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ON SOME NEW RETARDED INTEGRAL INEQUALITIES AND THEIR APPLICATIONS

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Abstract. The aim of this paper is to establish explicit bounds on some more general retarded integral inequalities which can be used as convenient tools in some applications. The two independent variable generalizations of the main results and some applications are also given.

1. Introduction

Integral inequalities which provide explicit bounds on unknown functions have played a fundamental role in the development of the theory of differential and integral equations. Over the years, various investigators have discovered many useful integral inequalities in order to achieve diversity of desired goals, see [1]-[8] and the references given therein. In a recent paper [6], X.Q.Zhao and F.W.Meng have given some inequalities which can be used to study the qualitative behavior of the solutions of certain classes of differential and integral equations. However, the integral inequalities available in the literature do not apply directly in certain general situations and it is desirable to find integral inequalities useful in some new applications. The main purpose of the present paper is to establish explicit bounds on more general retarded integral

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inequalities which can be used as tools in the qualitative study of certain retarded integro-differential equations. Some immediate applications of one of the result to convey the importance of our results to the literature are also given.

2. Lemmas

The following lemmas are useful in our main results.

Lemma 2.1. [6] *Assume that* $p \geq q > 0$, $a \geq 0$, *then*

$$a^{\frac{q}{p}} \leq \frac{q}{p} k^{\frac{q-p}{p}} a + \frac{p-q}{p} k^{\frac{q}{p}}, \tag{2.1}$$

for any $k > 0$.

Lemma 2.2. *Let* $u(x,y), \varphi(x,y), f(x,y) \in C(R_+ \times R_+, R_+)$ *and* $a(x,y,t,s)$, $b(x,y,t,s)$ *be continuous nondecreasing in* x *and* y *for each* t, s. *If* $0 \leq \alpha(x) \leq x$, $0 \leq \beta(y) \leq y$, $\alpha'(x), \beta'(y) \geq 0$ *are real valued continuous functions defined for* $x \geq 0$, $y \geq 0$, *such that*

$$\begin{aligned} u(x,y) \leq \varphi(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} \{ & a(x,y,t,s) u(t,s) \\ & + f(t,s) \int_{\alpha(x_0)}^{t} \int_{\beta(y_0)}^{s} b(t,s,\sigma_1,\sigma_2) u(\sigma_1,\sigma_2) d\sigma_2 d\sigma_1 \} ds dt, \end{aligned} \tag{2.2}$$

then

$$\begin{aligned} u(x,y) \leq \varphi(x,y) + & e^{\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} (a(x,y,t,s) + f(t,s) \int_{\alpha(x_0)}^{t} \int_{\beta(y_0)}^{s} b(t,s,\sigma_1,\sigma_2) d\sigma_2 d\sigma_1 ds dt} \\ & \times (\int_{x_0}^{x} e^{-\int_{\alpha(x_0)}^{\alpha(l)} \int_{\beta(y_0)}^{\beta(y)} (a(l,y,t,s) + f(t,s) \int_{\alpha(x_0)}^{t} \int_{\beta(y_0)}^{s} b(t,s,\sigma_1,\sigma_2) d\sigma_2 d\sigma_1) ds dt} \\ & \times \frac{\partial}{\partial l} (\int_{\alpha(x_0)}^{\alpha(l)} \int_{\beta(y_0)}^{\beta(y)} (a(l,y,t,s) \varphi(t,s) \\ & + f(t,s) \int_{\alpha(x_0)}^{t} \int_{\beta(y_0)}^{s} b(t,s,\sigma_1,\sigma_2) \varphi(\sigma_1,\sigma_2) d\sigma_2 d\sigma_1) ds dt) dl). \end{aligned} \tag{2.3}$$

Proof. Let

$$\begin{aligned} z(x,y) = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} (& a(x,y,t,s) u(t,s) \\ & + f(t,s) \int_{\alpha(x_0)}^{t} \int_{\beta(y_0)}^{s} b(t,s,\sigma_1,\sigma_2) u(\sigma_1,\sigma_2) d\sigma_2 d\sigma_1) ds dt. \end{aligned} \tag{2.4}$$

