Nonlinear Functional Analysis and Applications
Vol. 24, No. 4 (2019), pp. 715-733
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

PRICING VOLATILITY SWAPS UNDER DOUBLE HESTON STOCHASTIC VOLATILITY MODEL WITH REGIME SWITCHING

Yun Huang[1] and Ming-hui Wang[2]

[1]Department of Mathematics
Sichuan University, Chengdu, Sichuan, 610064, P.R. China
e-mail: hy20122128@163.com

[2]Department of Mathematics
Sichuan University, Chengdu, Sichuan, 610064, P.R. China
e-mail: wmhfranklin@foxmail.com

Abstract. In this paper, we consider a continuous-time Heston stochastic volatility model with regime-switching, the asset and volatility dynamics are closely related to the value of Markovian modulated process. Compared with the previous literatures, our propose is to extend the one-stochastic volatility model to the double-stochastic volatility model. We are interested in finding solutions to pricing the discretely-sampled volatility swaps under Heston's framework. We also get a closed-form solution by deriving the characteristic function of the lognormal asset price via a system of partial differential equations.

1. Introduction

Volatility, in the field of financial mathematics, refers to the variability of financial assets over a certain period of time. Many investors have developed great interest in volatility-related products. Volatility-related products can diversify asset risks in the stock market, and the volatility swaps of such products can be directly linked to volatility, thus obtaining widespread attention. In the last decades, with the rapid growth of transaction volatility swaps,

[0]Received February 19, 2019. Revised April 22, 2019.
[0]2010 Mathematics Subject Classification: 91B24; 91B25; 91G80.
[0]Keywords: Regime-switching, Esscher transform, Heston model, characteristic function, stochastic volatility.
[0]Corresponding Author: Ming-hui Wang (wmhfranklin@foxmail.com).

correctly calculating volatility swaps becomes an active research topic and many scholars have studied it. Heston [9] derived a closed-form solution for the price of the European call option under the stochastic volatility model in 1993. Grunbichler and Longstaff [8] presented several closed-form expressions for volatility futures and option prices. They also examined their implications for the characteristics of these securities in 1996. Howison et al. [10] considered the pricing of volatility derivatives and gave a approximate solution to a general partial differential equation (PDE) for volatility products in 2004. Based on Heston's model, Swishchuk [14] proposed a probabilistic approach to study variance and volatility swaps in financial markets in 2004. Biswas et al. [1] found the locally risk minimizing price of European type vanilla option under the stochastic volatility model where the stock volatility dynamics to be a semi-Markovian modulated square root mean reverting process in 2017.

Most papers with respect to pricing the volatility swaps just imply one-factor stochastic volatility. It can capture the slope of the smirk while cannot explain such largely independent fluctuations in its level and slope over time such that the double-stochastic volatility is more suitable for the description of markets. In 2008, Siu et al. [13] investigated the valuation of the European and American currency options under a double Markovian-modulated stochastic volatility model. In their paper, the first stochastic volatility component was driven by a lognormal diffusion process, the second independent stochastic volatility component was driven by a continuous-time finite-state Markovian chain model. Christoffersen et al. [2] presented a double-stochastic volatility model to illustrate some critical differences between one- and two-factor models in 2009. Zhu and Lian [16] found a closed-form exact solution for the PDE system in the Heston's double stochastic volatility model in 2011. Recently, Mehrdoust [11] presented an efficient Monte Carlo simulation scheme with the variance reduction methods and evaluated arithmetic average Asian options under the double Heston's stochastic volatility model with jumps. Zhang and Sun [15] presented an extension of double Heston's stochastic volatility model by the stock price process, they also derived the characteristic function of the lognormal price of a asset which can be substituted in the valuation of the forward starting options.

Regime-switching is also a critical character of the financial market. In an incomplete market, the regime-switching Esscher transform provides market practitioners in a convenient and flexible method to determine an equivalent martingale measure. Regime-switching has been considered in many literatures. In 2005, Elliott et al. [5] considered the option pricing problem under an incomplete market with Markov-modulated geometric Brownian motion model and adopted a regime-switching random Esscher transform to determine an equivalent martingale pricing measure. Elliott and Siu [6] used a continuous-time Markovian regime-switching financial model to solve a risk minimization problem in 2010. Elliott and Lian [7] examined the effect of ignoring regime switching on pricing variance and volatility swaps in 2013. Recently, Zhu et al. [17] applied a hidden Markovian regime-switching model with a stochastic interest rate and volatility to evaluate a standard European option.

Inspired by the models developed by [2] and [12], we extent the one-factor stochastic volatility model to the two-factor stochastic volatility model to pricing volatility swaps in the regime-switching environment. It has two advantages in pricing volatility swaps. Firstly, it can describe different economic conditions which can reflect the state of the economy and the mood of the investors. Secondly, it can capture much more information of the market than one-factor stochastic volatility model. Thus, our model can describe the market more appropriately than other exist models. To our best knowledge, there are still a few works on pricing volatility swaps with the double stochastic volatility model in regime-switching environment. We also determine an equivalent martingale measure in an incomplete market and get risk-neutral conditions via the regime-switching Esscher transform. The characteristic function of the lognormal price of an asset can be easily derived.

The rest of our paper is organized as follows. In section 2, we establish a brand new model based on a continuous-time Markovian-modulated version of Heston's double stochastic volatility model. We also get the risk-neutral conditions by using the regime-switching Esscher transform. In section 3, we give the closed-form solution to pricing the discretely-sampled volatility-average swaps under the Heston's double-stochastic volatility framework. Finally, the conclusion of the paper is reached.

2. Preliminaries

In this section, based on a continuous-time Markovian-modulated stochastic volatility model, we use the PDE approach for the valuation of volatility derivatives. Our model can be considered as the regime-switching augmentation modulated by [14] for pricing volatility swaps. This model can describe the consequences for the asset price and volatility dynamics of the transitions of observable macroeconomic factor's states. The observable macroeconomic factor can affect the asset prices and volatility dynamics, such as observable economic indicators of business cycles or the sovereign ratings of the region by some international rating agencies. In particular, the parameters of asset price dynamics and stochastic volatility are determined by an economic index, described by an observable Markovian chain.

We assume that the continuous-time financial model includes a risk-free bound and a risk stock. Let $\mathcal{P}$ be the real probability measure with respect to a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and $\mathcal{T}$ denote the time index set with interval $[0, \infty)$. Consider a continuous-time finite state observable Markovian chain $X := \{X_t\}_{t\in\mathcal{T}}$ on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, which states describe the states of an observable economic indicator. Let $\{s_1, s_2, \ldots, s_N\}$ be the set of the state space $\mathcal{S}$ with $s_i \in \mathcal{R}^N, i = 1, 2, \ldots, N$. In general, the unit vectors $\{e_1, e_2, \ldots, e_N\} \in \mathcal{R}^N$ can be used to identify the state space of the Markovian chain X.

Let $\Pi(t)$ be the generator, the dynamics of the Markovian chain are generated by the matrix $\Pi(t) = [\pi_{ij}(t)]_{i,j=1,2,\ldots,N}$ under $\mathcal{P}$. For $i \neq j$, π_{ij} is the intensity of the transition of X from state e_i to state e_j in a small interval of time, satisfying π_{ij} for $i \neq j$ and $\sum_{i=1}^N \pi_{ij} = 0$. According to the semi-martingale representation theorem in [4], the process X can be written as

$$X_t = X_0 + \int_0^t \Pi(s) X_s ds + M_t. \tag{2.1}$$

Here $\{M_t\}_{t\in\mathcal{T}}$ is an $\mathcal{R}^N$-valued martingale increment process with respect to the natural filtration generated by X under $\mathcal{P}$.

