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CONVERGENCE ANALYSIS OF VISCOSITY IMPLICIT RULES OF NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, the study of implicit viscosity approximation methods for nonexpansive mappings in Banach spaces is explored. A new iterative sequence is introduced for the class of nonexpansive mappings in Banach spaces. Suitable conditions are imposed on the control parameters to prove a strong convergence theorem. Moreover, the strong convergence of the newly introduced sequence to a fixed point of a nonexpansive mapping is obtained which also solves the variational inequality problem. These results are improvement and extension of some recent corresponding results announced.

1. Introduction

Following the idea of Attouch [3], the viscosity approximation method for nonexpansive mappings in Hilbert spaces was introduced in 2000 by Moudafi [10].

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$, K be a nonempty, closed and convex subset of H. Let $G : K \to K$ be a contraction ($i.e.$, $\|G(u) - G(v)\| \leq c\|u - v\|$ for all $u, v \in K$ and for some $c \in [0, 1)$), and

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let $T : K \to K$ be a nonexpansive mapping (i.e., $\|Tu - Tv\| \leq \|u - v\|$ for all $u, v \in K$). The set of fixed points of T will be denoted by $F(T)$. Recently, Xu *et al.* [16] proposed the implicit midpoint procedure:

$$x_{n+1} = \lambda_n G(x_n) + (1 - \lambda_n) T \left(\frac{x_n + x_{n+1}}{2} \right), \ n \in \mathbb{N}, \tag{1.1}$$

where $\{\lambda_n\}_{n=1}^{\infty} \subset [0, 1]$. Under certain conditions imposed on the control parameter, it was established that the implicit midpoint procedure (1.1) converges to a fixed point p of T which also solves the variational inequality:

$$\langle (I - G)p, x - p \rangle \geq 0, \ \forall \ x \in F(T). \tag{1.2}$$

Ke and Ma [5] introduced generalized viscosity implicit rules which extend the results of Xu *et al.* [16]. The generalized viscosity implicit procedures are given by

$$x_{n+1} = \lambda_n G(x_n) + (1 - \lambda_n) T \left(\delta_n x_n + (1 - \delta_n) x_{n+1} \right), \ n \in \mathbb{N}, \tag{1.3}$$

and

$$y_{n+1} = \lambda_n G(y_n) + \beta_n y_n + \gamma_n T \left(\delta_n y_n + (1 - \delta_n) y_{n+1} \right), \ n \in \mathbb{N}, \tag{1.4}$$

where $\{\lambda_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}, \{\gamma_n\}_{n=1}^{\infty} \subset [0, 1]$ with $\lambda_n + \beta_n + \gamma_n = 1$. Suitable conditions were imposed on the control parameters to show that the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point p of the nonexpansive mapping T, which is also the unique solution of the variational inequality (1.2). In other words, p is the unique fixed point of the contraction $P_{F(T)}G$, that is, $P_{F(T)}G(p) = p$. Replacement of strict contractions in (1.4) by the generalized contractions and extension to uniformly smooth Banach spaces was considered by Yan *et al.* [17]. Under certain conditions on imposed on the parameters which are involved, the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point p of the nonexpansive mapping T, which is also the unique solution of the variational inequality

$$\langle (I - G)p, J(x - p) \rangle \geq 0, \ \forall \, x \in F(T), \tag{1.5}$$

where J is the normalized duality mapping.

Inspired by the previous works in this direction, we propose a new implicit iterative algorithm. Precisely, for a nonempty closed convex subset K of a uniformly smooth Banach space E and for real sequences $\left\{ \{\lambda_n^i\}_{n=1}^{\infty} \right\}_{i=1}^{3} \subset [0, 1]$ and $\{\delta_n\}_{n=1}^{\infty} \subset (0, 1)$, the implicit iterative scheme is defined from an arbitrary $x_1 \in K$ by

$$x_{n+1} = \lambda_n^1 G_1(x_n) + \lambda_n^2 x_n + \lambda_n^3 T \left((1 - \delta_n) G_2(x_n) + \delta_n x_{n+1} \right), \tag{1.6}$$

where $T : K \to K$ is a nonexpansive mapping and $G_i : K \to K$ is a generalized contraction mapping for each $i = 1, 2$.

2. Preliminaries

Let E be a real Banach space with dual E^* and denotes the norm on E by $\|.\|$. The normalized duality mapping $J : E \to 2^{E^*}$ is defined as

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \|x\| = \|f\|\},$$

where $\langle \cdot, \cdot\rangle$ is the duality pairing between E and E^*. Let B_E denotes the unit ball of E. The modulus of convexity of E is defined as

$$\delta_E(\epsilon) = \inf \left\{1 - \frac{\|x + y\|}{2} : x, y \in B_E, \|x - y\| \geq \epsilon\right\},\ 0 \leq \epsilon \leq 2.$$

E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0, 2]$. E is said to be smooth (or Gáteaux differentiable) if the limit

$$\lim_{t\to 0^+} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in B_E$. E is said to have uniformly Gâteaux differentiable norm if for each $y \in B_E$, the limit is attained uniformly for $x \in B_E$ and uniformly smooth if it is smooth and the limit is attained uniformly for each $x, y \in B_E$. Recall that if E is smooth, then J is single-valued and onto if E is reflexive. Furthermore, the normalized duality mapping J is uniformly continuous on bounded subsets of E from the strong topology of E to the weak-star topology of E^* if E is a Banach space with a uniformly Gâteaux differentiable norm.

Let T be a self-mapping of K. $T : K \to K$ is said to be L-Lipschitzian if there exists a constant $L > 0$, such that for all $u, v \in K$,

$$\|Tu - Tv\| \leq L\|u - v\|.$$

Let (X, d) be a metric space and K a subset of X. A mapping $G : K \to K$ is said to be a Meir-Keeler contraction if for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for each $u, v \in K$, with $\epsilon \leq d(u, v) < \epsilon + \delta$, we have

$$d(G(u), G(v)) < \epsilon.$$

Let $\mathbb{N}$ be the set of all positive integers and $\mathbb{R}^+$ the set of all positive real numbers. A mapping $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be an L-function if $\psi(0) = 0$, $\psi(t) > 0$ for all $t > 0$ and for every $s > 0$, there exists $u > s$ such that $\psi(t) \leq s$ for each $t \in [s, u]$. A mapping $G : E \to E$ is called a (ψ, L)-contraction if $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is an L-function and

$$d(G(x), G(y)) < \psi(d(x, y)),$$

for all $x, y \in E,\ x \neq y$.

The following interesting results about the Meir-Keeler contraction are well known.

Proposition 2.1. ([9]) *Let (X,d) be a complete metric space and let G be a Meir-Keeler contraction on X. Then G has a unique fixed point in X.*

Remark 2.2. If K is a nonempty closed (convex) subset of a complete metric space (X,d), then the conclusion of Proposition 2.1 is still true.

Proposition 2.3. ([13]) *Let E be a Banach space, K a convex subset of E and $G : K \to K$ a Meir-Keeler contraction. Then for all $\epsilon > 0$, there exists a $c \in (0,1)$ such that*

$$\|G(u) - G(v)\| \leq c\|u - v\| \tag{2.1}$$

for all $u, v \in K$ with $\|u - v\| \geq \epsilon$.

Proposition 2.4. ([8]) *Let K be a nonempty convex subset of a Banach space E, $T : K \to K$ a nonexpansive mapping and $G : K \to K$ a Meir-Keeler contraction. Then TG and $GT : K \to K$ are Meir-Keeler contractions.*

The following lemmas are also needed in the sequel.

Lemma 2.5. ([11]) *Let K be a nonempty closed and convex subset of a uniformly smooth Banach space E. Let $T : K \to K$ be a nonexpansive mapping such that $F(T) \neq \emptyset$ and $G : K \to K$ be a generalized contraction mapping. Then $\{x_t\}$ defined by*

$$x_t = tG(x_t) + (1-t)Tx_t$$

for $t \in (0,1)$, converges strongly to $p \in F(T)$, which solves the variational inequality:

$$\langle G(p) - p, J(z-p)\rangle \leq 0, \ \ \forall\, z \in F(T).$$

Lemma 2.6. ([11]) *Let K be a nonempty closed and convex subset of a uniformly smooth Banach space E. Let $T : K \to K$ be a nonexpansive mapping such that $F(T) \neq \emptyset$ and $G : K \to K$ be a generalized contraction mapping. Assume that $\{x_t\}$ defined by*

$$x_t = tG(x_t) + (1-t)Tx_t$$

for $t \in (0,1)$, converges strongly to $p \in F(T)$ as $t \to 0$. Suppose that $\{x_n\}$ is a bounded sequence such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. Then

$$\limsup_{n\to\infty} \langle G(p) - p, J(x_n - p)\rangle \leq 0.$$

Lemma 2.7. ([12]) *Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be bounded sequences in a Banach space E and $\{t_n\}_{n=1}^{\infty}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} t_n \leq \limsup_{n\to\infty} t_n < 1$. Suppose that for all $n \geq 0$,*

$$u_{n+1} = (1-t_n)u_n + t_n v_n$$

and

$$\limsup_{n\to\infty} \left(\|u_{n+1} - u_n\| - \|v_{n+1} - v_n\|\right) \leq 0.$$

Then $\lim_{n\to\infty} \|u_n - v_n\| = 0.$

Lemma 2.8. ([15]) *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers satisfying the following relations:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n,\ n \in \mathbb{N},$$

where

(i) $\{\alpha\}_n \subset (0,1),\ \sum_{n=1}^{\infty} \alpha_n = \infty;$

(ii) $\limsup_{n\to\infty} \sigma_n \leq 0;$

(iii) $\gamma_n \geq 0,\ \sum_{n=1}^{\infty} \gamma_n < \infty.$

Then, $a_n \to 0$ *as* $n \to \infty$.