Then $z(x_0,y)=z(x,y_0)=0$. Our assumption on a,b,u,f,α and β imply that z is nondecreasing positive function for $x\geq 0$, $y\geq 0$. We have

$$\begin{aligned}
z_x(x,y) &= \alpha'(x)\int_{\beta(y_0)}^{\beta(y)}(a(x,y,\alpha(x),s)u(\alpha(x),s)\\
&+ f(\alpha(x),s)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{s} b(\alpha(x),s,\sigma_1,\sigma_2)u(\sigma_1,\sigma_2)d\sigma_2 d\sigma_1)ds\\
&+ \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}\partial_x a(x,y,t,s)u(t,s)dsdt\\
&\leq \alpha'(x)\int_{\beta(y_0)}^{\beta(y)}(a(x,y,\alpha(x),s)\varphi(\alpha(x),s)\\
&+ f(\alpha(x),s)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{s} b(\alpha(x),s,\sigma_1,\sigma_2)\varphi(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2)ds \qquad (2.5)\\
&+ \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}\partial_x a(x,y,t,s)\varphi(t,s)dsdt\\
&+ z(x,y)(\alpha'(x)\int_{\beta(y_0)}^{\beta(y)}(a(x,y,\alpha(x),s)\\
&+ f(\alpha(x),s)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{s} b(\alpha(x),s,\sigma_1,\sigma_2)d\sigma_1 d\sigma_2)ds\\
&+ \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}\partial_x a(x,y,t,s)dsdt),
\end{aligned}$$

that is,

$$\begin{aligned}
& z_x(x,y) - z(x,y)\frac{\partial}{\partial x}(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}(a(x,y,t,s)\\
&+ f(t,s)\int_{\alpha(x_0)}^{t}\int_{\beta(y_0)}^{s} b(t,s,\sigma_1,\sigma_2)d\sigma_1 d\sigma_2)dsdt)\\
&\leq \frac{\partial}{\partial x}(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}(a(x,y,t,s)\varphi(t,s)\\
&+ f(t,s)\int_{\alpha(x_0)}^{t}\int_{\beta(y_0)}^{s} b(t,s,\sigma_1,\sigma_2)\varphi(\sigma_1,\sigma_2)d\sigma_1 d\sigma_2)dsdt).
\end{aligned} \qquad (2.6)$$

Multiplying both sides of (2.6) by

$$e^{-\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}(a(x,y,t,s)+f(t,s)\int_{\alpha(x_0)}^{t}\int_{\beta(y_0)}^{s}b(t,s,\sigma_1,\sigma_2)d\sigma_1 d\sigma_2)dsdt}, \qquad (2.7)$$

we get

$$\begin{aligned}
&\frac{\partial}{\partial x}(z(x,y)e^{-\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}(a(x,y,t,s)+f(t,s)\int_{\alpha(x_0)}^{t}\int_{\beta(y_0)}^{s}b(t,s,\sigma_1,\sigma_2)d\sigma_2 d\sigma_1)dsdt})\\
&\le e^{-\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}(a(x,y,t,s)+f(t,s)\int_{\alpha(x_0)}^{t}\int_{\beta(y_0)}^{s}b(t,s,\sigma_1,\sigma_2)d\sigma_2 d\sigma_1)dsdt}\\
&\times\frac{\partial}{\partial x}(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}(a(x,y,t,s)\varphi(t,s)\\
&+f(t,s)\int_{\alpha(x_0)}^{t}\int_{\beta(y_0)}^{s}b(t,s,\sigma_1,\sigma_2)\varphi(\sigma_1,\sigma_2)d\sigma_2 d\sigma_1)dsdt).
\end{aligned}\tag{2.8}$$

Now integrating (2.8) on the interval $[x_0,x]$ for x respectively, we obtain

$$\begin{aligned}
z(x,y)\le\ & e^{\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}(a(x,y,t,s)+f(t,s)\int_{\alpha(x_0)}^{t}\int_{\beta(y_0)}^{s}b(t,s,\sigma_1,\sigma_2)d\sigma_2 d\sigma_1)dsdt}\\
&\times(\int_{x_0}^{x}e^{-\int_{\alpha(x_0)}^{\alpha(l)}\int_{\beta(y_0)}^{\beta(y)}(a(l,y,t,s)+f(t,s)\int_{\alpha(x_0)}^{t}\int_{\beta(y_0)}^{s}b(t,s,\sigma_1,\sigma_2)d\sigma_2 d\sigma_1)dsdt}\\
&\times\frac{\partial}{\partial l}(\int_{\alpha(x_0)}^{\alpha(l)}\int_{\beta(y_0)}^{\beta(y)}(a(l,y,t,s)\varphi(t,s)\\
&+f(t,s)\int_{\alpha(x_0)}^{t}\int_{\beta(y_0)}^{s}b(t,s,\sigma_1,\sigma_2)\varphi(\sigma_1,\sigma_2)d\sigma_2 d\sigma_1)dsdt)dl).
\end{aligned}\tag{2.9}$$