Let $W^1 := \{W_t^1\}_{t\in\mathcal{T}}, W^2 := \{W_t^2\}_{t\in\mathcal{T}}, W^3 := \{W_t^3\}_{t\in\mathcal{T}}$ and $W^4 := \{W_t^4\}_{t\in\mathcal{T}}$ be four standard Brownian motions on $(\Omega, \mathcal{F}, \mathcal{P})$, where W^1 is independent with W^2 and W^4, and W^3 is independent with W^2 and W^4. We also let $\widetilde{W}_t^3 := (W_t^1, W_t^3)$ and $\widetilde{W}_t^4 := (W_t^2, W_t^4)$. The correction of these Brownian motions are

$$Cov(dW_t^1, dW_t^3) = \rho_1 dt, \quad Cov(dW_t^2, dW_t^4) = \rho_2 dt,$$

where, ρ_1 and ρ_2 are two constants between 0 and 1. We also assume that X is independent with $\{W^i\}, i = 1, 2, 3, 4$.

Let $r(t, X_t)_{t\in\mathcal{T}}$ be the instantaneous market interest rate of the bond depending on the states of the economic indicator X so that

$$r(t, X_t) = \langle r, X_t \rangle, \quad t \in \mathcal{T},$$

where $r = (r_1, r_2, \ldots, r_N)$ with $r_i > 0$ for $i = 1, 2, \ldots, N$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathcal{R}^N$. For simplicity, we write r_t for $r(t, X_t)$. Thus, the dynamics of the price process $\{B_t\}_{t\in\mathcal{T}}$ for the bond can be written as follows:

$$\begin{cases} dB_t = r_t B_t dt, \\ B_0 = 1. \end{cases}$$

Similarly, let the expected appreciation rate $\{\mu_t\}_{t\in\mathcal{T}}$ which depends on states of economic indicator X be the rate of the risky stock S and can be described by

$$\mu_t := \mu(t, X_t) = \langle \mu, X_t \rangle,$$

where $\mu := (\mu_1, \mu_2, \ldots, \mu_N)$, with $\mu_i \in \mathcal{R}$, for $i = 1, 2, \ldots, N$.

Let $\{\theta_{it}\}_{t\in\mathcal{T}}$ be the long-term mean of the volatility for $i=1,2$. We also suppose that θ_{it} depends on the states of the economic indicator X and

$$\begin{cases} \theta_{1t} := \theta_1(t, X_t) = \langle \theta_1, X_t\rangle, \\ \theta_{2t} := \theta_2(t, X_t) = \langle \theta_2, X_t\rangle, \end{cases}$$

where $\theta_i = (\theta_{i1}, \theta_{i2}, \dots, \theta_{iN})$ with $\theta_{ig} > 0$ for $g = 1, 2, \dots, N$.

The parameters κ_1, σ_1 are the mean-reverting speed and the volatility of volatility in the instantaneous volatility process $V_1(t)$, respectively. The parameters κ_2, σ_2 determine the speed of mean reversion and the volatility of the volatility process $V_2(t)$. To simplify our model, we suppose the above four parameters are constants. By combining the dynamics of the price process $\{S(t)\}_{t\in\mathcal{T}}$ and the volatility processes $V_1(t)$ and $V_2(t)$, we can get the following system of stochastic differential equations (SDEs):

$$\begin{cases} dS(t) = \mu_t S(t)dt + \sqrt{V_1(t)}S(t)dW_t^1 + \sqrt{V_2(t)}S(t)dW_t^2, \\ dV_1(t) = \kappa_1(\theta_{1t}^2 - V_1(t))dt + \sigma_1\sqrt{V_1(t)}dW_t^3, \\ dV_2(t) = \kappa_2(\theta_{2t}^2 - V_2(t))dt + \sigma_2\sqrt{V_2(t)}dW_t^4. \end{cases} \tag{2.2}$$

Let $\widetilde{\rho}_i = \sqrt{1-\rho_i^2}$ for $i = 1, 2$. Let $\widetilde{W}^3 := \{\widetilde{W}_t^3\}_{t\in\mathcal{T}}, \widetilde{W}^4 := \{\widetilde{W}_t^4\}_{t\in\mathcal{T}}$ be standard Brownian motions which are independent of W^1, W^2 and X. Thus, we can rewrite equations (2.2) as follows:

$$\begin{cases} dS(t) = \mu_t S(t)dt + \sqrt{V_1(t)}S(t)dW_t^1 + \sqrt{V_2(t)}S(t)dW_t^2, \\ dV_1(t) = \kappa_1(\theta_{1t}^2 - V_1(t))dt + \rho_1\sigma_1\sqrt{V_1(t)}dW_t^1 + \widetilde{\rho}_1\sigma_1\sqrt{V_1(t)}d\widetilde{W}_t^3, \\ dV_2(t) = \kappa_2(\theta_{2t}^2 - V_2(t))dt + \rho_2\sigma_2\sqrt{V_2(t)}dW_t^2 + \widetilde{\rho}_2\sigma_2\sqrt{V_2(t)}d\widetilde{W}_t^4. \end{cases}$$

Let Y_t be the logarithmic return $\ln(S(t)/S(0))$ over the internal $[0, t]$. Then, we have

$$Y_t = \int_0^t \left(\mu_u - \frac{1}{2}V_1(u) - \frac{1}{2}V_2(u)\right)du + \int_0^t \sqrt{V_1(u)}dW_u^1 + \int_0^t \sqrt{V_2(u)}dW_u^2.$$

In our model, there are five sources of randomness: X, W^1, W^2, W^3 and W^4. Let $\mathcal{F}^X := \{\mathcal{F}_t^X\}_{t\in\mathcal{T}}$, $\mathcal{F}^{W^1} := \{\mathcal{F}_t^{W^1}\}_{t\in\mathcal{T}}, \mathcal{F}^{W^2} := \{\mathcal{F}_t^{W^2}\}_{t\in\mathcal{T}}, \mathcal{F}^{W^3} := \{\mathcal{F}_t^{W^3}\}_{t\in\mathcal{T}}$ and $\mathcal{F}^{W^4} := \{\mathcal{F}_t^{W^4}\}_{t\in\mathcal{T}}$ be the $\mathcal{P}$-augmentation of the natural filtrations generated by $\{X_t\}_{t\in\mathcal{T}}, \{W_t^1\}_{t\in\mathcal{T}}, \{W_t^2\}_{t\in\mathcal{T}}, \{W_t^3\}_{t\in\mathcal{T}}$ and $\{W_t^4\}_{t\in\mathcal{T}}$, respectively. Let $\mathcal{F}^S = \{\mathcal{F}_t^S\}_{t\in\mathcal{T}}$ denote the $\mathcal{P}$-augmentation of the natural filtration generated by $\{S_t\}_{t\in\mathcal{T}}$.

In an incomplete market, there are infinite numbers of equivalent martingale pricing measures. So we need utilize the regime-switching Esscher transform to establish an equivalent martingale pricing measure according to the volatility swaps.

Denote $\mathcal{G}_t := \sigma(\mathcal{F}_t^X \bigcup \mathcal{F}_t^{W^1} \bigcup \mathcal{F}_t^{W^2} \bigcup \mathcal{F}_t^{W^3} \bigcup \mathcal{F}_t^{W^4})$ to be the σ-algebra augmented by $\mathcal{F}_t^X$ and $\mathcal{F}_t^{W^i}(i = 1, 2, 3, 4)$, for each $t \in \mathcal{T}$.