In this paper, the generalized contraction mappings refer to Meir-Keeler contractions or (ψ, L)-contractions. It is assumed from the definition of (ψ, L)-contraction that L-function is continuous, strictly increasing and $\lim_{t\to\infty} \phi(t) = \infty$, where $\phi(t) = t - \psi(t)$ for all $t \in \mathbb{R}^+$. Whenever there is no confusion, $\phi(t)$ and $\psi(t)$ will be written as $\phi\ t$ and $\psi\ t$, respectively.

3. Main results

Assumption 3.1. *Let* K *be a nonempty closed convex subset of a uniformly smooth Banach space* E. *Let* $G_i : K \to K$ *be generalized contraction mappings and* T *a nonexpansive self-mapping defined on* K *with* $F(T) \neq \emptyset$, *for each* $i = 1, 2$. *The real sequences* $\{\{\lambda_n^i\}_{n=1}^{\infty}\}_{i=1}^{3} \subset [0,1]$ *and* $\{\delta_n\}_{n=1}^{\infty} \subset (0,1)$ *are assumed to satisfy the following conditions:*

(i) $\sum_{i=1}^{3} \lambda_n^i = 1;$

(ii) $\lim_{n\to\infty} (1 - \lambda_n^2 - \lambda_n^3\delta_n) = 0,\ \sum_{n=1}^{\infty} (1 - \lambda_n^2 - \lambda_n^3\delta_n) = \infty;$

(iii) $0 < \liminf_{n\to\infty} \lambda_n^2 \leq \limsup_{n\to\infty} \lambda_n^2 < 1;$

(iv) $\lim_{n\to\infty} \lambda_n^3 = 0,\ \sum_{n=1}^{\infty} \lambda_n^3(1-\delta_n) < \infty;$

(v) $0 < \epsilon \leq \delta_n \leq \delta_{n+1} \leq \delta < 1,\ \ \forall n \in \mathbb{N}.$

The convergence of the iterative scheme (1.6) is being considered under the conditions (i)-(v) of Assumption 3.1 stated above.

First, it is observed that for all $\omega \in K$, the mapping defined by

$$u \mapsto T_\omega(u) : \quad = \quad \lambda_n^1 G_1(\omega) + \lambda_n^2 \omega + \lambda_n^3 T((1-\delta_n)G_2(\omega) + \delta_n u), \quad (3.1)$$

for all $u \in K$, where $\left\{\{\lambda_n^i\}_{n=1}^{\infty}\right\}_{i=1}^{3} \subset [0,1]$, $\{\delta_n\}_{n=1}^{\infty} \subset (0,1)$, is a contraction with the contractive constant $\delta \in (0,1)$.

Indeed, for all $u, v \in K$,

$$\begin{aligned} \|T_\omega(u) - T_\omega(v)\| &= \lambda_n^3 \|T((1-\delta_n)G_2(\omega) + \delta_n u) - T((1-\delta_n)G_2(\omega) + \delta_n v)\| \\ &\leq \lambda_n^3 \|(1-\delta_n)G_2(\omega) + \delta_n u - (1-\delta_n)G_2(\omega) - \delta_n v\| \\ &\leq \lambda_n^3 \delta_n \|u - v\| \\ &\leq \delta_n \|u - v\| \\ &\leq \delta \|u - v\|. \end{aligned} \quad (3.2)$$

Therefore, T_ω is a contraction. Thus, (1.6) is well defined since every contraction in a Banach space has a fixed point.

The proof of the following lemmas which are useful in establishing our main result are given as below.

Lemma 3.2. *Let K be a nonempty closed convex subset of a uniformly smooth Banach space E. Let $G_i : K \to K$ be a generalized contraction mapping and T a nonexpansive self-mapping defined on K with $F(T) \neq \emptyset$ for each $i = 1, 2$. For an arbitrary $x_1 \in K$, define the iterative sequence $\{x_n\}_{n=1}^{\infty}$ by* (1.6). *Then the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded under the conditions* (i)-(v) *of Assumption* 3.1.

Proof. It is shown that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded. Let $\psi = \max\{\psi_1, \psi_2\}$ and $G = \max\{\|G_1(p) - p\|, \ \|G_2(p) - p\|\}$. For $p \in F(T)$,

$$\begin{aligned} \|x_{n+1} - p\| &= \|\lambda_n^1 G_1(x_n) + \lambda_n^2 x_n + \lambda_n^3 T((1-\delta_n)G_2(x_n) + \delta_n x_{n+1}) - p\| \\ &\leq \lambda_n^1 \|G_1(x_n) - p\| + \lambda_n^2 \|x_n - p\| \\ &\quad + \lambda_n^3 \|T((1-\delta_n)G_2(x_n) + \delta_n x_{n+1}) - p\| \\ &\leq \lambda_n^1 \|G_1(x_n) - G_1(p)\| + \lambda_n^1 \|G_1(p) - p\| + \lambda_n^2 \|x_n - p\| \\ &\quad + \lambda_n^3 \|(1-\delta_n)G_2(x_n) + \delta_n x_{n+1} - p\| \\ &= \lambda_n^1 \|G_1(x_n) - G_1(p)\| + \lambda_n^1 \|G_1(p) - p\| + \lambda_n^2 \|x_n - p\| \\ &\quad + \lambda_n^3 \|(1-\delta_n)(G_2(x_n) - p) + \delta_n (x_{n+1} - p)\| \\ &\leq \lambda_n^1 \|G_1(x_n) - G_1(p)\| + \lambda_n^1 \|G_1(p) - p\| + \lambda_n^2 \|x_n - p\| \\ &\quad + \lambda_n^3 (1-\delta_n) \|G_2(x_n) - G_2(p)\| + \lambda_n^3 (1-\delta_n) \|G_2(p) - p\| \\ &\quad + \lambda_n^3 \delta_n \|x_{n+1} - p\| \end{aligned}$$

$$
\begin{aligned}
&\leq \lambda_n^1\psi_1\|x_n-p\| + \lambda_n^1\|G_1(p)-p\| + \lambda_n^2\|x_n-p\| \\
&\quad + \lambda_n^3(1-\delta_n)\psi_2\|x_n-p\| + \lambda_n^3(1-\delta_n)\|G_2(p)-p\| \\
&\quad + \lambda_n^3\delta_n\|x_{n+1}-p\| \\
&\leq \left(\lambda_n^1\psi + \lambda_n^2 + \lambda_n^3(1-\delta_n)\psi\right)\|x_n-p\| \\
&\quad + \left(\lambda_n^1 + \lambda_n^3(1-\delta_n)\right)G + \lambda_n^3\delta_n\|x_{n+1}-p\| \\
&= \left(\psi + \lambda_n^2(1-\psi) - \lambda_n^3\delta_n\psi\right)\|x_n-p\| \\
&\quad + \left(1-\lambda_n^2-\lambda_n^3\delta_n\right)G + \lambda_n^3\delta_n\|x_{n+1}-p\|.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\|x_{n+1}-p\| &\leq \frac{\psi + \lambda_n^2(1-\psi) - \lambda_n^3\delta_n\psi}{1-\lambda_n^3\delta_n}\|x_n-p\| + \frac{1-\lambda_n^2-\lambda_n^3\delta_n}{1-\lambda_n^3\delta_n}G \\
&= \left(1 - \frac{(1-\lambda_n^2-\lambda_n^3\delta_n)\phi}{1-\lambda_n^3\delta_n}\right)\|x_n-p\| + \frac{(1-\lambda_n^2-\lambda_n^3\delta_n)\phi}{1-\lambda_n^3\delta_n}\phi^{-1}G \\
&\leq \max\left\{\|x_n-p\|,\ \phi^{-1}G\right\}. \qquad (3.3)
\end{aligned}
$$

Then by induction,

$$
\|x_{n+1}-p\| \leq \max\left\{\|x_1-p\|,\ \phi^{-1}G\right\}.
$$

This shows that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded and hence $\{\{G_i(x_n)\}_{n=1}^{\infty}\}_{i=1}^{2}$ and $\{T((1-\delta_n)G_2(x_n)+\delta_n x_{n+1})\}_{n=1}^{\infty}$ are bounded. Certainly, for $p \in F(T)$,

$$
\begin{aligned}
\|G_1(x_n)\| &\leq \|G_1(x_n)-G_1(p)\| + \|G_1(p)\| \\
&\leq \psi_1\|x_n-p\| + \|G_1(p)\| \\
&\leq \max\left\{\psi_1\|x_1-p\|,\ \psi_1\phi^{-1}G\right\} + \|G_1(p)\| \text{ (by induction).}
\end{aligned}
$$

