^{n-1}\right]. \tag{2.7}$$

From (2.5) in (2.7), we get

$$\sum_{k=0}^{n}\binom{n}{k}k\ p^k = pn(1+p)^{n-1}. \tag{2.8}$$

Using (2.8) in (2.4), we get

$$g(u) = (1-u)^n\left[pn(1+p)^{n-1} + (1+p)^n\right]$$
$$= (1-u)^n\left\{n\ \left(\frac{u}{1-u}\right)\left(\frac{1}{1-u}\right)^{n-1} + \left(\frac{1}{1-u}\right)^n\right\}$$
$$= un + 1. \tag{2.9}$$

From (2.3) and (2.9), we get

$$|K_n(t)| = \frac{1}{2\pi}\left[\left\{\int_0^1 (un+1)du\right\}\right]$$
$$= O(n+1).$$

□

Lemma 2.2. *For* $t \in \left[\frac{1}{n+1}, \pi\right]$, $|K_n(t)| = O\left(\frac{1}{t}\right)$.

Proof. For $t \in \left[\frac{1}{n+1}, \pi\right]$, $\sin(t/2) \geq t/\pi$ and $\sup_{0\leq k\leq 1} |\gamma^{'}(u)| = N$,

$$|K_n(t)| = \frac{1}{2\pi}\left|\left[\left\{\int_0^1 \sum_{k=0}^{n}\binom{n}{k}u^k(1-u)^{n-k}\gamma(u)\right\}\left\{\sum_{\nu=0}^{k} c_{k,\nu}\frac{\sin\left(\nu+\frac{1}{2}\right)t}{\sin(t/2)}\right\}\right]\right|. \tag{2.10}$$

First, we solve the following:

$$\begin{aligned}\left|\sum_{\nu=0}^{k} c_{k,\nu}\frac{\sin\left(\nu+\frac{1}{2}\right)t}{\sin(t/2)}\right| &\leq \sum_{\nu=0}^{k} c_{k,\nu}\frac{\left|\sin\left(\nu+\frac{1}{2}\right)t\right|}{\left|\sin(t/2)\right|}\\ &\leq \frac{\pi}{t}\left|\sum_{\nu=0}^{k} c_{k,\nu}\ \mathrm{Im}\ e^{i\left(\nu+\frac{1}{2}\right)t}\right|\\ &\leq \frac{\pi}{t}\left|\sum_{\nu=0}^{k} c_{k,\nu}\ \mathrm{Im}\ e^{i\nu t}\right|\left|e^{i\frac{t}{2}}\right|\\ &\leq \frac{\pi}{t}\left|\sum_{\nu=0}^{\tau-1} c_{k,\nu}\ \mathrm{Im}\ e^{i\nu t}\right| + \frac{\pi}{t}\left|\sum_{\nu=\tau}^{k} c_{k,\nu}\ \mathrm{Im}\ e^{i\nu t}\right|. \qquad (2.11)\end{aligned}$$

Now considering first term of (2.11):

$$\begin{aligned}\frac{\pi}{t}\left|\sum_{\nu=0}^{\tau-1} c_{k,\nu}\ \mathrm{Im}\ e^{i\nu t}\right| &\leq \frac{\pi}{t}\left|\sum_{\nu=0}^{\tau-1} c_{k,\nu}\right|\left|e^{i\nu t}\right|\\ &\leq \frac{\pi}{t}\left|\sum_{\nu=0}^{\tau-1} c_{k,\nu}\right|. \qquad (2.12)\end{aligned}$$

Now considering the second term of (2.11) and using Abel's Lemma, we get

$$\begin{aligned}\frac{\pi}{t}\left|\sum_{\nu=\tau}^{k} c_{k,\nu}\ \mathrm{Im}\ e^{i\nu l}\right| &\leq \frac{\pi}{t}\sum_{\nu=\tau}^{k} c_{k,\nu}\max_{0\leq m\leq \nu}\left|e^{imt}\right|\\ &\leq \frac{\pi}{t}\sum_{\nu=\tau}^{k} c_{k,\nu}. \qquad (2.13)\end{aligned}$$