Let $\Theta_t := \Theta\left(t, X_t, \sqrt{V_1(t)}, \sqrt{V_2(t)}\right)$ be a regime switching Esscher process, which can be written as:

$$\Theta_t = \Theta\left(t, X_t, \sqrt{V_1(t)}, \sqrt{V_2(t)}\right) = \left\langle \Theta\left(t, \sqrt{V_1(t)}, \sqrt{V_2(t)}\right), X_t \right\rangle,$$

where

$$\begin{aligned} &\Theta\left(t, \sqrt{V_1(t)}, \sqrt{V_2(t)}\right) \\ = &\left(\Theta\left(t, \sqrt{V_1(t)}, \sqrt{V_2(t)}, e_1\right), \ldots, \Theta\left(t, \sqrt{V_1(t)}, \sqrt{V_2(t)}, e_N\right)\right). \end{aligned}$$

We denote $\mathcal{G}_t^1 := \sigma(\mathcal{F}_t^{W^3} \bigcup \mathcal{F}_t^{W^4})$ to be the σ-algebra augmented by $\mathcal{F}_t^{W^3}$ and $\mathcal{F}_t^{W^4}$, so $\Theta\left(t, \sqrt{V_1(t)}, \sqrt{V_2(t)}, e_i\right)$ is $\mathcal{G}_t^1$-measurable, for each $i = 1, 2, \ldots, N$. Denote $\mathcal{G}_t^2 := \sigma(\mathcal{F}_t^X \bigcup \mathcal{F}_t^{W^3} \bigcup \mathcal{F}_t^{W^4})$. Then

$$\Theta\left(t, X_t, \sqrt{V_1(t)}, \sqrt{V_2(t)}\right)$$

is an N-dimensional $\mathcal{G}_t^2$-measurable random vector.

Definition 2.1. Following the idea developed in [5], the regime-switching Esscher transform $\mathcal{Q}_\Theta \sim \mathcal{P}$ on $\mathcal{G}_t$ with respect to a family of parameters $\{\Theta_u\}_{u \in [0,t]}$ can be defined by

$$\left.\frac{d\mathcal{Q}_\Theta}{d\mathcal{P}}\right|_{\mathcal{G}_t} = \frac{\exp(\int_0^t \Theta_u dY_u)}{E_{\mathcal{P}}\left[\exp(\int_0^t \Theta_u dY_u) | \mathcal{G}_t^2\right]}, \quad \forall t \in \mathcal{T}. \tag{2.3}$$

After $\mathcal{G}_t^2$ is given, the Radon-Nikodym derivative of the regime-switching Esscher transform can be written as

$$\begin{aligned} \left.\frac{d\mathcal{Q}_\Theta}{d\mathcal{P}}\right|_{\mathcal{G}_t} = &\exp\left(\int_0^t \Theta_u \sqrt{V_1(u)} dW_u^1\right. \\ &\left. - \frac{1}{2}\int_0^t \Theta_u^2 V_1(u) du + \int_0^t \Theta_u \sqrt{V_2(u)} dW_u^2 - \frac{1}{2}\int_0^t \Theta_u^2 V_2(u) du\right). \end{aligned}$$

Denote a process $\tilde{\Theta} = \{\tilde{\Theta}_t\}_{t \in \mathcal{T}}$ to be the risk-neutral regime-switching Esscher parameters. On the condition of the process X, W^3 and W^4, the martingale condition is characterized by considering an enlarged filtration and requiring

$$S_0 = E^{\mathcal{Q}_{\tilde{\Theta}}}\left[\exp(-\int_0^t r_s ds) S(t) \left|\mathcal{G}_t^2\right.)\right].$$

This condition can be interpreted that the market's agent acquires information about the Markovian chain and the stochastic volatility process in advance.

Similar to the argument in [5], we know that $\tilde{\Theta}_t := \tilde{\Theta}\left(t, X_t, \sqrt{V_1(t)}, \sqrt{V_2(t)}\right)$ can be obtained by martingale condition (2.3) as follows

$$\begin{aligned}\tilde{\Theta}_t &= \frac{r(t,X_t)-\mu(t,X_t)}{V_1(t)} + \frac{r(t,X_t)-\mu(t,X_t)}{V_2(t)} \\ &= \frac{\lambda\left(t, X_t, \sqrt{V_1(t)}, \sqrt{V_2(t)}\right)}{\sqrt{V_1(t)}} + \frac{\lambda\left(t, X_t, \sqrt{V_1(t)}, \sqrt{V_2(t)}\right)}{\sqrt{V_2(t)}},\end{aligned}$$

where $\lambda_t := \lambda\left(t, X_t, \sqrt{V_1(t)}, \sqrt{V_2(t)}\right) \in \mathcal{G}_t$ is the market price of risk at time t.

Then, we can obtain

$$\tilde{\Theta}_t = \left\langle \tilde{\Theta}\left(t, \sqrt{V_1(t)}, \sqrt{V_2(t)}\right), X_t \right\rangle,$$

where

$$\tilde{\Theta}\left(t, \sqrt{V_1(t)}, \sqrt{V_2(t)}\right) = \left(\frac{r_1-\mu_1}{\sqrt{V_1(t)}} + \frac{r_1-\mu_1}{\sqrt{V_2(t)}}, \cdots, \frac{r_N-\mu_N}{\sqrt{V_1(t)}} + \frac{r_N-\mu_N}{\sqrt{V_2(t)}}\right).$$

It is an N-dimensional $\mathcal{G}_t^1$-measurable random vector. Thus, the Radon-Nikodym derivative of $\mathcal{Q}_{\tilde{\Theta}}$ with respect to $\mathcal{P}$ can be written as

$$\begin{aligned}\left.\frac{d\Theta_{\tilde{\Theta}}}{dP}\right|_{\mathcal{G}_t} &= \exp\left[\int_0^t \frac{r_u-\mu_u}{\sqrt{V_1(u)}} dW_u^1 + \int_0^t \frac{r_u-\mu_u}{\sqrt{V_2(u)}} dW_u^2 \right. \\ &\quad \left. - \frac{1}{2}\int_0^t \left(\frac{r_u-\mu_u}{\sqrt{V_1(u)}}\right)^2 du - \frac{1}{2}\int_0^t \left(\frac{r_u-\mu_u}{\sqrt{V_2(u)}}\right)^2 du \right].\end{aligned}$$

According to Girsanov's theorem, we know that

$$\widetilde{W}_t^1 = W_t^1 + \int_0^t \frac{r_s-\mu_s}{\sqrt{V_1(s)}} ds$$

and

$$\widetilde{W}_t^2 = W_t^2 + \int_0^t \frac{r_s-\mu_s}{\sqrt{V_2(s)}} ds$$

are two standard Brownian motions with respect to $\{\mathcal{G}_t\}_{t\in\mathcal{T}}$ under $\mathcal{Q}_{\widetilde{\Theta}}$. Since $\widetilde{W}^3$ and X are independent of W^1 and $\widetilde{W}^4$, X is independent of W^2, we can see that $\widetilde{W}^3$ and $\widetilde{W}^4$ are two standard Brownian motions under $\mathcal{Q}_{\widetilde{\Theta}}$. Note that X remains unchanged under the change of the probability measure from $\mathcal{P}$ to $\mathcal{Q}_{\widetilde{\Theta}}$. Let

$$\widetilde{\theta}_{1t}^2 = \theta_{1t}^2 - \rho_1\sigma_1(r_t-\mu_t), \quad \widetilde{\theta}_{1t}^2 = \theta_{2t}^2 - \rho_2\sigma_2(r_t-\mu_t).$$