Similarly,

$$
\|G_2(x_n)\| \leq \max\left\{\psi_1\|x_1-p\|,\ \psi_1\phi^{-1}G\right\} + \|G_2(p)\|.
$$

Furthermore,

$$
\begin{aligned}
&\|T((1-\delta_n)G_2(x_n)+\delta_n x_{n+1})\| \\
&= \|T((1-\delta_n)G_2(x_n)+\delta_n x_{n+1}) - p + p\| \\
&\leq \|T((1-\delta_n)G_2(x_n)+\delta_n x_{n+1}) - Tp\| + \|p\| \\
&\leq \|(1-\delta_n)G_2(x_n)+\delta_n x_{n+1} - p\| + \|p\| \\
&\leq (1-\delta_n)\|G_2(x_n)-p\| + \delta_n\|x_{n+1}-p\| + ||p|| \\
&\leq (1-\delta_n)\|G_2(x_n)-G_2(p)\| + (1-\delta_n)\|G_2(p)-p\| + \delta_n\|x_{n+1}-p\| + ||p|| \\
&\leq (1-\delta_n)\psi_2\|x_n-p\| + \delta_n\|x_{n+1}-p\| + (1-\delta_n)\|G_2(p)-p\| + ||p|| \\
&\leq (1-\epsilon)\psi_2\|x_n-p\| + \delta\|x_{n+1}-p\| + (1-\epsilon)\|G_2(p)-p\| + ||p||.
\end{aligned}
$$

Therefore, we have

$$\begin{aligned}&\|T((1-\delta_n)G_2(x_n)+\delta_n x_{n+1})\|\\&\le \max\left\{(1+\delta-\epsilon)\psi\|x_1-p\|,(1+\delta-\epsilon)\psi\phi^{-1}G\right\}\\&\quad+(1-\epsilon)|G_2(p)-p\|+||p|| \text{ (by induction).}\end{aligned}$$

□

Lemma 3.3. *Let K be a nonempty closed convex subset of a uniformly smooth Banach space E. Let $G : K \to K$ be a generalized contraction mapping and T a nonexpansive self-mapping defined on K with $F(T) \neq \emptyset$. Suppose that $\{\delta_n\}_{n=1}^{\infty}$ is a real sequence in $(0,1)$ and $\{x_n\}_{n=1}^{\infty} \subset K$. Set*

$$v_n = (1-\delta_n)G(x_n) + \delta_n x_{n+1}.$$

Then, we have

$$\begin{aligned}\|Tv_{n+1}-Tv_n\| \le{}& (1-\delta_{n+1})\psi\|x_{n+1}-x_n\| + (\delta_{n+1}-\delta_n)\|x_{n+1}-G(x_n)\|\\&+\delta_{n+1}\|x_{n+2}-x_{n+1}\|.\end{aligned}$$

Proof.

$$\begin{aligned}&\|Tv_{n+1}-Tv_n\|\\&=\|T((1-\delta_{n+1})G(x_{n+1})+\delta_{n+1}x_{n+2})-T((1-\delta_n)G(x_n)+\delta_n x_{n+1})\|\\&\le\|(1-\delta_{n+1})G(x_{n+1})+\delta_{n+1}x_{n+2}-(1-\delta_n)G(x_n)-\delta_n x_{n+1}\|\\&=\|(1-\delta_{n+1})G(x_{n+1})-(1-\delta_{n+1})G(x_n)\\&\quad+(1-\delta_{n+1})G(x_n)-(1-\delta_n)G(x_n)\\&\quad+\delta_{n+1}x_{n+2}-\delta_{n+1}x_{n+1}+\delta_{n+1}x_{n+1}-\delta_n x_{n+1}\|\\&=\|(1-\delta_{n+1})(G(x_{n+1})-G(x_n))-(\delta_{n+1}-\delta_n)G(x_n)\\&\quad+\delta_{n+1}(x_{n+2}-x_{n+1})+(\delta_{n+1}-\delta_n)x_{n+1}\|\\&=\|(1-\delta_{n+1})(G(x_{n+1})-G(x_n))+(\delta_{n+1}-\delta_n)(x_{n+1}-G(x_n))\\&\quad+\delta_{n+1}(x_{n+2}-x_{n+1})\|\\&\le(1-\delta_{n+1})\|G(x_{n+1})-G(x_n)\|+(\delta_{n+1}-\delta_n)\|x_{n+1}-G(x_n)\|\\&\quad+\delta_{n+1}\|x_{n+2}-x_{n+1}\|\\&\le(1-\delta_{n+1})\psi\|x_{n+1}-x_n\|+(\delta_{n+1}-\delta_n)\|x_{n+1}-G(x_n)\|\\&\quad+\delta_{n+1}\|x_{n+2}-x_{n+1}\|. \qquad (3.4)\end{aligned}$$

□

Theorem 3.4. *Let K be a nonempty closed convex subset of a uniformly smooth Banach space E. Let $G_i : K \to K$ be generalized contraction mapping and T a nonexpansive self-mapping defined on K with $F(T) \neq \emptyset$, for each*

$i = 1, 2$. *Assume that the conditions* $(i) - (v)$ *of Assumption* 3.1 *are satisfied. Then the iterative sequence* $\{x_n\}_{n=1}^{\infty}$ *which is defined from an arbitrary* $x_1 \in K$ *by* (1.6), *converges strongly to a fixed point* p *of* T, *which solves the variational inequality*

$$\langle (I - G_1)p, J(x - p)\rangle \geq 0, \quad \forall\, x \in F(T). \tag{3.5}$$

Proof. Set $u_n = \frac{x_{n+1} - \lambda_n^2 x_n}{1 - \lambda_n^2}$ and $v_n = (1 - \delta_n)G_2(x_n) + \delta_n x_{n+1}$. Then it could be obtained that,

$$\begin{aligned}
u_{n+1} - u_n &= \frac{x_{n+2} - \lambda_{n+1}^2 x_{n+1}}{1 - \lambda_{n+1}^2} - \frac{x_{n+1} - \lambda_n^2 x_n}{1 - \lambda_n^2} \\
&= \frac{\lambda_{n+1}^1 G_1(x_{n+1}) + \lambda_{n+1}^3 T(y_{n+1})}{1 - \lambda_{n+1}^2} - \frac{\lambda_n^1 G_1(x_n) + \lambda_n^3 T(y_n)}{1 - \lambda_n^2} \\
&= \frac{\lambda_{n+1}^1}{1 - \lambda_{n+1}^2}(G_1(x_{n+1}) - G_1(x_n)) + \left(\frac{\lambda_{n+1}^1}{1 - \lambda_{n+1}^2} - \frac{\lambda_n^1}{1 - \lambda_n^2}\right) G_1(x_n) \\
&\quad + \frac{\lambda_{n+1}^3}{1 - \lambda_{n+1}^2}(T(y_{n+1}) - T(y_n)) + \left(\frac{\lambda_{n+1}^3}{1 - \lambda_{n+1}^2} - \frac{\lambda_n^3}{1 - \lambda_n^2}\right) T(y_n) \\
&= \frac{\lambda_{n+1}^1}{1 - \lambda_{n+1}^2}(G_1(x_{n+1}) - G_1(x_n)) - \left(\frac{\lambda_{n+1}^3}{1 - \lambda_{n+1}^2} - \frac{\lambda_n^3}{1 - \lambda_n^2}\right) G_1(x_n) \\
&\quad + \frac{\lambda_{n+1}^3}{1 - \lambda_{n+1}^2}(T(y_{n+1}) - T(y_n)) + \left(\frac{\lambda_{n+1}^3}{1 - \lambda_{n+1}^2} - \frac{\lambda_n^3}{1 - \lambda_n^2}\right) T(y_n) \\
&= \frac{\lambda_{n+1}^1}{1 - \lambda_{n+1}^2}(G_1(x_{n+1}) - G_1(x_n)) \\
&\quad + \left(\frac{\lambda_{n+1}^3}{1 - \lambda_{n+1}^2} - \frac{\lambda_n^3}{1 - \lambda_n^2}\right)(T(y_n) - G_1(x_n)) \\
&\quad + \frac{\lambda_{n+1}^3}{1 - \lambda_{n+1}^2}(T(y_{n+1}) - T(y_n)).
\end{aligned}$$