Combine (2.9) with $u(t)\le k(t)+z(t)$ to get (2.3) and with this, the proof is complete. $\square$

3. Main theorems

Theorem 3.1. *Let* $u(x,y),\varphi(x,y),f(x,y)\in C(R_+\times R_+,R_+)$ *and* $a(x,y,t,s)$, $b(x,y,t,s)$ *be continuous nondecreasing in* x *and* y *for each* t, s. *Let* $0\le\alpha(x)\le x$, $0\le\beta(y)\le y$, $\alpha'(x)$, $\beta'(y)\ge 0$ *be real valued continuous functions defined for* $x\ge 0$, $y\ge 0$, *and* $p\ge q>0$, $p\ge r>0$, p, q, *and* r *are constants. If*

$$\begin{aligned}
u^p(x,y)\le\varphi(x,y)+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}&(a(x,y,t,s)u^q(t,s)\\
&+f(t,s)\int_{\alpha(x_0)}^{t}\int_{\beta(y_0)}^{s}b(t,s,\sigma_1,\sigma_2)u^r(\sigma_1,\sigma_2)d\sigma_2 d\sigma_1)dsdt,
\end{aligned}\tag{3.1}$$

then

$$
\begin{aligned}
u(x,y) \le (\varphi(x,y) &+ F(x,y) \\
&+ e^{\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}(Aa(x,y,t,s)+Bf(t,s)\int_{\alpha(x_0)}^{t}\int_{\beta(y_0)}^{s} b(t,s,\sigma_1,\sigma_2)d\sigma_2 d\sigma_1)dsdt} \\
&\times (\int_{x_0}^{x} e^{-\int_{\alpha(x_0)}^{\alpha(l)}\int_{\beta(y_0)}^{\beta(y)}(Aa(l,y,t,s)+Bf(t,s)\int_{\alpha(x_0)}^{t}\int_{\beta(y_0)}^{s} b(t,s,\sigma_1,\sigma_2)d\sigma_2 d\sigma_1)dsdt} \\
&\times \frac{\partial}{\partial l}(\int_{\alpha(x_0)}^{\alpha(l)}\int_{\beta(y_0)}^{\beta(y)}(Aa(l,y,t,s)\varphi(t,s) \\
&+ Bf(t,s)\int_{\alpha(x_0)}^{t}\int_{\beta(y_0)}^{s} b(t,s,\sigma_1,\sigma_2)\varphi(\sigma_1,\sigma_2)d\sigma_2 d\sigma_1)dsdt)dl))^{\frac{1}{p}},
\end{aligned}
\tag{3.2}
$$

for $t \ge t_0 \ge 0$ *and any* $k > 0$*, where*

$$
\begin{aligned}
F(x,y) = &\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(x,y,t,s)(A\varphi(t,s) + \frac{p-q}{p}k^{\frac{q}{p}})dsdt \\
&+ \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(t,s)\int_{x_0}^{t}\int_{y_0}^{s} b(t,s,\sigma_1,\sigma_2) \\
&\times (B\varphi(\sigma_1,\sigma_2) + \frac{p-r}{p}k^{\frac{r}{p}})d\sigma_2 d\sigma_1 dsdt, \\
A = &\frac{q}{p}k^{\frac{q-p}{p}}, \qquad B = \frac{r}{p}k^{\frac{r-p}{p}}.
\end{aligned}
\tag{3.3}
$$

Proof. Define a function $z(x,y)$ by

$$
\begin{aligned}
z(x,y) = &\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}(a(x,y,t,s)u^q(t,s) \\
&+ f(t,s)\int_{\alpha(x_0)}^{t}\int_{\beta(y_0)}^{s} b(t,s,\sigma_1,\sigma_2)u^r(\sigma_1,\sigma_2)d\sigma_2 d\sigma_1)dsdt,
\end{aligned}
\tag{3.4}
$$