Then the dynamics (2.2) can be written as

$$\begin{cases} dS(t) = r_t S(t)dt + \sqrt{V_1(t)}S(t)d\widetilde{W}_t^1 + \sqrt{V_2(t)}S(t)d\widetilde{W}_t^2, \\ dV_1(t) = \kappa_1(\widetilde{\theta}_{1t}^2 - V_1(t))dt + \rho_1\sigma_1\sqrt{V_1(t)}d\widetilde{W}_t^1 + \widetilde{\rho}_1\sigma_1\sqrt{V_1(t)}d\widetilde{W}_t^3, \\ dV_2(t) = \kappa_2(\widetilde{\theta}_{2t}^2 - V_2(t))dt + \rho_2\sigma_2\sqrt{V_2(t)}d\widetilde{W}_t^2 + \widetilde{\rho}_2\sigma_2\sqrt{V_2(t)}d\widetilde{W}_t^4. \end{cases} \tag{2.4}$$

Let $W_t^{3V} := \rho_1\widetilde{W}_t^1 + \widetilde{\rho}_1\widetilde{W}_t^3$ and $W_t^{4V} := \rho_2\widetilde{W}_t^2 + \widetilde{\rho}_2\widetilde{W}_t^4$. Then

$$\begin{cases} dV_1(t) = \kappa_1(\widetilde{\theta}_{1t}^2 - V_1(t))dt + \sigma_1\sqrt{V_1(t)}dW_t^{3V}, \\ dV_2(t) = \kappa_2(\widetilde{\theta}_{2t}^2 - V_2(t))dt + \sigma_2\sqrt{V_2(t)}dW_t^{4V}. \end{cases} \tag{2.5}$$

We remark that, if there is no regime-switching in the risk-neutral dynamics (2.5) under $\mathcal{Q}_{\widetilde{\Theta}}$, then it reduces to the risk-neutral dynamics in [9]. We also need the following lemma developed in [7].

Lemma 2.2. *If X is a regime-switching Markovian chain which has the dynamics (2.1), then the characteristic function denoted by $f_t(\phi)$ with respect to the stochastic variable $\int_t^T \langle \nu, X_s\rangle u(s)ds \in R$ is given by*

$$f_t(\phi) = E^{Q_{\widetilde{\Theta}}}\left[\exp\left(\phi\int_t^T \langle \nu, X_s\rangle u(s)ds\right)\middle| \mathcal{F}_t^X\right] = \langle \Phi(t,T;\nu)X_t, I\rangle,$$

where $I = (1, 1, \dots, 1) \in R^N$ and

$$\Phi(t,T;\nu) = \exp\left(\int_t^T \left(\Pi' + \phi u(s) diag[\nu]\right) ds\right),$$

here Π' denotes the transposition of Π.

3. Main Results

In this section, we adopt a PDE approach to pricing discretely-sampled volatility-average swaps. Volatility swaps are essentially annualized forward contracts with realized volatility, which provide investors with a simple way to trade future realized volatility against current implied volatility. A volatility swap is a forward contract that relates to the historical fluctuations in the specified equity index. The amount paid at expiration is based on the notional amount multiplied by the difference between the realized volatility and the implied volatility. Assume that the value of the volatility swap at expiry can be written as $(RV(0, N, T_e) - K_{vol}) \times L$ while the current time is 0, where $RV(0, N, T_e)$ is the annualized realized volatility over the contract life $[0, T_e]$, K_{vol} is the annualized delivery price for the volatility swap, and L is the notional amount of the swap in dollars per annualized volatility point squared. When the contract is entered, K_{vol} can let the value of a volatility swap be zero for both long and short positions. To a certain extent, it reflects the market expectations of future realized volatility.

At the beginning of the contract of the volatility swap, it clearly defines the details of the calculation of the volatility $RV(0, N, T_e)$. The important factors for calculating the realized volatility include the underlying assets, the observation frequency of the price of the assets, the annualization factors, the lifetime of contracts, and the method of calculating the volatility.

Definition 3.1. According to [10], two different measures of realized volatility can be defined as follows:

$$RV_{d1}(0, N, T_e) = \sqrt{\frac{AF}{N}\sum_{k=1}^{N}\left(\frac{S(t_k) - S(t_{k-1})}{S(t_{k-1})}\right)^2} \times 100 \tag{3.1}$$

and

$$RV_{d2}(0, N, T_e) = \sqrt{\frac{\pi}{2NT_e}}\sum_{k=1}^{N}\left|\frac{S(t_k) - S(t_{k-1})}{S(t_{k-1})}\right| \times 100, \tag{3.2}$$

where $t_k,\ k = 0 \dots N$, is the kth observation time of the realized volatility in the pre-specified time period $[0, T_e]$, $t_0 = 0$, $T_N = T_e$ and AF is the annualized factor that converts this expression to annualized variance.

For most trading variance swaps or even over-the-counter transactions, the sampling period is usually constant, which makes calculation of the realized volatility easier. Therefore, in this paper we assume the discrete observational values of equal intervals $[0, T_e]$. Then, the annualized coefficient is a simple expression $AF = \frac{1}{\Delta t} = \frac{N}{T_e}$.

In order to make sure that the variance is always positive, we require that $2\kappa_i \widetilde{\theta}_{it}^2 \geq V_i(t),\ (i = 1, 2)$. Then we will begin our analysis on the risk-neutral Heston's model. The conditional expectation at time t is denoted by

$$E_t^{\mathcal{Q}_{\tilde{\Theta}}} = E^{\mathcal{Q}_{\tilde{\Theta}}}[\,\cdot\,|\mathcal{F}_t],$$

where $\mathcal{F}_t$ is the filtration up to time t. The fair delivery price can be presented by

$$K_{vol} = E_0^{\mathcal{Q}_{\tilde{\Theta}}}[RV(0, N, T_e)]$$

in the risk-neutral world as a given definition of realized volatility $RV(0, N, T_e)$ via (2.4). In what follows, we illustrate a PDE approach to get a closed-form solution to obtain the fair strike price of a discretely-sampled volatility-average swap whose realized volatility depends on a specific measurement based on (3.2).

From the definition of $RV_{d2}(0,N,T_e)$, we can simplify (3.2) as

$$\begin{aligned} K_{vol} &= E_0^{\mathcal{Q}_{\tilde{\Theta}}}[RV_{d2}(0,N,T_e)] \\ &= E_0^{\mathcal{Q}_{\tilde{\Theta}}}\left[\sqrt{\frac{\pi}{2NT_e}}\sum_{k=1}^{N}\left|\frac{S(t_i)-S(t_{k-1})}{S(t_{k-1})}\right|\right]\times 100 \\ &= \sqrt{\frac{\pi}{2NT_e}}\sum_{k=1}^{N}E_0^{\mathcal{Q}_{\tilde{\Theta}}}\left[\left|\frac{S(t_k)-S(t_{k-1})}{S(t_{k-1})}\right|\right]\times 100, \end{aligned} \tag{3.3}$$

where N denotes the whole sampling times of the swap contract. The problem can be reduced to

$$E_0^{\mathcal{Q}_{\tilde{\Theta}}}\left[\left|\frac{S(t_k)-S(t_{k-1})}{S(t_{k-1})}\right|\right], \tag{3.4}$$

for some fixed equal time interval Δt and N different tensors $t_k = k\Delta t$, $k = 1\ldots N$. We consider all the sampling points t_k are known constants once the specific discretization along the time $[0,T_e]$ is made.