Let

$$\begin{aligned}
M_n^1 &= \sup_n \{\|T(y_n) - G_1(x_n)\|\}, \\
M_n^2 &= \sup_n \{\|x_n - G_1(x_n)\|\}, \\
M_n^3 &= \sup_n \{\|x_{n+1} - G_2(x_n)\|\}
\end{aligned}$$

and $M = \max\left\{M_n^1,\ M_n^2,\ M_n^3\right\}$. Put $\psi = \max\{\psi_1,\ \psi_2\}$. Then, it can be obtained from (3.4) that

$$
\begin{aligned}
\|u_{n+1}-u_n\| \leq\ & \frac{\lambda^1_{n+1}}{1-\lambda^2_{n+1}}\|G_1(x_{n+1})-G_1(x_n)\| \\
&+\left|\frac{\lambda^3_{n+1}}{1-\lambda^2_{n+1}}-\frac{\lambda^3_n}{1-\lambda^2_n}\right| \ \|T(y_n)-G_1(x_n)\| \\
&+\frac{\lambda^3_{n+1}}{1-\lambda^2_{n+1}}\|T(y_{n+1})-T(y_n)\| \\
\leq\ & \frac{\lambda^1_{n+1}}{1-\lambda^2_{n+1}}\psi_1\|x_{n+1}-x_n\| \\
&+\left|\frac{\lambda^3_{n+1}}{1-\lambda^2_{n+1}}-\frac{\lambda^3_n}{1-\lambda^2_n}\right| \|T(y_n)-G_1(x_n)\| \\
&+\frac{\lambda^3_{n+1}}{1-\lambda^2_{n+1}}\ [(1-\delta_{n+1})\psi_2\|x_{n+1}-x_n\| \\
&+(\delta_{n+1}-\delta_n)\|x_{n+1}-G_2(x_n)\|+\delta_{n+1}\|x_{n+2}-x_{n+1}\|\ ] \\
\leq\ & \frac{\lambda^1_{n+1}\psi+\lambda^3_{n+1}(1-\delta_{n+1})\psi}{1-\lambda^2_{n+1}}\|x_{n+1}-x_n\| \\
&+\left(\left|\frac{\lambda^3_{n+1}}{1-\lambda^2_{n+1}}-\frac{\lambda^3_n}{1-\lambda^2_n}\right|+\frac{\lambda^3_{n+1}(\delta_{n+1}-\delta_n)}{1-\lambda^2_{n+1}}\right)M \\
&+\frac{\lambda^3_{n+1}\delta_{n+1}}{1-\lambda^2_{n+1}}\|x_{n+2}-x_{n+1}\|. \qquad (3.6)
\end{aligned}
$$

Next is to evaluate $\|x_{n+1}-x_n\|$.

$$
\begin{aligned}
x_{n+2}-x_{n+1} =\ & \lambda^1_{n+1}G_1(x_{n+1})+\lambda^2_{n+1}x_{n+1}+\lambda^3_{n+1}Ty_{n+1} \\
&-\left(\lambda^1_nG_1(x_n)+\lambda^2_nx_n+\lambda^3_nTy_n\right) \\
=\ & \lambda^1_{n+1}(G_1(x_{n+1})-G_1(x_n))+\lambda^2_{n+1}(x_{n+1}-x_n) \\
&+\lambda^3_{n+1}(Ty_{n+1}-Ty_n)+(\lambda^1_{n+1}-\lambda^1_n)G_1(x_n) \\
&+(\lambda^2_{n+1}-\lambda^2_n)x_n+(\lambda^3_{n+1}-\lambda^3_n)Ty_n \\
=\ & \lambda^1_{n+1}(G_1(x_{n+1})-G_1(x_n))+\lambda^2_{n+1}(x_{n+1}-x_n) \\
&+\lambda^3_{n+1}(Ty_{n+1}-Ty_n) \\
&+((\lambda^2_n-\lambda^2_{n+1})+(\lambda^3_n-\lambda^3_{n+1}))G_1(x_n) \\
&+(\lambda^2_{n+1}-\lambda^2_n)x_n+(\lambda^3_{n+1}-\lambda^3_n)Ty_n \\
=\ & \lambda^1_{n+1}(G_1(x_{n+1})-G_1(x_n))+\lambda^2_{n+1}(x_{n+1}-x_n) \\
&+\lambda^3_{n+1}(Ty_{n+1}-Ty_n)+(\lambda^2_{n+1}-\lambda^2_n)(x_n-G_1(x_n)) \\
&+(\lambda^3_{n+1}-\lambda^3_n)(Ty_n-G_1(x_n)).
\end{aligned}
$$

Then, from (3.4)) it leads to

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &\leq \lambda^1_{n+1}\psi\|x_{n+1}-x_n\| + \lambda^2_{n+1}\|x_{n+1}-x_n\| \\
&\quad +\lambda^3_{n+1}\|Ty_{n+1}-Ty_n\| \\
&\quad +|\lambda^2_{n+1}-\lambda^2_n|\|x_n-G_1(x_n)\| \\
&\quad +|\lambda^3_{n+1}-\lambda^3_n|\|Ty_n-G_1(x_n)\| \\
&\leq \lambda^1_{n+1}\psi\|x_{n+1}-x_n\| + \lambda^2_{n+1}\|x_{n+1}-x_n\| \\
&\quad +\lambda^3_{n+1}[(1-\delta_{n+1})\psi\|x_{n+1}-x_n\| \\
&\quad +(\delta_{n+1}-\delta_n)\|x_{n+1}-G_2(x_n)\| + \delta_{n+1}\|x_{n+2}-x_{n+1}\|\ ] \\
&\quad +|\lambda^2_{n+1}-\lambda^2_n|\|x_n-G_1(x_n)\| \\
&\quad +|\lambda^3_{n+1}-\lambda^3_n|\|Ty_n-G_1(x_n)\| \\
&= \left(\lambda^2_{n+1}+(\lambda^1_{n+1}+\lambda^3_{n+1})\psi-\lambda^3_{n+1}\delta_{n+1}\psi\right)\|x_{n+1}-x_n\| \\
&\quad +\lambda^3_{n+1}\delta_{n+1}\|x_{n+2}-x_{n+1}\| \\
&\quad +\left(|\lambda^2_{n+1}-\lambda^2_n|+|\lambda^3_{n+1}-\lambda^3_n|+\lambda^3_{n+1}(\delta_{n+1}-\delta_n)\right)M \\
&= \left(\lambda^2_{n+1}+(1-\lambda^2_{n+1})\psi-\lambda^3_{n+1}\delta_{n+1}\psi\right)\|x_{n+1}-x_n\| \\
&\quad +\lambda^3_{n+1}\delta_{n+1}\|x_{n+2}-x_{n+1}\| \\
&\quad +\left(|\lambda^2_{n+1}-\lambda^2_n|+|\lambda^3_{n+1}-\lambda^3_n|+\lambda^3_{n+1}(\delta_{n+1}-\delta_n)\right)M \\
&= \left(\psi+\lambda^2_{n+1}(1-\psi)-\lambda^3_{n+1}\delta_{n+1}\psi\right)\|x_{n+1}-x_n\| \\
&\quad +\lambda^3_{n+1}\delta_{n+1}\|x_{n+2}-x_{n+1}\| \\
&\quad +\left(|\lambda^2_{n+1}-\lambda^2_n|+|\lambda^3_{n+1}-\lambda^3_n|+\lambda^3_{n+1}(\delta_{n+1}-\delta_n)\right)M \\
&= \left(\lambda^2_{n+1}(1-\psi)+(1-\lambda^3_{n+1}\delta_{n+1})\psi\right)\|x_{n+1}-x_n\| \\
&\quad +\lambda^3_{n+1}\delta_{n+1}\|x_{n+2}-x_{n+1}\| \\
&\quad +\left(|\lambda^2_{n+1}-\lambda^2_n|+|\lambda^3_{n+1}-\lambda^3_n|+\lambda^3_{n+1}(\delta_{n+1}-\delta_n)\right)M.
\end{aligned}$$

Putting $d_n = \left(|\lambda^2_{n+1}-\lambda^2_n|+|\lambda^3_{n+1}-\lambda^3_n|+\lambda^3_{n+1}(\delta_{n+1}-\delta_n)\right)$, it could be obtained that,

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &\leq \frac{\lambda^2_{n+1}(1-\psi)+(1-\lambda^3_{n+1}\delta_{n+1})\psi}{1-\lambda^3_{n+1}\delta_{n+1}}\|x_{n+1}-x_n\| \\
&\quad +\frac{d_nM}{1-\lambda^3_{n+1}\delta_{n+1}}.
\end{aligned} \tag{3.7}$$

Let $S_n = \left|\frac{\lambda^3_{n+1}}{1-\lambda^2_{n+1}}-\frac{\lambda^3_n}{1-\lambda^2_n}\right| + \frac{\lambda^3_{n+1}(\delta_{n+1}-\delta_n)}{1-\lambda^2_{n+1}}$ and substitute (3.7) into (3.6) to obtain