Then (3.1) can be restated as

$$u^p(x,y) \le \varphi(x,y) + z(x,y). \tag{3.5}$$

By (3.5), we have

$$u(x,y) \le (\varphi(x,y) + z(x,y))^{\frac{1}{p}}. \tag{3.6}$$

Thus from (3.4) and (3.6), we have

$$
\begin{aligned}
z(x,y) \le F(x,y) + &\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}(Aa(x,y,t,s)z(t,s) \\
&+ Bf(t,s)\int_{\alpha(x_0)}^{t}\int_{\beta(y_0)}^{s} b(t,s,\sigma_1,\sigma_2)z(\sigma_1,\sigma_2)d\sigma_2 d\sigma_1)dsdt,
\end{aligned}
\tag{3.7}
$$

where $F(x,y)$ is defined by (3.3). It is easy to see that $F(x,y)$ is nonnegative and continuous in x and y for $(x,y)\in R_+\times R_+$. Using Lemma 2.2 to (3.7), we get

$$\begin{aligned} z(x,y) \le{}& F(x,y) + e^{\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}(Aa(x,y,t,s)+Bf(t,s)\int_{\alpha(x_0)}^{t}\int_{\beta(y_0)}^{s}b(t,s,\sigma_1,\sigma_2)d\sigma_2 d\sigma_1)dsdt} \\ &\times (\int_{x_0}^{x} e^{-\int_{\alpha(x_0)}^{\alpha(l)}\int_{\beta(y_0)}^{\beta(y)}(Aa(l,y,t,s)+Bf(t,s)\int_{\alpha(x_0)}^{t}\int_{\beta(y_0)}^{s}b(t,s,\sigma_1,\sigma_2)d\sigma_2 d\sigma_1)dsdt} \\ &\times \frac{\partial}{\partial l}(\int_{\alpha(x_0)}^{\alpha(l)}\int_{\beta(y_0)}^{\beta(y)}(Aa(l,y,t,s)\varphi(t,s) \\ &+ Bf(t,s)\int_{\alpha(x_0)}^{t}\int_{\beta(y_0)}^{s} b(t,s,\sigma_1,\sigma_2)\varphi(\sigma_1,\sigma_2)d\sigma_2 d\sigma_1)dsdt)dl). \end{aligned} \tag{3.8}$$

The desired inequality (3.2) follows from (3.6) and (3.8).

□

Theorem 3.2. *Let $u(x,y),\varphi(x,y),f(x,y)\in C(R_+\times R_+,R_+)$ and $a(x,y,t,s)$, $b(x,y,t,s)$ be continuous nondecreasing in x and y for each t,s. Let $0\le\alpha(x)\le x$, $0\le\beta(y)\le y$, $\alpha'(x),\beta'(y)\ge 0$ be real valued continuous functions defined for $x\ge 0$, $y\ge 0$, and $p\ge q>0$, p and q are constants. If*

$$\begin{aligned} u^p(x,y) \le{}& \varphi(x,y) + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}(a(x,y,t,s)u^p(t,s) \\ &+ f(t,s)\int_{\alpha(x_0)}^{t}\int_{\beta(y_0)}^{s} b(t,s,\sigma_1,\sigma_2)u^q(\sigma_1,\sigma_2)d\sigma_2 d\sigma_1)dsdt, \end{aligned} \tag{3.9}$$

then

$$\begin{aligned} u(x,y) \le{}& (\varphi(x,y)+B(x,y)+F(x,y) \\ &+ e^{\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}Af(t,s)\int_{\alpha(x_0)}^{t}\int_{\beta(y_0)}^{s}b(t,s,\sigma_1,\sigma_2)d\sigma_2 d\sigma_1 dsdt} \\ &\times (\int_{x_0}^{x} e^{-\int_{\alpha(x_0)}^{\alpha(l)}\int_{\beta(y_0)}^{\beta(y)}Af(t,s)\int_{\alpha(x_0)}^{t}\int_{\beta(y_0)}^{s}b(t,s,\sigma_1,\sigma_2)d\sigma_2 d\sigma_1 dsdt} \\ &\times \frac{\partial}{\partial l}(\int_{\alpha(x_0)}^{\alpha(l)}\int_{\beta(y_0)}^{\beta(y)}Af(t,s) \\ &\times \int_{\alpha(x_0)}^{t}\int_{\beta(y_0)}^{s} b(t,s,\sigma_1,\sigma_2)F(\sigma_1,\sigma_2)d\sigma_2 d\sigma_1 dsdt)dl))^{\frac{1}{p}}, \end{aligned} \tag{3.10}$$

for $t \geq t_0 \geq 0$ and any $k > 0$, where

$$\begin{aligned} F(x,y) = &\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(t,s) \int_{\alpha(x_0)}^{t} \int_{\beta(y_0)}^{s} b(t,s,\sigma_1,\sigma_2) \\ &\times (A(\varphi(x,y) + B(x,y)) + \frac{p-q}{p} k^{\frac{q}{p}}) d\sigma_2 d\sigma_1 dsdt, \end{aligned} \tag{3.11}$$