We would like to point out that $S(t_k)$ and $S(t_{k-1})$ are the underlying prices at two future sampling points t_k and t_{k-1} so that the two stochastic variable $S(t_k)$ and $S(t_{k-1})$ concurrently exist inside of the expectation operator, which informs the difficulty to our pricing problem. To deal with such difficulty and obtain the forward characteristic function, we suppose the current time to be t and $y_T = \ln S(T+\Delta) - \ln S(T)$ defines the forward characteristic function (FCF) $f(\phi; t, T, \Delta, V_1(t), V_2(t))$ of y_T as the Fourier transform of the probability density function of y_T , i.e.,

$$f(\phi; t, T, \Delta, V_1(t), V_2(t)) = E^{Q_{\tilde{\Theta}}}\left[e^{\phi y_T}\middle| y_t, V_1(t), V_2(t), \mathcal{F}_t^X\right], \qquad t < T. \tag{3.5}$$

Note that imaginary unit $i = \sqrt{-1}$ has been absorbed into the parameter ϕ. For simplicity, the explicit exposition of i does not alter the essence of this function, so we still call it the FCF. In what follows, we will give the detail about how to obtain the explicit exposition of the condition value. To get the characteristic function, we firstly consider the evaluation of the condition value, or price, of a derivative given the information about the sample path of the Markovian chain X from time 0 to $T+\Delta$ as well as $\mathcal{F}_{T+\triangle}^X$. After giving the filtration $\mathcal{F}_{T+\triangle}^X$ and a realization path of X_t, the parameters r_t and $\widetilde{\theta}_{jt}^2$, $j = 1, 2$ can be considered to be time-dependent deterministic functions.

The conditional characteristic function is given by the following theorem.

Theorem 3.2. *If the value of the underlying asset is given by (2.5), then the FCF of the stochastic variable* $y_T = \ln S(T+\Delta) - \ln S(T)$ *can be obtained by*

$$\begin{aligned} & f(\phi; t, T, \Delta, V_1(t), V_2(t)|\mathcal{F}^X_{T+\Delta}) = E^{Q_{\tilde{\Theta}}}[e^{\phi y_T}|y_t, V_1(t), V_2(t), \mathcal{F}^X_{T+\Delta}] \\ & = e^{C(\phi,T)} g\left(\sum_{j=1}^{2} D_j(\phi, T); t, T, V_1(t), V_2(t)\right), j = 1, 2, \end{aligned} \tag{3.6}$$

where

$$\begin{cases} C(\phi, t) = \int_t^{T+\Delta} \left\langle r\phi + \sum_{j=1}^{2} \kappa_j \widetilde{\theta}_j^2 D_j(\phi, s), X_s \right\rangle ds, \\ D_j(\phi, t) = \dfrac{a_j + b_j}{\sigma_j^2} \dfrac{1 - e^{b_j(T+\Delta-t)}}{1 - g_j e^{b_j(T+\Delta-b)}}, \\ a_j = \kappa_j - \rho_j\sigma_j\phi, \quad b_j = \sqrt{a_j^2 + \sigma_j^2(\phi - \phi^2)}, \quad g_j = \dfrac{a_j + b_j}{a_j - b_j}, \quad j = 1, 2. \end{cases} \tag{3.7}$$

and

$$\begin{cases} g(\phi; t, T, V_1(t), V_2(t)) = e^{\sum_{j=1}^{2} F_j(\phi,t) + \sum_{j=1}^{2} G_j(\phi,t) V_j(t)}, \\ F_j(\phi, t) = \int_t^T \left\langle \kappa_j \widetilde{\theta}_j^2 G_j(\phi, s), X_s \right\rangle ds, \\ G_j(\phi, t) = \dfrac{2\kappa_j \phi}{\sigma_j^2 \phi + (2\kappa_j - \sigma_j^2 \phi) e^{k_j(T-t)}}, \quad j = 1, 2. \end{cases} \tag{3.8}$$

Proof. Assuming the current time is $t (t < T)$. Let

$$y_T = \ln S(T+\Delta) - \ln S(T),$$

where $S(t)$ is the underlying price according to the Heston's model. When given the filtration $\mathcal{F}^X_{T+\Delta}$, the parameters r_t and $\widetilde{\theta}^2_{jt} (j = 1, 2)$ can be considered to be time-dependent deterministic functions, and the FCF of y_T can be defined as

$$f(\phi; t, T, \Delta, V_1(t), V_2(t)) = E^{\mathcal{Q}_{\tilde{\Theta}}}[e^{\phi y_T}|y_t, V_1(t), V_2(t)]. \tag{3.9}$$

Then, by using the tower rule of expectation, one has

$$\begin{aligned} & f(\phi; t, T, \Delta, V_1(0), V_2(0)) \\ = \; & E^{\mathcal{Q}_{\tilde{\Theta}}}\left[E^{\mathcal{Q}_{\tilde{\Theta}}}[e^{\phi y_T}|y_T, V_1(T), V_2(T)]|y_t, V_1(t), V_2(t)\right]. \end{aligned} \tag{3.10}$$

The inner expectation $E^{\mathcal{Q}_{\tilde{\Theta}}}[e^{\phi y_T}|y_T, V_1(T), V_2(T)]$ can be solved by utilizing the Feynman-Kac theorem, which has been presented in [9]. We define

$U(\phi;t,X,V_1(t),V_2(t)) = E^{\mathcal{Q}_{\tilde{\Theta}}}[e^{\phi y_T}|y_T,V_1(T),V_2(T)]$ with $T \le t \le T+\triangle$, such that

$$\begin{cases} \frac{\partial U}{\partial t}+[\kappa_1(\widetilde{\theta_{1t}^2}-V_1(t))]\frac{\partial U}{\partial V_1(t)}+[\kappa_2(\widetilde{\theta_{2t}^2}-V_2(t))]\frac{\partial U}{\partial V_2(t)} \\ +\frac{1}{2}\sigma_1^2V_1(t)\frac{\partial^2 U}{\partial V_1(t)^2}+\frac{1}{2}\sigma_2^2V_2(t)\frac{\partial^2 U}{\partial V_2(t)^2} \\ +[r-\frac{1}{2}(V_1(t)+V_2(t))]\frac{\partial U}{\partial X}+\frac{1}{2}(V_1(t)+V_2(t))\frac{\partial^2 U}{\partial X^2} \\ +\sigma_1V_1(t)\rho_1\frac{\partial^2 U}{\partial X\partial V_1(t)}+\sigma_2V_2(t)\rho_2\frac{\partial^2 U}{\partial X\partial V_2(t)}=0, \\ U(\phi;T+\Delta,X,V_1(t),V_2(t))=e^{\phi X}, \end{cases} \tag{3.11}$$

where $X=\ln S_t-\ln S_T(T<t<T+\Delta)$. Inspired by the solution proposed by [9], we assume that the solution has the following form:

$$U(\phi;t,X,V_1(t),V_2(t))=e^{C(\phi,t)+D_1(\phi,t)V_1(t)+D_2(\phi,t)V_2(t)+\phi X}, \tag{3.12}$$

then we can get

$$\begin{aligned} &C'+D_1'V_1(t)+D_2'V_2(t)+[\kappa_1(\widetilde{\theta_{1t}^2}-V_1(t))]D_1+[\kappa_2(\widetilde{\theta_{2t}^2}-V_2(t))]D_2 \\ &+\frac{1}{2}\sigma_1^2V_1(t)D_1^2+\frac{1}{2}\sigma_2^2V_2(t)D_2^2+\phi[r(t)-\frac{1}{2}(V_1(t)+V_2(t))] \\ &+\frac{1}{2}\phi^2(V_1(t)+V_2(t))+\phi D_1\sigma_1\rho_1V_1(t)+\phi D_2\sigma_2\rho_2V_2(t)=0. \end{aligned}$$