$$
\begin{aligned}
&\|u_{n+1}-u_n\| \\
\leq\ & [\frac{\lambda^1_{n+1}\psi+\lambda^3_{n+1}(1-\delta_{n+1})\psi}{1-\lambda^2_{n+1}}+\frac{\lambda^3_{n+1}\delta_{n+1}}{1-\lambda^2_{n+1}} \\
&\times\frac{\lambda^2_{n+1}(1-\psi)+(1-\lambda^3_{n+1}\delta_{n+1})\psi}{1-\lambda^3_{n+1}\delta_{n+1}}\ ]\|x_{n+1}-x_n\| \\
&+S_nM+\frac{\lambda^3_{n+1}\delta_{n+1}}{1-\lambda^2_{n+1}}\times\frac{d_nM}{1-\lambda^3_{n+1}\delta_{n+1}} \\
=\ & [\frac{\lambda^1_{n+1}\psi+\lambda^3_{n+1}(1-\delta_{n+1})\psi-\lambda^3_{n+1}\delta_{n+1}(\lambda^1_{n+1}\psi+\lambda^3_{n+1}(1-\delta_{n+1})\psi)}{[1-\lambda^2_{n+1}][1-\lambda^3_{n+1}\delta_{n+1}]} \\
&+\frac{\lambda^3_{n+1}\delta_{n+1}(\lambda^2_{n+1}(1-\psi)+(1-\lambda^3_{n+1}\delta_{n+1})\psi)}{[1-\lambda^2_{n+1}][1-\lambda^3_{n+1}\delta_{n+1}]}\ ]\|x_{n+1}-x_n\| \\
&+\left(S_n+\frac{d_n\lambda^3_{n+1}\delta_{n+1}}{[1-\lambda^2_{n+1}][1-\lambda^3_{n+1}\delta_{n+1}]}\right)M \\
=\ & [\frac{\lambda^1_{n+1}\psi+\lambda^3_{n+1}(1-\delta_{n+1})\psi-\lambda^3_{n+1}\delta_{n+1}(\lambda^1_{n+1}\psi+\lambda^3_{n+1}\psi-\lambda^3_{n+1}\delta_{n+1}\psi)}{[1-\lambda^2_{n+1}][1-\lambda^3_{n+1}\delta_{n+1}]} \\
&+\frac{\lambda^3_{n+1}\delta_{n+1}(\lambda^2_{n+1}-\lambda^2_{n+1}\psi+\psi-\lambda^3_{n+1}\delta_{n+1}\psi)}{[1-\lambda^2_{n+1}][1-\lambda^3_{n+1}\delta_{n+1}]}\ ]\|x_{n+1}-x_n\| \\
&+\left(S_n+\frac{d_n\lambda^3_{n+1}\delta_{n+1}}{[1-\lambda^2_{n+1}][1-\lambda^3_{n+1}\delta_{n+1}]}\right)M \\
=\ & \Big[\frac{\lambda^1_{n+1}\psi+\lambda^3_{n+1}(1-\delta_{n+1})\psi-\lambda^3_{n+1}\delta_{n+1}((1-\lambda^2_{n+1})\psi-\lambda^3_{n+1}\delta_{n+1}\psi)}{[1-\lambda^2_{n+1}][1-\lambda^3_{n+1}\delta_{n+1}]} \\
&+\frac{\lambda^3_{n+1}\delta_{n+1}(\lambda^2_{n+1}+(1-\lambda^2_{n+1})\psi-\lambda^3_{n+1}\delta_{n+1}\psi)}{[1-\lambda^2_{n+1}][1-\lambda^3_{n+1}\delta_{n+1}]}\ \Big]\|x_{n+1}-x_n\| \\
&+\left(S_n+\frac{d_n\lambda^3_{n+1}\delta_{n+1}}{[1-\lambda^2_{n+1}][1-\lambda^3_{n+1}\delta_{n+1}]}\right)M \\
=\ & \frac{\lambda^1_{n+1}\psi+\lambda^3_{n+1}(1-\delta_{n+1})\psi+\lambda^3_{n+1}\delta_{n+1}\lambda^2_{n+1}}{[1-\lambda^2_{n+1}][1-\lambda^3_{n+1}\delta_{n+1}]}\|x_{n+1}-x_n\| \\
&+\left(S_n+\frac{d_n\lambda^3_{n+1}\delta_{n+1}}{[1-\lambda^2_{n+1}][1-\lambda^3_{n+1}\delta_{n+1}]}\right)M \\
=\ & \frac{(1-\lambda^2_{n+1})\psi-\lambda^3_{n+1}\delta_{n+1}\psi+\lambda^3_{n+1}\delta_{n+1}\lambda^2_{n+1}}{[1-\lambda^2_{n+1}][1-\lambda^3_{n+1}\delta_{n+1}]}\|x_{n+1}-x_n\| \\
&+\left(S_n+\frac{d_n\lambda^3_{n+1}\delta_{n+1}}{[1-\lambda^2_{n+1}][1-\lambda^3_{n+1}\delta_{n+1}]}\right)M
\end{aligned}
$$

$$
\begin{aligned}
&= \left(1 - \frac{(1-\lambda_{n+1}^2)(1-\psi) - \lambda_{n+1}^3\delta_{n+1}(1-\psi)}{[1-\lambda_{n+1}^2][1-\lambda_{n+1}^3\delta_{n+1}]}\right)\|x_{n+1}-x_n\| \\
&\quad + \left(S_n + \frac{d_n\lambda_{n+1}^3\delta_{n+1}}{[1-\lambda_{n+1}^2][1-\lambda_{n+1}^3\delta_{n+1}]}\right)M \\
&= \left(1 - \frac{(1-\lambda_{n+1}^2)\phi - \lambda_{n+1}^3\delta_{n+1}\phi}{[1-\lambda_{n+1}^2][1-\lambda_{n+1}^3\delta_{n+1}]}\right)\|x_{n+1}-x_n\| \\
&\quad + \left(S_n + \frac{d_n\lambda_{n+1}^3\delta_{n+1}}{[1-\lambda_{n+1}^2][1-\lambda_{n+1}^3\delta_{n+1}]}\right)M \\
&= \left(1 - \frac{(1-\lambda_{n+1}^2 - \lambda_{n+1}^3\delta_{n+1})\phi}{[1-\lambda_{n+1}^2][1-\lambda_{n+1}^3\delta_{n+1}]}\right)\|x_{n+1}-x_n\| \\
&\quad + \left(S_n + \frac{d_n\lambda_{n+1}^3\delta_{n+1}}{[1-\lambda_{n+1}^2][1-\lambda_{n+1}^3\delta_{n+1}]}\right)M \\
&\leq \left(1 - \frac{(1-\lambda_{n+1}^2 - \lambda_{n+1}^3\delta_{n+1})\phi}{1-\lambda_{n+1}^2}\right)\|x_{n+1}-x_n\| \\
&\quad + \left(S_n + \frac{d_n\lambda_{n+1}^3\delta_{n+1}}{[1-\lambda_{n+1}^2][1-\lambda_{n+1}^3\delta_{n+1}]}\right)M.
\end{aligned}
$$

It then follows that

$$
\begin{aligned}
\|u_{n+1}-u_n\| - \|x_{n+1}-x_n\| \quad \leq \quad & -\frac{(1-\lambda_{n+1}^2 - \lambda_{n+1}^3\delta_{n+1})\phi}{1-\lambda_{n+1}^2}\|x_{n+1}-x_n\| \\
& + \left(S_n + \frac{d_n\lambda_{n+1}^3\delta_{n+1}}{(1-\lambda_{n+1}^2)(1-\lambda_{n+1}^3\delta_{n+1})}\right)M,
\end{aligned}
$$

and thus,

$$
\limsup_{n\to\infty}\left(\|u_{n+1}-u_n\| - \|x_{n+1}-x_n\|\right) \leq 0. \tag{3.8}
$$

Invoking Lemma 2.7 gives

$$
\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{3.9}
$$

Consequently,

$$
\begin{aligned}
\|x_{n+1}-x_n\| &= \|(1-\lambda_n^2)u_n + \lambda_n^2 x_n - x_n\| \\
&= \|(1-\lambda_n^2)u_n - (1-\lambda_n^2)x_n\| \\
&= \|(1-\lambda_n^2)(u_n - x_n)\| \\
&\leq (1-\lambda_n^2)\|u_n - x_n\| \to 0 \text{ as } \ n\to\infty.
\end{aligned} \tag{3.10}
$$

Next is to show that $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$. From (1.6), we could have that

$$\begin{aligned}
\|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T(x_n)\| \\
&\leq \|x_{n+1} - x_n\| + \|\lambda_n^1 G_1(x_n) + \lambda_n^2 x_n + \lambda_n^3 T(v_n) - T(x_n)\| \\
&\leq \|x_{n+1} - x_n\| + \lambda_n^1 \|G_1(x_n) - T(x_n)\| + \lambda_n^2 \|x_n - T(x_n)\| \\
&\quad + \lambda_n^3 \|T(v_n) - T(x_n)\| \\
&\leq \|x_{n+1} - x_n\| + \lambda_n^1 \|G_1(x_n) - T(x_n)\| + \lambda_n^2 \|x_n - T(x_n)\| \\
&\quad + \lambda_n^3 \|v_n - x_n\| \\
&\leq \|x_{n+1} - x_n\| + \lambda_n^1 \|G_1(x_n) - T(x_n)\| + \lambda_n^2 \|x_n - T(x_n)\| \\
&\quad + \lambda_n^3 \|(1-\delta_n) G_2(x_n) + \delta_n x_{n+1} - x_n\| \\
&\leq \|x_{n+1} - x_n\| + \lambda_n^1 \|G_1(x_n) - T(x_n)\| + \lambda_n^2 \|x_n - T(x_n)\| \\
&\quad + \lambda_n^3 (1-\delta_n) \|x_n - G_2(x_n)\| + \lambda_n^3 \delta_n \|x_{n+1} - x_n\| \\
&= (1 + \lambda_n^3 \delta_n) \|x_{n+1} - x_n\| + (\lambda_n^1 + \lambda_n^3 (1-\delta_n)) M \\
&\quad + \lambda_n^2 \|x_n - T(x_n)\| \\
&= (1 + \lambda_n^3 \delta_n) \|x_{n+1} - x_n\| + (1 - \lambda_n^3 \delta_n - \lambda_n^2) M \\
&\quad + \lambda_n^2 \|x_n - T(x_n)\|.
\end{aligned}$$

From $0 < \liminf_{n\to\infty} \lambda_n^2 \leq \limsup_{n\to\infty} \lambda_n^2 < 1$, let $0 < \eta \leq \lambda_n^2 < 1$. Then

$$\begin{aligned}
\|x_n - Tx_n\| &\leq \frac{1 + \lambda_n^3 \delta_n}{1 - \lambda_n^2} \|x_{n+1} - x_n\| + \frac{1 - \lambda_n^2 - \lambda_n^3 \delta_n}{1 - \lambda_n^2} M \\
&\leq \frac{1 + \lambda_n^3 \delta_n}{1 - \eta} \|x_{n+1} - x_n\| + \frac{1 - \lambda_n^2 - \lambda_n^3 \delta_n}{1 - \eta} M, \qquad (3.11)
\end{aligned}$$

which goes to zero as $n \to \infty$ by (3.10) and condition (ii) of Assumption 3.1.