$$\begin{aligned} B(x,y) = &e^{\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(x,y,t,s)dsdt} \times (\int_{x_0}^{x} e^{-\int_{\alpha(x_0)}^{\alpha(l)} \int_{\beta(y_0)}^{\beta(y)} a(l,y,t,s)dsdt} \\ &\times \frac{\partial}{\partial l} (\int_{\alpha(x_0)}^{\alpha(l)} \int_{\beta(y_0)}^{\beta(y)} a(l,y,t,s) z(t,s) dsdt) dl), \ A = \frac{q}{p} k^{\frac{q-p}{p}}. \end{aligned} \tag{3.12}$$

Proof. Define a function $z(x,y)$ by

$$\begin{aligned} z(x,y) = &\varphi(x,y) \\ &+ \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(t,s) \int_{\alpha(x_0)}^{t} \int_{\beta(y_0)}^{s} b(t,s,\sigma_1,\sigma_2) u^q(\sigma_1,\sigma_2) d\sigma_2 d\sigma_1 dsdt. \end{aligned} \tag{3.13}$$

Then (3.9) can be restated as

$$u^p(x,y) \leq z(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(x,y,t,s) u^p(t,s) dsdt. \tag{3.14}$$

Clearly, $z(x,y)$ is nonnegative and continuous for $x \geq x_0 \geq 0$, $y \geq y_0 \geq 0$. Using Lemma 2.2 to (3.14), we have

$$u^p(x,y) \leq z(x,y) + B(x,y), \tag{3.15}$$

where $B(x,y)$ is defined by (3.12). From (3.13) and (3.15) we obtain

$$u^p(x,y) \leq \varphi(x,y) + v(x,y) + B(x,y), \tag{3.16}$$

where

$$v(x,y) = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(t,s) \int_{\alpha(x_0)}^{t} \int_{\beta(y_0)}^{s} b(t,s,\sigma_1,\sigma_2) u^q(\sigma_1,\sigma_2) d\sigma_2 d\sigma_1 dsdt. \tag{3.17}$$

From (3.16), we have

$$u(x,y) \leq (\varphi(x,y) + v(x,y) + B(x,y))^{\frac{1}{p}}, \ x \geq x_0 \geq 0, \ y \geq y_0 \geq 0. \tag{3.18}$$

From (3.17), (3.18) and Lemma 2.1, we get

$$\begin{aligned} v(x,y) \leq &F(x,y) \\ &+ A \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(t,s) \int_{\alpha(x_0)}^{t} \int_{\beta(y_0)}^{s} b(t,s,\sigma_1,\sigma_2) v(\sigma_1,\sigma_2) d\sigma_2 d\sigma_1 dsdt. \end{aligned} \tag{3.19}$$

It is easy to see that $F(x,y)$ is nonnegative and continuous for $x \ge x_0 \ge 0, y \ge y_0 \ge 0$. Using Lemma 2.2 to (3.19), we have

$$\begin{aligned} v(x,y) \le F(x,y) &+ e^{\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} Af(t,s) \int_{\alpha(x_0)}^{t} \int_{\beta(y_0)}^{s} b(t,s,\sigma_1,\sigma_2) d\sigma_2 d\sigma_1 ds dt} \\ &\times (\int_{x_0}^{x} e^{-\int_{\alpha(x_0)}^{\alpha(l)} \int_{\beta(y_0)}^{\beta(y)} Af(t,s) \int_{\alpha(x_0)}^{t} \int_{\beta(y_0)}^{s} b(t,s,\sigma_1,\sigma_2) d\sigma_2 d\sigma_1 ds dt} \\ &\times \frac{\partial}{\partial l} (\int_{\alpha(x_0)}^{\alpha(l)} \int_{\beta(y_0)}^{\beta(y)} Af(t,s) \\ &\times \int_{\alpha(x_0)}^{t} \int_{\beta(y_0)}^{s} b(t,s,\sigma_1,\sigma_2) F(\sigma_1,\sigma_2) d\sigma_2 d\sigma_1 ds dt) dl). \end{aligned} \tag{3.20}$$

The desired inequality (3.8) follows from (3.18) and (3.20). The proof is complete.