Thus, the PDE (3.11) reduces to the ordinary differential equations (ODEs):

$$\begin{cases} -\frac{\partial C}{\partial t}=r(t)\phi+\kappa_1\widetilde{\theta_{1t}^2}D_1+\kappa_2\widetilde{\theta_{2t}^2}D_2, \\ \frac{\partial D_1}{\partial t}=\frac{1}{2}(\phi-1)\phi+(\sigma_1\rho_1\phi-\kappa_1)D_1+\frac{1}{2}\sigma_1^2D_1^2, \\ \frac{\partial D_2}{\partial t}=\frac{1}{2}(\phi-1)\phi+(\sigma_2\rho_2\phi-\kappa_2)D_2+\frac{1}{2}\sigma_2^2D_2^2, \end{cases}$$

with initial conditions

$$\begin{cases} C(\phi,T+\Delta)=0, \\ D_1(\phi,T+\Delta)=0, \\ D_2(\phi,T+\Delta)=0. \end{cases}$$

These ODEs can be solved by the solutions

$$U(\phi;t,X,V_1(t),V_2(t))=e^{C(\phi,t)+D_1(\phi,t)V_1(t)+D_2(\phi,t)V_2(t)+\phi X}, \quad T\le t\le T+\Delta,$$

where,

$$\begin{cases} C(\phi,t) = \displaystyle\int_t^{T+\Delta}\left[r(s)\phi + \sum_{j=1}^{2}\kappa_j\widetilde{\theta}_j^2(s)D_j(s)\right]ds, \\ D_j(\phi,t) = \dfrac{a_j+b_j}{\sigma_j^2}\dfrac{1-e^{b_j(T+\Delta-t)}}{1-g_je^{b_j(T+\Delta-b)}}, \\ a_j = \kappa_j - \rho_j\sigma_j\phi, \quad b_j = \sqrt{a_j^2+\sigma_j^2(\phi-\phi^2)}, \quad g_j = \dfrac{a_j+b_j}{a_j-b_j}, \quad j=1,2. \end{cases} \tag{3.13}$$

At time T, one has $X = \ln S(T) - \ln S(T) = 0$. Thus, we can obtain

$$\begin{aligned} E^{\mathcal{Q}_{\tilde{\Theta}}}[e^{\phi y_T}|y_T, V_1(T), V_2(T)] &= U(\phi; T, X, V_1(T), V_2(T)) \\ &= e^{C(\phi,T)+D_1(\phi,T)V_1(T)+D_2(\phi,T)V_2(T)}. \end{aligned}$$

Let

$$\begin{cases} g_1(\phi,t,T,V_1(t)) = E^{\mathcal{Q}_{\tilde{\Theta}}}[e^{\phi V_1(T)}|y_T, V_1(T)], \\ g_2(\phi,t,T,V_2(t)) = E^{\mathcal{Q}_{\tilde{\Theta}}}[e^{\phi V_2(T)}|y_T, V_2(T)], \\ g(\phi,t,T,V_1(t),V_2(t)) = g_1(\phi,t,T,V_1(t))g_2(\phi,t,T,V_2(t)). \end{cases}$$

Applying the characteristic function $g(\phi,t,T,V_1(t),V_2(t))$ of the stochastic variable $V_1(T)$ and $V_2(T)$, the affine-form solution facilitates the calculation of the exterior expectation ($0 \le t \le T$). Utilizing the Feynman-Kac theorem again, we can find the solution of $g(\phi,t,T,V_1(t),V_2(t))$ satisfies the following PDEs:

$$\begin{cases} \frac{\partial g_1}{\partial t} + \frac{1}{2}\sigma_1^2 V_1 \frac{\partial g_1^2}{\partial V_1^2} + [\kappa_1(\widetilde{\theta}_{1t}^2 - V_1)\frac{\partial g_1}{\partial V_1}] = 0, \\ g_1(\phi,T,T,V_1) = e^{\phi V_1} \end{cases}$$

and

$$\begin{cases} \frac{\partial g_2}{\partial t} + \frac{1}{2}\sigma_2^2 V_2 \frac{\partial g_2^2}{\partial V_2^2} + [\kappa_2(\widetilde{\theta}_{2t}^2 - V_2)\frac{\partial g_2}{\partial V_2}] = 0, \\ g_2(\phi,T,T,V_2) = e^{\phi V_2}. \end{cases}$$

Following the solution procedure developed by [9], we can solve the two PDEs by firstly guessing that the solution might be the form

$$\begin{cases} g_1(\phi,t,T,V_1) = e^{F_1(\phi,t)+G_1(\phi,t)V_1(t)}, \\ g_2(\phi,t,T,V_2) = e^{F_2(\phi,t)+G_2(\phi,t)V_2(t)}. \end{cases}$$

The functions $G_1(\phi,t), G_2(\phi,t)$ and $F_1(\phi,t), F_1(\phi,t)$ can be found by solving the two Riccati ODEs:

$$\begin{cases} -\dfrac{dG_1}{dt} = -\kappa_1 G_1 + \dfrac{1}{2}\sigma_1^2 G_1^2, \\ -\dfrac{dF}{dt} = \kappa_1 \widetilde{\theta}_1^2 G_1 \end{cases}$$

and

$$\begin{cases} -\dfrac{dG_2}{dt} = -\kappa_2 G_2 + \dfrac{1}{2}\sigma_2^2 G_2^2, \\ -\dfrac{dF}{dt} = \kappa_2 \widetilde{\theta}_2^2 G_2 \end{cases}$$

with initial conditions

$$F_1(\phi,T) = 0, \quad F_2(\phi,T) = 0, \quad G_1(\phi,T) = \phi, \quad G_2(\phi,T) = \phi.$$

Then we can get the solutions

$$\begin{cases} F_1(\phi,t) = \displaystyle\int_t^T \kappa_1 \widetilde{\theta}_1^2(s) G_1(s) ds, \\ G_1(\phi,t) = \dfrac{2\kappa_1\phi}{\sigma_1^2\phi + (2\kappa_1 - \sigma_1^2\phi)e^{\kappa_1(T-t)}} \end{cases}$$

and

$$\begin{cases} F_2(\phi,t) = \displaystyle\int_t^T \kappa_2 \widetilde{\theta}_2^2(s) G_2(s) ds, \\ G_2(\phi,t) = \dfrac{2\kappa_2\phi}{\sigma_2^2\phi + (2\kappa_2 - \sigma_2^2\phi)e^{\kappa_2(T-t)}}. \end{cases}$$

Thus,

$$g(\phi,t,T,V_1(t),V_2(t)) = g_1 g_2 = e^{F_1(\phi,t)+G_1(\phi,t)V_1(t)+F_2(\phi,t)+G_2(\phi,t)V_2(t)}.$$