Let a net $\{x_t\}$ be defined by $x_t = tG_1(x_t) + (1-t)Tx_t$ for $t \in (0,1)$. It is known by Lemma 2.5 that $\{x_t\}$ converges strongly to $p \in F(T)$, which solves the variational inequality:

$$\langle G_1(p) - p, J(x-p) \rangle \leq 0, \quad \forall\, x \in F(T),$$

which is equivalent to

$$\langle (I - G_1)p, J(x-p) \rangle \geq 0, \quad \forall\, x \in F(T).$$

It is claimed that

$$\limsup_{n\to\infty} \langle G_1(p) - p,\ J(x_{n+1} - p) \rangle \leq 0, \qquad (3.12)$$

where $p \in F(T)$ is the unique fixed point of the generalized contraction $P_{F(T)}G_1(p)$ (Proposition 2.4), that is, $p = P_{F(T)}G_1(p)$.

By (3.11), $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. So it follows from Lemma 2.6 that

$$\limsup_{n\to\infty} \langle G_1(p) - p, J(x_n - p)\rangle \leq 0.$$

Due to the norm-to-weak* uniform continuity on bounded sets of the duality map and the fact that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$ by (3.10), we obtain that,

$$\begin{aligned}
&\limsup_{n\to\infty} \langle G_1(p) - p,\ J(x_{n+1} - p)\rangle \\
&= \limsup_{n\to\infty} \langle G_1(p) - p,\ J(x_{n+1} - x_n + x_n - p)\rangle \\
&= \limsup_{n\to\infty} \langle G_1(p) - p,\ J(x_n - p)\rangle \leq 0. \qquad (3.13)
\end{aligned}$$

Lastly, it is established that $x_n \to p \in F(T)$ as $n \to \infty$. Suppose that the sequence $\{x_n\}_{n=1}^{\infty}$ does not converge strongly to $p \in F(T)$. Then there exists $\epsilon > 0$ and a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\|x_{n_k} - p\| \geq \epsilon$, for all $k \in \mathbb{N}$. Therefore, for this ϵ, there exists $c_i \in (0, \frac{1}{2})$ such that

$$\|G_i(x_{n_k}) - G_i(p)\| \leq c_i \|x_{n_k} - p\|,\ i = 1, 2.$$

Let $c = \max\{c_1, c_2\}$. Then,

$$\begin{aligned}
||x_{n_{k+1}} - p||^2 &= \lambda_{n_k}^1 \left\langle G_1(x_{n_k}) - p,\ J(x_{n_{k+1}} - p)\right\rangle \\
&\quad + \lambda_{n_k}^2 \left\langle x_{n_k} - p,\ J(x_{n_{k+1}} - p)\right\rangle \\
&\quad + \lambda_{n_k}^3 \left\langle T(y_{n_k}) - p,\ J(x_{n_{k+1}} - p)\right\rangle \\
&= \lambda_{n_k}^1 \left\langle G_1(x_{n_k}) - G_1(p),\ J(x_{n_{k+1}} - p)\right\rangle \\
&\quad + \lambda_n^1 \left\langle G_1(p) - p,\ J(x_{n_{k+1}} - p)\right\rangle \\
&\quad + \lambda_{n_k}^2 \left\langle x_{n_k} - p,\ J(x_{n_{k+1}} - p)\right\rangle \\
&\quad + \lambda_{n_k}^3 \left\langle T(y_{n_k}) - p, J(x_{n_{k+1}} - p)\right\rangle \\
&\leq c\lambda_{n_k}^1 \|x_{n_k} - p\|\ \|x_{n_{k+1}} - p\| \\
&\quad + \lambda_n^1 \left\langle G_1(p) - p,\ J(x_{n_{k+1}} - p)\right\rangle \\
&\quad + \lambda_{n_k}^2 \|x_{n_k} - p\|\ \|x_{n_{k+1}} - p\| \\
&\quad + \lambda_{n_k}^3 ||(1 - \delta_{n_k})G_2(x_{n_k}) + \delta_{n_k} x_{n_{k+1}} - p||\ ||x_{n_{k+1}} - p|| \\
&\leq c\lambda_{n_k}^1 \|x_{n_k} - p\|\ \|x_{n_k+1} - p\|
\end{aligned}$$

$$
\begin{aligned}
& +\lambda_n^1 \left\langle G_1(p)-p,\ J(x_{n_{k+1}}-p)\right\rangle \\
& +\lambda_{n_k}^2 \|x_{n_k}-p\|\ \|x_{n_{k+1}}-p\| \\
& +\lambda_{n_k}^3 (1-\delta_{n_k}) ||G_2(x_{n_k})-p||\ ||x_{n_{k+1}}-p|| \\
& +\lambda_{n_k}^3 \delta_{n_k} \|x_{n_{k+1}}-p\|^2 \\
\leq\ & c\lambda_{n_k}^1 \|x_{n_k}-p\|\ \|x_{n_{k+1}}-p\| + \lambda_n^1 \left\langle G_1(p)-p,\ J(x_{n_{k+1}}-p)\right\rangle \\
& +\lambda_{n_k}^2 \|x_{n_k}-p\|\ \|x_{n_{k+1}}-p\| + c\lambda_{n_k}^3 (1-\delta_{n_k}) ||x_{n_k}-p||\ ||x_{n_{k+1}}-p|| \\
& +\lambda_{n_k}^3 (1-\delta_{n_k}) ||G_2(p)-p||\ ||x_{n_{k+1}}-p|| + \lambda_{n_k}^3 \delta_{n_k} \|x_{n_{k+1}}-p\|^2 \\
=\ & \left(c\lambda_{n_k}^1 + \lambda_{n_k}^2 + c\lambda_{n_k}^3 (1-\delta_{n_k})\right) \|x_{n_k}-p\|\ \|x_{n_{k+1}}-p\| \\
& +\lambda_n^1 \left\langle G_1(p)-p,\ J(x_{n_{k+1}}-p)\right\rangle \\
& +\lambda_{n_k}^3 (1-\delta_{n_k}) ||G_2(p)-p||\ ||x_{n_{k+1}}-p|| + \lambda_{n_k}^3 \delta_{n_k} \|x_{n_{k+1}}-p\|^2 \\
\leq\ & \frac{1}{2}\left(c\lambda_{n_k}^1 + \lambda_{n_k}^2 + c\lambda_{n_k}^3 (1-\delta_{n_k})\right) \left(\|x_{n_k}-p\|^2 + \|x_{n_{k+1}}-p\|^2\right) \\
& +\lambda_n^1 \left\langle G_1(p)-p,\ x_{n_{k+1}}-p\right\rangle + \lambda_{n_k}^3 \delta_{n_k} \|x_{n_{k+1}}-p\|^2 \\
& +\frac{1}{2}\lambda_{n_k}^3 (1-\delta_{n_k}) \left(\|G_2(p)-p\|^2 + \|x_{n_{k+1}}-p\|^2\right) \\
=\ & \frac{1}{2}\left(c(\lambda_{n_k}^1 + \lambda_{n_k}^3 (1-\delta_{n_k})) + \lambda_{n_k}^2\right) \|x_{n_k}-p\|^2 \\
& +\lambda_n^1 \left\langle G_1(p)-p,\ J(x_{n_{k+1}}-p)\right\rangle \\
& +\frac{1}{2}\left(c(\lambda_{n_k}^1 + \lambda_{n_k}^3 (1-\delta_{n_k})) + \lambda_{n_k}^2 + 2\lambda_{n_k}^3 \delta_{n_k} + \lambda_{n_k}^3 (1-\delta_{n_k})\right) \|x_{n_{k+1}}-p\|^2 \\
& +\frac{1}{2}\lambda_{n_k}^3 (1-\delta_{n_k}) \|G_2(p)-p\|^2 \\
=\ & \frac{1}{2}\left(c(\lambda_{n_k}^1 + \lambda_{n_k}^3 (1-\delta_{n_k})) + \lambda_{n_k}^2\right) \|x_{n_k}-p\|^2 \\
& +\lambda_n^1 \left\langle G_1(p)-p,\ J(x_{n_{k+1}}-p)\right\rangle \\
& +\frac{1}{2}\left(c(\lambda_{n_k}^1 + \lambda_{n_k}^3 (1-\delta_{n_k})) + \lambda_{n_k}^2 + \lambda_{n_k}^3 (1+\delta_{n_k})\right) \|x_{n_{k+1}}-p\|^2 \\
& +\frac{1}{2}\lambda_{n_k}^3 (1-\delta_{n_k}) \|G_2(p)-p\|^2 \\
=\ & \frac{1}{2}\left(c(1-\lambda_{n_k}^2 - \lambda_{n_k}^3 \delta_{n_k}) + \lambda_{n_k}^2\right) \|x_{n_k}-p\|^2 \\
& +\lambda_n^1 \left\langle G_1(p)-p,\ J(x_{n_{k+1}}-p)\right\rangle \\
& +\frac{1}{2}\left(c(1-\lambda_{n_k}^2 - \lambda_{n_k}^3 \delta_{n_k}) + \lambda_{n_k}^2 + \lambda_{n_k}^3 (1+\delta_{n_k})\right) \|x_{n_{k+1}}-p\|^2 \\
& +\frac{1}{2}\lambda_{n_k}^3 (1-\delta_{n_k}) \|G_2(p)-p\|^2. \qquad (3.14)
\end{aligned}
$$