Combining (2.11), (2.12) and (2.13), we get

$$\begin{aligned}\left|\sum_{\nu=0}^{k} c_{k,\nu}\frac{\sin\left(\nu+\frac{1}{2}\right)l}{\sin(t/2)}\right| &\leq \frac{\pi}{t}\sum_{\nu=0}^{\tau-1} c_{k,\nu} + \frac{\pi}{t}\sum_{\nu=\tau}^{k} c_{k,\nu}\\ &= O\left(\frac{1}{t}\right). \qquad (2.14)\end{aligned}$$

From (2.10) and (2.14), we get

$$\begin{aligned}|K_n(t)| =&O\left[\frac{1}{2\pi t}\left\{\int_0^1\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)u^k(1-u)^{n-k}du\right\}\right]\\ =&O\left[\frac{1}{2\pi t}\left\{\int_0^1(u+1-u)^n du\right\}\right]\\ =&O\left(\frac{1}{t}\right).\end{aligned}$$

□

3. Main results

Theorem 3.1. *Error estimation of a function f (2π- periodic) in $W\left(L_r,\xi\left(t\right)\right)$, $r>1$, class by Hausdorff-Matrix ($\wedge T$) means of its Fourier series is given by*

$$\left\|K_n^{\wedge T}(f)-f\left(x\right)\right\|_r = O\left[(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{(n+1)}\right)\right]$$

provided a positive increasing function $\xi\left(t\right)$ satisfies the following conditions:

$$\frac{\xi\left(t\right)}{t}\textit{is non-increasing}, \tag{3.1}$$

$$\left\{\int_0^{\frac{1}{n+1}}\left(\frac{|\phi\left(t\right)|\sin^{\beta}(t/2)}{\xi\left(t\right)}\right)^r dt\right\}^{\frac{1}{r}} = O\left(\frac{1}{(n+1)^{1/r}}\right), \tag{3.2}$$

$$\left\{\int_{\epsilon}^{\frac{1}{n+1}}\left(\frac{\xi(t)}{\sin^{\beta}(t/2)}\right)^r dt\right\}^{\frac{1}{r}} = O(n+1)^{\beta-1+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right) \tag{3.3}$$

and

$$\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}\left|\phi\left(t\right)\right|}{\xi\left(t\right)}\right)^r dt\right\}^{\frac{1}{r}} = O\left\{\,(n+1)^{\delta}\right\}, \tag{3.4}$$

where δ is an arbitrary positive number such that $0<\delta<\beta+\frac{1}{r}$, $0<\beta\leq 1-\frac{1}{r}$ and $\frac{1}{r}+\frac{1}{s}=1$. The conditions (3.2) *and* (3.4) *hold uniformly in x.*

Proof. In view of the fact that $\phi(t) \in W(L_r, \xi(t))$ and $\frac{\phi(t)}{\xi(t)}$ is bounded, (3.2) and (3.4) can be verified. Moreover, in view of mean value theorem, (3.3) is obvious. Following [29], we have

$$s_n(f;x) - f(x) = \frac{1}{2\pi}\int_0^\pi \phi(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt.$$

Hausdorff-Matrix ($\wedge T$) transform of $s_n(f;x)$ is given by

$$\begin{aligned}
&f(x) - t_n^{\wedge T}(x)\\
&= \frac{1}{2\pi}\int_0^\pi \phi(t)\left[\left\{\sum_{k=0}^{n}\binom{n}{k}u^k(1-u)^{n-k}d\gamma(u)\right\}\left\{\sum_{\nu=0}^{k}c_{k,\nu}\frac{\sin\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right]dt\\
&= \int_0^\pi \phi(t)K_n(t)dt\\
&= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right]\phi(t)\ K_n(t)\ dt\\
&= I_1 + I_2 \text{ (say).}
\end{aligned} \tag{3.5}$$