This completes the proof. □

In the sequel, we will get the expectation of (3.6), where r_t and $\widetilde{\theta}_{jt}^2$ depend on the path of X process up to time $T+\Delta$. Firstly, we have

$$\begin{aligned} & f(\phi;t,T,\Delta,V_1(t),V_2(t)) \\ = \; & E^{Q_{\tilde{\Theta}}}\left[e^{\phi y_T} | y_t, V_1(t), V_2(t), \mathcal{F}_t^X\right] \\ = \; & E^{Q_{\tilde{\Theta}}}\left[E^{Q_{\tilde{\Theta}}}\left[e^{\phi y_T} | y_t, V_1(t), V_2(t), \mathcal{F}_{T+\Delta}^X\right] | y_t, V_1(t), V_2(t), \mathcal{F}_t^X\right] \end{aligned} \tag{3.14}$$

$$
\begin{aligned}
&= E^{Q_{\widetilde{\Theta}}}\left[e^{C(\phi,T)}\, g\left(\sum_{j=1}^{2} D_j(\phi,T);t,T,V_1(t),V_2(t)\right)\middle| y_t, V_1(t), V_2(t), \mathcal{F}_t^X\right]\\
&= E^{Q_{\widetilde{\Theta}}}\left[\exp\left(\int_T^{T+\Delta}\left\langle r\phi + \sum_{j=1}^{2}\kappa_j\widetilde{\theta}_j^2 D_j(\phi,s), X_s\right\rangle ds\right.\right.\\
&\quad + \int_t^T\left\langle\sum_{j=1}^{2}\kappa_j\widetilde{\theta}_j^2 G_j(D_j(\phi,T),s), X_s\right\rangle ds\\
&\quad \left.\left. + \sum_{j=1}^{2}\frac{2\kappa_j D_j(\phi,T)}{\sigma_j^2 D_j(\phi,T) + (2\kappa_j - \sigma_j^2 D_j(\phi,T))e^{\kappa_j(T-t)}}V_j(t)\right)\middle| y_t, V_1(t), V_2(t), \mathcal{F}_t^X\right]\\
&= E^{Q_{\widetilde{\Theta}}}\left[\exp\left(\int_t^{T+\Delta}\langle J_1(s)+J_2(s), X_s\rangle ds\right)\middle| y_t, V_1(t), V_2(t), \mathcal{F}_t^X\right]\\
&\quad \times \exp\left(\sum_{j=1}^{2} V_j(t) G_j(D_j(\phi,T),t)\right).
\end{aligned}
$$

Here, functions $J_1(t)$ and $J_2(t)$ are given by

$$
\begin{cases}
J_1(t) = \kappa_1\widetilde{\theta}_1^2 G_1(D_1(\phi,T),t)(1-H_T(t)) + (r\phi + \kappa_1\widetilde{\theta}_1^2 D_1(\phi,t))H_T(t),\\
J_2(t) = \kappa_2\widetilde{\theta}_2^2 G_2(D_2(\phi,T),t)(1-H_T(t)) + (r\phi + \kappa_2\widetilde{\theta}_2^2 D_2(\phi,t))H_T(t),
\end{cases}
$$

and $H_T(t)$ is a Heaviside unit step function, when $t \geq T$, $H_T(t) = 1$, or else $H_T(t) = 0$. The core calculation involved in equation (3.14) can be simply represented in the form

$$
E^{Q_{\widetilde{\Theta}}}\left[\exp\left(\int_t^T \langle \nu, X_s\rangle u(s) ds\right)\middle|\mathcal{F}_t^X\right],
$$

where ν is an R^N vector and $u(s)$ is a general deterministic integrable function.

Utilizing Lemma 2.2 and all the derivation procedures above, we can obtain the characteristic function of $y_T = \ln S(T+\triangle) - \ln S(T)$ under the regime-switching environment as follows

$$
\begin{aligned}
f(\phi;t,T,\Delta,V_1(t),V_2(t) &= E^{Q_{\widetilde{\Theta}}}\left[e^{\phi y_T}|y_t, V_1(t), V_2(t), \mathcal{F}_t^X\right]\\
&= \exp\left(\sum_{j=1}^{2} V_j(t) G_j(D_j(\phi,T),t)\right)\langle\Phi(t,T+\Delta;J_1+J_2)X_t, I\rangle,
\end{aligned}
$$

where

$$D_j(\phi, t) = \frac{a_j + b_j}{\sigma_j^2} \frac{1 - e^{b_j(T+\Delta-t)}}{1 - g_j e^{b_j(T+\Delta-t)}}$$

and

$$G_j(\phi, t) = \frac{2\kappa_j \phi}{\sigma_j^2 \phi + (2\kappa_j - \sigma_j^2 \phi) e^{\kappa_j (T-t)}}, \quad j = 1, 2.$$

Here

$$a_j = \kappa_j - \rho_j \sigma_j \phi, \quad b_j = \sqrt{a_j^2 + \sigma_j^2(\phi - \phi^2)}, \quad g_j = \frac{a_j + b_j}{a_j - b_j}$$

and

$$\phi(t, T+\Delta; J_1 + J_2) = \exp\left(\int_t^{T+\Delta} (2\Pi' + diag[J_1(s)] + diag[J_2(s)]) ds\right)$$

with

$$J_j(t) = \kappa_j \widetilde{\theta}_j^2 G_j(D_j(\phi, T)t)(1 - H_T(t)) + \left(r\phi + \kappa_j \widetilde{\theta}_j^2 D_j(\phi, t)\right) H_T(t), \quad j = 1, 2.$$

In the sequel, we will obtain the closed-form pricing formula for the volatility swaps. By utilizing the FCF, we denote $p(y_{t_k-1,t_k})$ as the probability density function of the stochastic variable $y_{t_k-1,t_k} = \ln S(t_k) - \ln S(t_k - 1)$ which can be obtained by presenting the inverse Fourier transform. We also let Q_k:=prob$(y_{t_k-1,t_k} > 0)$ be the probability of the event $y_{t_k-1,t_k} > 0$. Based on the relationship between the characteristic function and the cumulative function, we can write Q_k as follows:

$$\begin{aligned} Q_k &= \int_0^\infty P(y_{t_{k-1},t_k}) dy_{t_{k-1},t_k} \\ &= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{f(\phi i; t_{k-1}, t_k, \Delta t, V_1(0), V_2(0))}{\phi i}\right] d\phi. \end{aligned} \tag{3.15}$$

We indicate that the function

$$q(y_{t_k-1,t_k}) = e^{(y_{t_k-1,t_k} - r\Delta t)} p(y_{t_k-1,t_k}) (\Delta t = t_k - t_{k-1})$$

satisfies the following two properties via equation (3.15)

(i) $q(y_{t_k-1,t_k}) \geq 0$;
(ii) $\int_{-\infty}^{+\infty} q(y_{t_{k-1},t_k}) dy_{t_{k-1},t_k} = 1$.

Therefore, it can be concluded that the function $q(y_{t_k-1,t_k})$ is a probability density function of $q(y_{t_k-1,t_k})$, whose characteristic function

$$\widetilde{f}(\phi; t_{k-1}, t_k, \Delta t, V_1(0), V_2(0))$$

can be defined as follows:

$$\begin{aligned}\widetilde{f}(\phi; t_{k-1}, t_k, \Delta t, V_1(0), V_2(0)) &= \mathcal{F}[e^{(y_{t_{k-1},t_k} - r\Delta t)} p(y_{t_{k-1},t_k})] \\ &= e^{-r\Delta t} \mathcal{F}[e^{y_{t_{k-1},t_k}} p(y_{t_{k-1},t_k})] \\ &= e^{-r\Delta t} f(\phi i + 1; t_{k-1}, t_k, \Delta t, V_1(0), V_2(0)).\end{aligned} \tag{3.16}$$

We note that

$$f(\phi i; t_{k-1}, t_k, \Delta t, V_1(0), V_2(0)) = \mathcal{F}[p(y_{t_{k-1},t_k})]$$

and the Fourier transform is defined as

$$\mathcal{F}[\Psi(x)] = \int_{-\infty}^{+\infty} e^{i\phi x} \Psi(x) dx.$$

Similarly, we can get the probability

$$\begin{aligned}\widetilde{Q_k} &= \int_0^{+\infty} e^{(y_{t_{k-1},t_k^{-\Delta t}})} p(y_{t_{k-1},t_k}) dy_{t_{k-1},t_k} \\ &= \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \mathrm{Re}\left[\frac{e^{-r\Delta t} f(\phi i + 1; t_{k-1}, t_k, \Delta t, V_1(0), V_2(0))}{\phi i}\right] d\phi.\end{aligned}$$