Observe that

$$\begin{aligned}
&2 - c(1 - \lambda_{n_k}^2 - \lambda_{n_k}^3 \delta_{n_k}) - \lambda_{n_k}^2 - \lambda_{n_k}^3 (1 + \delta_{n_k}) \\
&= 2 - c + c\lambda_{n_k}^2 + c\lambda_{n_k}^3 \delta_{n_k} - \lambda_{n_k}^2 - \lambda_{n_k}^3 - \lambda_{n_k}^3 \delta_{n_k} \\
&= 2 - c - (1-c)\lambda_{n_k}^2 - (1-c)\lambda_{n_k}^3 \delta_{n_k} - \lambda_{n_k}^3 \\
&= 1 - c - (1-c)\lambda_{n_k}^2 - (1-c)\lambda_{n_k}^3 \delta_{n_k} + 1 - \lambda_{n_k}^3 \\
&= 1 + (1-c)\left(1 - \lambda_{n_k}^2 - \lambda_{n_k}^3 \delta_{n_k}\right) - \lambda_{n_k}^3
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
\lambda_{n_k}^1 &= 1 - \lambda_{n_k}^2 - \lambda_{n_k}^3 \\
&\leq 1 - \lambda_{n_k}^2 - \lambda_{n_k}^3 \delta_{n_k} \text{ (since } \delta_{n_k} \in (0,1)).
\end{aligned} \tag{3.16}$$

Simplifying (3.14) by 2 gives

$$\begin{aligned}
&||x_{n_{k+1}} - p||^2 \\
&\leq \frac{c(1 - \lambda_{n_k}^2 - \lambda_{n_k}^3 \delta_{n_k}) + \lambda_{n_k}^2}{1 + (1-c)\left(1 - \lambda_{n_k}^2 - \lambda_{n_k}^3 \delta_{n_k}\right) - \lambda_{n_k}^3} \|x_{n_k} - p\|^2 \\
&\quad + \frac{\lambda_n^1}{1 + (1-c)\left(1 - \lambda_{n_k}^2 - \lambda_{n_k}^3 \delta_{n_k}\right) - \lambda_{n_k}^3} \left\langle G_1(p) - p,\ J(x_{n_{k+1}} - p)\right\rangle \\
&\quad + \frac{\lambda_{n_k}^3 (1 - \delta_{n_k})}{1 + (1-c)\left(1 - \lambda_{n_k}^2 - \lambda_{n_k}^3 \delta_{n_k}\right) - \lambda_{n_k}^3} \|G_2(p) - p\|^2 \\
&= \left(1 - \frac{(1-2c)(1 - \lambda_{n_k}^2 - \lambda_{n_k}^3 \delta_{n_k}) + \lambda_{n_k}^1}{1 + (1-c)\left(1 - \lambda_{n_k}^2 - \lambda_{n_k}^3 \delta_{n_k}\right) - \lambda_{n_k}^3}\right) \|x_{n_k} - p\|^2 \\
&\quad + \frac{\lambda_{n_k}^1}{1 + (1-c)\left(1 - \lambda_{n_k}^2 - \lambda_{n_k}^3 \delta_{n_k}\right) - \lambda_{n_k}^3} \left\langle G_1(p) - p,\ J(x_{n_{k+1}} - p)\right\rangle \\
&\quad + \frac{\lambda_{n_k}^3 (1 - \delta_{n_k})}{1 + (1-c)\left(1 - \lambda_{n_k}^2 - \lambda_{n_k}^3 \delta_{n_k}\right) - \lambda_{n_k}^3} \|G_2(p) - p\|^2 \\
&\leq \left(1 - \frac{(1-2c)(1 - \lambda_{n_k}^2 - \lambda_{n_k}^3 \delta_{n_k})}{1 + (1-c)\left(1 - \lambda_{n_k}^2 - \lambda_{n_k}^3 \delta_{n_k}\right) - \lambda_{n_k}^3}\right) \|x_{n_k} - p\|^2 \\
&\quad + \frac{(1-2c)(1 - \lambda_{n_k}^2 - \lambda_{n_k}^3 \delta_{n_k})}{1 + (1-c)\left(1 - \lambda_{n_k}^2 - \lambda_{n_k}^3 \delta_{n_k}\right) - \lambda_{n_k}^3} \frac{1}{1-2c} \left\langle G_1(p) - p,\ J(x_{n_{k+1}} - p)\right\rangle \\
&\quad + \frac{\lambda_{n_k}^3 (1 - \delta_{n_k})}{1 + (1-c)\left(1 - \lambda_{n_k}^2 - \lambda_{n_k}^3 \delta_{n_k}\right) - \lambda_{n_k}^3} \|G_2(p) - p\|^2 \text{ (By (3.16))}.
\end{aligned}$$

By taking $\alpha_n = (1-2c)(1-\lambda^2_{n_k} - \lambda^3_{n_k}\delta_{n_k}),\ \sigma_n = \langle G_1(p) - p,\ J(x_{n_{k+1}} - p)\rangle$ and $\gamma_n = \lambda^3_{n_k}(1-\delta_{n_k})$ in Lemma 2.8, it shows that $x_{n_k} \to p$ as $k \to \infty$, which is a contradiction. Hence, $\{x_n\}_{n=1}^{\infty}$ converges strongly to $p \in F(T)$. □

The next result shows that under suitable conditions, the implicit iterative sequences (1.4) and (1.6) converge to the same fixed point.

Theorem 3.5. *Let K be a nonempty closed convex subset of a uniformly smooth Banach space E. Let $G_i : K \to K$ be a c-contraction mapping and T be a nonexpansive self-mapping defined on K with $F(T) \neq \emptyset$ for each $i = 1, 2$. Let $\{\{\lambda^i_n\}_{n=1}^{\infty}\}_{i=1}^{3} \subset [0,1]$ and $\{\delta_n\}_{n=1}^{\infty} \subset (0,1)$ be real sequences such that $\sum_{i=1}^{3} \lambda^i_n = 1$. Suppose that G in (1.4) is the same as G_1 in (1.6) and $\lim_{n\to\infty} \frac{\lambda^3_n}{(1-\lambda^2_n - \lambda^3_n\delta_n)} = 0$. Then $\{x_n\}_{n=1}^{\infty}$ defined by (1.6) converges to p if and only if $\{y_n\}_{n=1}^{\infty}$ defined by (1.4) converges to p.*

Proof. Let $c = \max\{c_1,\ c_2\}$.

$$
\begin{aligned}
&\|x_{n+1} - y_{n+1}\| \\
&= ||\lambda^1_n G_1(x_n) + \lambda^2_n x_n + \lambda^3_n T((1-\delta_n)G_2(x_n) + \delta_n x_{n+1}) \\
&\quad - \left(\lambda^1_n G(y_n) + \lambda^2_n y_n + \lambda^3_n T(\delta_n y_n + (1-\delta_n)y_{n+1})\right)|| \\
&= \|\lambda^1_n (G_1(x_n) - G_1(y_n)) + \lambda^2_n (x_n - y_n) \\
&\quad + \lambda^3_n \left(T((1-\delta_n)G_2(x_n) + \delta_n x_{n+1}) - T(\delta_n y_n + (1-\delta_n)y_{n+1})\right)\| \\
&\leq \lambda^1_n ||G_1(x_n) - G_1(y_n)|| + \lambda^2_n \|x_n - y_n\| \\
&\quad + \lambda^3_n \|T((1-\delta_n)G_2(x_n) + \delta_n x_{n+1}) - T(\delta_n y_n + (1-\delta_n)y_{n+1})\| \\
&\leq \lambda^1_n c_1 ||x_n - y_n|| + \lambda^2_n \|x_n - y_n\| \\
&\quad + \lambda^3_n \|(1-\delta_n)(G_2(x_n) - y_{n+1}) + \delta_n(x_{n+1} - y_n)\| \\
&\leq \lambda^1_n c_1 ||x_n - y_n|| + \lambda^2_n \|x_n - y_n\| \\
&\quad + \lambda^3_n (1-\delta_n) \|G_2(x_n) - G_2(y_n) + G_2(y_n) - y_{n+1}\| \\
&\quad + \lambda^3_n \delta_n \|x_{n+1} - y_{n+1} + y_{n+1} - y_n\| \\
&\leq \lambda^1_n c_1 ||x_n - y_n|| + \lambda^2_n \|x_n - y_n\| + \lambda^3_n (1-\delta_n) c_2 \|x_n - y_n\| \\
&\quad + \lambda^3_n (1-\delta_n) \|y_{n+1} - G_2(y_n)\| + \lambda^3_n \delta_n \|x_{n+1} - y_{n+1}\| + \lambda^3_n \delta_n \|y_{n+1} - y_n\| \\
&= \left(\lambda^1_n c + \lambda^3_n (1-\delta_n) c + \lambda^2_n\right) ||x_n - y_n|| + \lambda^3_n \delta_n \|x_{n+1} - y_{n+1}\| \\
&\quad + \lambda^3_n (1-\delta_n) \|y_{n+1} - G_2(y_n)\| + \lambda^3_n \delta_n \|y_{n+1} - y_n\|.
\end{aligned}
$$