□

4. Some applications

In this section we present applications of the inequality (3.1) given in Theorem 3.1 to study the boundedness dependence of the solution of the initial value problem for retarded Volterra equations of the form

$$\begin{aligned} w^p(x,y) = \varphi(x,y) &+ \int_{x_0}^{x} \int_{y_0}^{y} \Phi(x,y,t,s,w(\alpha(t),\beta(s)), f(\alpha(t),\beta(s)) \\ &\times \int_{\alpha(x_0)}^{\alpha(t)} \int_{\beta(y_0)}^{\beta(s)} b(\alpha(t),\beta(s),\sigma_1,\sigma_2) w^r(\sigma_1,\sigma_2) d\sigma_2 d\sigma_1) ds dt, \end{aligned} \tag{4.1}$$

$$w(x_0,y_0) = \varphi(x_0,y_0), \tag{4.2}$$

where Φ is continuous, φ is continuous on $R_+ \times R_+$, $\alpha \in C^1(R_+,R_+)$ and $\alpha(x) \le x$, $\alpha'(x) > 0$ for $x \ge x_0 \ge 0; \beta \in C^1(R_+,R_+)$ and $\beta(y) \le y$, $\beta'(y) > 0$ for $y \ge y_0 \ge 0$.

First we give the bound on the solution of the problem (4.1)-(4.2).

Theorem 4.1. *Suppose that*

$$|\Phi(x,y,t,s,u,v)| \le a(x,y,t,s)|u|^q + |v|, \tag{4.3}$$

where a, f, b is as in Theorem 3.1. Let

$$M_1 = \max_{t \in R_+} (\alpha^{-1}(t))', \quad M_2 = \max_{t \in R_+} (\beta^{-1}(t))', \tag{4.4}$$

$$\bar{a}(x,y,t,s) = a(x,y,\alpha^{-1}(t),\beta^{-1}(s)). \tag{4.5}$$

If $w(x,y)$ is any solution of (4.1)-(4.2), then

$$\begin{aligned}
&w(x,y)\\
&\le (|\varphi(x,y)|+F(x,y)\\
&+e^{M_1M_2\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}(A\bar{a}(x,y,t,s)+Bf(t,s)\int_{\alpha(x_0)}^{t}\int_{\beta(y_0)}^{s}|b(t,s,\sigma_1,\sigma_2)|d\sigma_2 d\sigma_1)dsdt}\\
&\times(\int_{x_0}^{x} e^{-M_1M_2\int_{\alpha(x_0)}^{\alpha(l)}\int_{\beta(y_0)}^{\beta(y)}(A\bar{a}(l,y,t,s)+Bf(t,s)\int_{\alpha(x_0)}^{t}\int_{\beta(y_0)}^{s}|b(t,s,\sigma_1,\sigma_2)|d\sigma_2 d\sigma_1)dsdt}\\
&\times\frac{\partial}{\partial l}(M_1M_2\int_{\alpha(x_0)}^{\alpha(l)}\int_{\beta(y_0)}^{\beta(y)}(A\bar{a}(l,y,t,s)|\varphi(t,s)|\\
&+Bf(t,s)\int_{\alpha(x_0)}^{t}\int_{\beta(y_0)}^{s}|b(t,s,\sigma_1,\sigma_2)||\varphi(\sigma_1,\sigma_2)|d\sigma_2 d\sigma_1)dsdt)dl))^{\frac{1}{p}},
\end{aligned}\tag{4.6}$$

for $t\ge t_0\ge 0$ and any $k>0$, where

$$\begin{aligned}
F(x,y)=&\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} M_1M_2\bar{a}(x,y,t,s)(A|\varphi(t,s)|+\frac{p-q}{p}k^{\frac{q}{p}})dsdt\\
&+M_1M_2\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(t,s)\int_{x_0}^{t}\int_{y_0}^{s}|b(t,s,\sigma_1,\sigma_2)|\\
&\times(B|\varphi(\sigma_1,\sigma_2)|+\frac{p-r}{p}k^{\frac{r}{p}})d\sigma_2 d\sigma_1 dsdt,\\
&A=\frac{q}{p}k^{\frac{q-p}{p}},\qquad B=\frac{r}{p}k^{\frac{r-p}{p}}.
\end{aligned}\tag{4.7}$$

Proposition 4.2. *Suppose that*

$$|\Phi(x,y,t,s,u,v)|\le a(x,y,t,s)|u|^p+|v|,\tag{4.8}$$

we can give the new boundedness on the solution of the problem (4.1)-(4.2) by Theorem 3.2.

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