Based on the above procedures, the expectation in (3.4) can be written as:

$$\begin{aligned}&E_0^{\mathcal{Q}_{\tilde{\Theta}}}\left[\left|\frac{S(t_k)}{S(t_{k-1})} - 1\right|\middle| y_0, V_1(0), V_2(0), \mathcal{F}_0^X)\right] \\ &= \int_{-\infty}^{+\infty} |e^{y_{t_{k-1},t_k}} - 1| p(y_{t_k-1,t_k}) dy_{t_k-1,t_k} \\ &= \int_0^{+\infty} (e^{y_{t_{k-1},t_k}} - 1) p(y_{t_k-1,t_k}) dy_{t_k-1,t_k} \\ &\quad + \int_{-\infty}^0 (1 - e^{y_{t_{k-1},t_k}}) p(y_{t_k-1,t_k}) dy_{t_k-1,t_k} \\ &= -\int_0^{+\infty} p(y_{t_k-1,t_k}) dy_{t_k-1,t_k} + \int_{-\infty}^0 p(y_{t_k-1,t_k}) dy_{t_k-1,t_k} \\ &\quad + e^{r\Delta t}\left(\int_0^{+\infty} p(y_{t_k-1,t_k}) dy_{t_k-1,t_k} - \int_{-\infty}^0 p(y_{t_k-1,t_k}) dy_{t_k-1,t_k}\right) \\ &= 1 - 2Q_k + e^{r\Delta t}(2\widetilde{Q_k} - 1) \\ &= \frac{2}{\pi} \int_0^{\infty} \mathrm{Re}[M_k] d\phi,\end{aligned}$$

where

$$M_k = \frac{f(\phi i + 1; t_{k-1}, t_k, \Delta t, V_1(0), V_2(0)) - f(\phi i; t_{k-1}, t_k, \Delta t, V_1(0), V_2(0))}{\phi i}.$$

Thus, (3.3) can be carried out all the way with k ranging from 1 to N which indicates to the final pricing formula for the volatility swaps as follows:

$$\begin{aligned} K_{vol} &= E_0^{\mathcal{Q}_{\tilde{\Theta}}}[RV_{d2}(0,N,T_e)] \\ &= E_0^{\mathcal{Q}_{\tilde{\Theta}}}\left[\sqrt{\frac{\pi}{2NT_e}}\sum_{k=1}^{N}\left|\frac{S(t_k)-S(t_{k-1})}{S(t_{k-1})}\right|\right]\times 100 \\ &= \sqrt{\frac{2}{\pi NT_e}}\int_0^{\infty}\sum_{k=1}^{N}\mathrm{Re}[M_k]d\phi\times 100. \end{aligned}$$

Therefore, we get the closed-form solution with respect to the fair strike price for the volatility swaps.

Remark 3.3. We would like to point out that the double stochastic volatility model can be expanded to a multi-stochastic volatility model, which can be derived to obtain the characteristic function to pricing the volatility swaps. As a result, it depicts market risks and other uncertainties more accurately.

4. Conclusion

In this paper, we develop a pricing volatility derivatives model to analyze the double-stochastic volatility swaps via a continuous-time Markovian-modulates version of Heston's stochastic volatility model. In general case, the market in the framework of Markovian-modulated model is incomplete with many equivalent martingale pricing measures. To overcome this difficult, we define a martingale pricing measure to value the volatility swaps via the regime switching Esscher transform by [5]. We also use the Heston's stochastic volatility model to describe the underlying asset's price and extend the one-factor volatility to the double stochastic volatility. Then, we obtain a closed-form solution to the discretely-sampled volatility swaps where the realized volatility is defined as the average value of the asset price of the absolute percentage increment. It can be considered as an appropriate progress in the field of pricing volatility swaps.

This study also shows that the proposed solution can be used to calculate the lower limit of the corresponding standard derived swaps, in which the volatility is defined as the square root of the average value of the variance. In addition, the efficiency of the calculation is improved significantly in helping the practitioners pricing volatility swaps by using the new analytical formula developed in this paper. However, we do not consider the case (3.2) in our model, which is also important in pricing volatility swaps and should be further investigated.

Acknowledgements: The authors are grateful to the editors and referees for their valuable comments and suggestions.

References

[1] A. Biswas, A. Goswami and L. Overbeck, *Option pricing in a regime switching stochastic volatility model,* Statistics and Probab. Letters, **138**(2018),116-126.

[2] P. Christoffersen, S. Heston and K. Jacobs, *The shape and term structure of the index option smirk: Why multifactor stochastic volatility models work so well*, Management Science, **55(12)**(2009),1914-1932.

[3] J.C. Cox, J.E. Ingersoll and S.A. Ross, *A theory of the term structure of interest rates,* Econometrica, **53(2)**(1985), 385-407.

[4] R.J. Elliott, L. Aggoun and J.B. Moore, Hidden Markov models: estimation and control, Springer-Verlag, 1995.

[5] R.J. Elliott, L. Chan and T.K. Siu, *Option pricing and esscher transform under regime switching,* Ann. of Finance, **1(4)**(2005), 423-432.

[6] R. J. Elliott and T. K. Siu, *On risk minimizing portfolios under a Markovian regime-switching Black-Scholes economy*, Ann. of Opera. Research, **176(1)**(2010), 271-291.

[7] R. J. Elliott and G. H. Lian, *Pricing variance and volatility swaps in a stochastic volatility model with regime switching: discrete observations case*, Quantitative Finance, **13(5)**(2013), 687-698.

[8] A. Grunbichler and F. Longstaff, *Valuing futures and options on volatility,* J. Banking and Finance, **20(6)**(1996), 985-1001.

[9] S.L. Heston, *A closed-form solution for options with stochastic volatility with applications to bond and currency options*, The Review of Financial Studies, **6(2)**(1993), 327-343.

[10] S. Howison, A. Rafailidis and H. Rasmussen, *On the pricing and hedging of volatility derivatives*, Appl. Math. Finance, **11(4)**(2004), 317-346.

[11] F. Mehrdoust and N. Saber, *Pricing arithmetic Asian option under a two-factor stochastic volatility model with jumps*, J. Statistical Comput. and Simul., **85(18)**(2015), 3811-3819.

[12] L.C. Robert, J. Eliott and T. K. Siu, *Pricing volatility swaps under heston's stochastic volatility model with regime switching*, Appl. Math. Finance, **14(1)**(2007), 41-62.

[13] T.K. Siu, H Yang and J.W. Lau, *Pricing currency options under two-factor Markov-modulated stochastic volatility models*, Insurance: Math. and Economics, **43(3)**(2008), 295-302.

[14] A. Swishchuk, *Modeling of variance and volatility swaps for financial markets with stochastic volatilities*, Wilmott Magazine, **2**(2004), 64-72.

[15] S. Zhang and Y. Sun, *Forward starting options pricing with double stochastic volatility, stochastic interest rates and double jumps*, J. Comput. Appl. Math., **325**(2017), 34-41.

[16] S.P. Zhu and G.H. Lian, *A closed-form exact solution for pricing variance swaps with stochastic volatility*, Math. Finance: An Inter. J. Math. Statistics and Finan. Economics, **21(2)**(2011), 233-256.

[17] D. M. Zhu, J. Lu , W. K. Ching and T.K. Siu, *Option pricing under a stochastic interest rate and volatility model with hidden Markovian regime-switching*, Comput. Economics, **53**(2019), 555-586.