Since $\{y_n\}_{n=1}^{\infty}$ and $\{G_2(y_n)\}_{n=1}^{\infty}$ are bounded [5], let

$$M_2 = \max\left\{\sup_n \|y_{n+1} - G_2(y_n)\|, \sup_n \|y_{n+1} - y_n\|\right\}.$$

Then

$$\begin{aligned}
&\|x_{n+1} - y_{n+1}\| \\
&\le \frac{\lambda_n^1 c + \lambda_n^3(1-\delta_n)c + \lambda_n^2}{1-\lambda_n^3\delta_n}||x_n - y_n|| + \frac{\lambda_n^3}{1-\lambda_n^3\delta_n}M_2 \\
&= \left(1 - \frac{(1-\lambda_n^2-\lambda_n^3\delta_n)(1-c)}{1-\lambda_n^3\delta_n}\right)||x_n - y_n|| + \frac{\lambda_n^3}{1-\lambda_n^3\delta_n}M_2 \\
&= \left(1 - \frac{(1-\lambda_n^2-\lambda_n^3\delta_n)(1-c)}{1-\lambda_n^3\delta_n}\right)||x_n - y_n|| + \frac{\lambda_n^3}{1-\lambda_n^3\delta_n}M_2 \\
&= (1-\beta_n)||x_n - y_n|| + \frac{\lambda_n^3}{(1-\lambda_n^2-\lambda_n^3\delta_n)(1-c)}\beta_n M_2, \qquad (3.17)
\end{aligned}$$

where $\beta_n = \frac{(1-\lambda_n^2-\lambda_n^3\delta_n)(1-c)}{1-\lambda_n^3\delta_n}$. From the given condition, it follows that

$$\limsup_{n\to\infty} \frac{\lambda_n^3}{(1-\lambda_n^2-\lambda_n^3\delta_n)} \le 0.$$

Apply Lemma 2.8 with $\gamma_n = 0$ to (3.17) to get that $||x_n - y_n|| \to 0$ as $n \to \infty$. Furthermore, suppose $||y_n - p|| \to 0$ as $n \to \infty$, it follows that,

$$\begin{aligned}
||x_n - p|| &= ||x_n - y_n + y_n - p|| \\
&\le ||x_n - y_n|| + ||y_n - p|| \\
&= ||y_n - p|| \\
&\to 0 \text{ (as } n \to \infty).
\end{aligned}$$

Similary, suppose $||x_n - p|| \to 0$ as $n \to \infty$, it follows that,

$$\begin{aligned}
||y_n - p|| &= ||y_n - x_n + x_n - p|| \\
&\le ||y_n - x_n|| + ||x_n - p|| \\
&= ||x_n - p|| \\
&\to 0 \text{ (as } n \to \infty).
\end{aligned}$$

□

Corollary 3.6. ([17]) *Let E be a uniformly smooth Banach space and K a nonempty closed convex subset of E. Let $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $G : K \to K$ a generalized contraction mapping. Pick any $x_0 \in K$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated by*

$$x_{n+1} = a_n G(x_n) + b_n x_n + c_n T\left(s_n x_n + (1-s_n)x_{n+1}\right), \qquad (3.18)$$

where $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ *and* $\{c_n\}_{n=1}^{\infty}$ *are three sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $a_n + b_n + c_n = 1$;

(ii) $\sum_{n=1}^{\infty} a_n = \infty, \ \lim_{n\to\infty} a_n = 0$;

(iii) $\sum_{n=1}^{\infty} |b_{n+1} - b_n| < \infty$ *and* $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$;

(iv) $0 < \epsilon \leq s_n \leq s_{n+1} < 1$ *for all* $n \in \mathbb{N}$.

Then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to a fixed point* p *of the nonexpansive mapping* T, *which is also the solution of the variational inequality* (1.5).

Proof. Observe that $\lambda_n^1 = a_n, \ \lambda_n^2 = b_n$ and $\lambda_n^3 = c_n$, by comparing (1.6) and (3.18). Taking $G_1 = G, \ \delta_n = 1 - s_n$ and G_2 to be the identity mapping of K in (1.6), we obtain (3.18). Hence, the conclusion follows from Theorem 3.4. □

Corollary 3.7. *Let* K *be a nonempty closed convex subset of a uniformly smooth Banach space* E. *Let* T *be a nonexpansive self-mapping defined on* K *with* $F(T) \neq \emptyset$. *Assume that the real sequences* $\{\lambda_n\}_{n=1}^{\infty} \subset (0,1)$ *and* $\{\delta_n\}_{n=1}^{\infty} \subset (0,1)$ *satisfy the conditions:*

(i) $\lim_{n\to\infty} \lambda_n = 0$;

(ii) $\sum_{n=1}^{\infty} \lambda_n = \infty$;

(iii) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$;

(iv) $0 < \epsilon \leq \delta_n \leq \delta_{n+1} < 1$ *for all* $n \in \mathbb{N}$.

Then the iterative sequence $\{x_n\}_{n=1}^{\infty}$ *which is defined from an arbitrary* $x_1 \in K$ *by*

$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n) T((1 - \delta_n) x_n + \delta_n x_{n+1}) \tag{3.19}$$

converges strongly to a fixed point p *of* T *which solves the variational inequality* (1.5).

Proof. The result follows from Theorem 3.4 by simply taking $G_1 = G_2$ to be the identity mappings of K in (1.6). □

Corollary 3.8. ([1]) *Let* E *be a uniformly smooth Banach space and* K *a nonempty closed convex subset of* E. *Let* $T : K \to K$ *be a nonexpansive mapping with* $F(T) \neq \emptyset$ *and* $G : K \to K$ *an* α*-contraction. Suppose that the real sequences* $\{a_n\} \subset (0,1), \ \{b_n\} \subset [0,1)$ *and* $\{c_n\} \subset (0,1)$ *are such that* $a_n + b_n + c_n = 1$, *for all* $n \in \mathbb{N}$ *and satisfy the following conditions:*

(i) $\lim_{n\to\infty} a_n = 0$;

(ii) $\sum_{n=1}^{\infty} a_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$;

(iv) $\lim_{n\to\infty} |b_{n+1} - b_n| = 0$.

For an arbitrary $x_1 \in K$, *define the iterative sequence* $\{x_n\}$ *by*

$$x_{n+1} = a_n G(x_n) + b_n x_n + c_n T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \in \mathbb{N}. \tag{3.20}$$

Then the sequence $\{x_n\}$ *converges in norm to a fixed point* p *of* T, *where* p *is the unique solution in* $F(T)$ *to the variational inequality* (1.5).

Proof. It is known that a generalized contraction is more broad that an α-contraction. Comparing (1.6) and (3.20), it is noted that $\lambda_n^1 = a_n,\ \lambda_n^2 = b_n$ and $\lambda_n^3 = c_n$. Taking G_2 to be the identity mappings of K and $\delta_n = 2$ for all $n \in \mathbb{N}$ in (1.6), it reduces to (3.20) with $G_1 = G$. Therefore, the desire result follows from Theorem 3.4. □

Corollary 3.9. *Let* K *be a nonempty closed convex subset of a uniformly smooth Banach space* E. *Let* T *be a nonexpansive self-mapping defined on* K *with* $F(T) \neq \emptyset$. *Assume that the real sequence* $\{\lambda_n\}_{n=1}^{\infty} \subset (0,1)$ *satisfies the following conditions:*

(i) $\lim_{n\to\infty} \lambda_n = 0$;

(ii) $\sum_{n=1}^{\infty} \lambda_n = \infty$;

(iii) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then the iterative sequence $\{x_n\}_{n=1}^{\infty}$ *which is defined from an arbitrary* $x_1 \in K$ *by*

$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n) T(\frac{x_n + x_{n+1}}{2}) \tag{3.21}$$

converges strongly to a fixed point p *of* T *which solves the variational inequality* (1.5).

Proof. The result follows from Theorem 3.4 by simply taking $G_1 = G_2$ to be the identity mappings of K and $\delta_n = 2$ for all $n \in \mathbb{N}$. Therfore, this improves and extend the results of Alghamdi et al. [2]. □

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