Now considering,

$$|I_1| \le \int_0^{\frac{1}{n+1}} |\phi(t)|\ |K_n(t)|\,dt.$$

Using Lemma 2.1 and Hölder's inequality, we obtain

$$\begin{aligned}
|I_1| &= O(n+1)\left[\int_0^{\frac{1}{n+1}} \frac{|\phi(t)|\sin^\beta(t/2)}{\xi(t)}.\frac{\xi(t)}{\sin^\beta(t/2)}dt\right]\\
&= O(n+1)\left[\int_0^{\frac{1}{n+1}}\left\{\frac{|\phi(t)|\sin^\beta(t/2)}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}}\left[\lim_{\epsilon\to 0}\int_\epsilon^{\frac{1}{n+1}}\left\{\frac{\xi(t)}{\sin^\beta(t/2)}\right\}^s dt\right]^{\frac{1}{s}}\\
&= O\left\{\frac{n+1}{(n+1)^{\frac{1}{r}}}(n+1)^{\beta-1+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\} \text{ by (3.2) and (3.3)}\\
&= O\left\{(n+1)^\beta\xi\left(\frac{1}{n+1}\right)\right\} \text{ since } \frac{1}{r}+\frac{1}{s} = 1.
\end{aligned} \tag{3.6}$$

Using Lemma 2.2, Holder's inequality and $\frac{1}{\sin(t/2)} \leq \frac{\pi}{t}$ for $0 < t < \pi$, we obtain

$$|I_2| = O\left[\int_{\frac{1}{n+1}}^{\pi} \frac{t^{-\delta}\,|\,\phi\,(t)\,|\sin^{\beta}(t/2)}{\xi\,(t)} . \frac{\xi(t)}{t^{-\delta+1}\sin^{\beta}(t/2)} dt\right]$$

$$= O\left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{t^{-\delta}\,|\,\phi\,(t)\,|\sin^{\beta}(t/2)}{\xi\,(t)}\right\}^{r} dt\right]^{\frac{1}{r}} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{\xi\,(t)}{t^{-\delta+1+\beta}}\right\}^{s} dt\right]^{\frac{1}{s}}.$$

Using (3.1), (3.4), mean value theorem for integrals and in view of $0 < \delta < \beta + \frac{1}{r}$, we get

$$= O\left[(n+1)^{\delta}\xi\left(\frac{1}{n+1}\right)(n+1)\left(\int_{\frac{1}{n+1}}^{\pi} t^{-(\beta-\delta)s} dt\right)^{\frac{1}{s}}\right]$$

$$= O\left[(n+1)^{\delta}\xi\left(\frac{1}{n+1}\right)(n+1)^{(\beta+1-\delta)-\frac{1}{s}}\right]$$

$$= O\left[(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right] \text{since} \frac{1}{r} + \frac{1}{s} = 1. \tag{3.7}$$

Thus combining (3.5), (3.6) and (3.7), we get

$$\left|f(x) - t_n^{\wedge T}(x)\right| = O\left[(n+1)^{\beta}\xi\left(\frac{1}{n+1}\right)\right] + O\left[(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right],$$

$$\left\|f(x) - t_n^{\wedge T}(x)\right\|_r = O\left[(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right]. \tag{3.8}$$

□

Theorem 3.2. *Error estimation of a function f (2π- periodic) in $W\,(L_1, \xi\,(t))$ class by Hausdorff-Matrix ($\wedge T$) means of its Fourier series is given by*

$$\left\|K_n^{\wedge T}(f) - f\,(x)\right\|_1 = O\left[(n+1)^{\beta+1}\,\xi\left(\frac{1}{(n+1)}\right)\right]$$

provided a positive increasing function $\xi\,(t)$ satisfies the following conditions:

$$\frac{\xi\,(t)}{t^{\delta}} \textit{ is non-decreasing,} \tag{3.9}$$

$$\int_{0}^{\frac{1}{n+1}} \frac{|\phi\,(t)|\sin^{\beta}(t/2)}{\xi\,(t)}\,dt = O\left(\frac{1}{n+1}\right), \tag{3.10}$$

$$\int_{\frac{1}{n+1}}^{\pi} \frac{t^{-\delta}\,|\phi\,(t)|}{\xi\,(t)}dt = O\left\{\,(n+1)^{\delta}\right\} \tag{3.11}$$

and

$$\frac{\xi\,(t)}{t^{\beta-\delta+1}}\textit{is non-increasing,} \tag{3.12}$$

where δ *is an arbitrary positive number such that* $0<\delta<\beta+1$ *and* $0 \leq \beta < 1$ *The conditions* (3.10) *and* (3.11) *hold uniformly in* x.

Proof. Following the proof of Theorem 3.1, for $r = 1$ *i.e.* $s = \infty$ and using Lemma 2.1, we obtain

$$\begin{aligned} I_1 &= O(n+1)\left[\int_0^{\frac{1}{n+1}} \frac{|\,\phi\,(t)\,|\sin^{\beta}(t/2)}{\xi\,(t)}dt\right] \text{ ess} \sup_{0<t<\frac{1}{n+1}}\left|\frac{\xi(t)}{\sin^{\beta}(t/2)}\right| \\ &= O\left(\text{ess} \sup_{0<t<\frac{1}{n+1}}\left|\frac{\xi(t)}{t^{\beta}}\right|\right) \text{ by (3.10)} \\ &= O\left(\left|\frac{\xi\left(\frac{1}{n+1}\right)}{\left(\frac{1}{n+1}\right)^{\beta}}\right|\right) \text{ by (3.9)} \\ &= O\left[(n+1)^{\beta}\xi\left(\frac{1}{n+1}\right)\right] \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} I_2 &= O\left[\int_{\frac{1}{n+1}}^{\pi} \frac{t^{-\delta}\,|\,\phi\,(t)\,|\sin^{\beta}(t/2)}{\xi\,(t)}dt\right] \text{ ess} \sup_{0<t<\frac{1}{n+1}}\left|\frac{\xi(t)}{t^{-\delta+1}\sin^{\beta}(t/2)}\right| \\ &= O\left((n+1)^{\delta} \text{ ess} \sup_{0<t<\frac{1}{n+1}}\left|\frac{\xi(t)}{t^{-\delta+\beta+1}}\right|\right) \text{ by (3.11)} \\ &= O\left\{(n+1)^{\delta}\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{1+\beta-\delta}\right\}\text{by (3.12)} \\ &= O\left\{(n+1)^{\beta+1}\,\xi\left(\frac{1}{n+1}\right)\right\}. \end{aligned} \tag{3.14}$$

Thus, combining (3.13) and (3.14), we get

$$\left\| f(x) - t_n^{\wedge T} \right\|_1 = O\left[(n+1)^{\beta+1} \xi \left(\frac{1}{(n+1)} \right) \right].$$

□

4. Corollaries

Corollary 4.1. *If $\beta = 0$ and $\xi(t) = t^{\alpha}$, then the degree of approximation of a function $f \in Lip(\alpha, r), 0 < \alpha \leq 1$, is given by*

$$\left\| f(x) - t_n^{\wedge T} \right\|_r = O\left[\left(\frac{1}{n+1} \right)^{\alpha - \frac{1}{r}} \right].$$

Corollary 4.2. *If $r \to \infty$ in Corollary 1, then $0 < \alpha \leq 1$*

$$\left\| f(x) - t_n^{\wedge T} \right\|_\infty = O\left[\frac{1}{(n+1)^{\alpha}} \right].$$

5. Particular cases

Remark 5.1. (i) If $a_{m,n} = \frac{1}{m+1}$, then in view of Remark 1.4 (case (ix)) for $d = 1$, Theorem 1 of [18] becomes a particular case of our main theorems.

(ii) In view of Remark 1.4 (case (vii)) for $m = 1$, Theorem 1 of [28] becomes a particular case of our main theorems.

(iii) If $a_{m,n} = \frac{1}{(1+d)^m} d^{m-n}$, then in view of Remark 1.4 (case (vii)) for $m = 1$, the result of [19] becomes a particular case of our main theorems.

(iv) If $a_{m,n} = \frac{p_{m-n}}{P_m}$, where $P_m = \sum_{n=0}^{m} p_n \neq 0$, then in view of Remark 1.4 (case (vii)) for $m = 1$, Theorem 1 of [14] becomes a particular case of our main theorems.

(v) If $a_{m,n} = \frac{p_{m-n}}{P_m}$, where $P_m = \sum_{n=0}^{m} p_n \neq 0$, then in view of Remark 1.4 (case (ix)) for $d = 1$, Theorem 3 of [20] becomes a particular case of our main theorems.

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