Nonlinear Functional Analysis and Applications
Vol. 24, No. 4 (2019), pp. 735-745
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

MODIFIED SP-ITERATION ALGORITHMS FOR SOLVING FIXED POINT PROBLEMS OF CONTINUOUS FUNCTIONS ON AN ARBITRARY INTERVAL

Prasit Cholamjiak[1], Tanakit Thianwan[2]
Lanchakorn Kittiratanawasin[3] and Chonjaroen Chairatsiripong[4]

[1]School of Science, University of Phayao
Phayao 56000, Thailand
e-mail: prasitch2008@yahoo.com

[2]School of Science, University of Phayao
Phayao 56000, Thailand
e-mail: tanakit.th@up.ac.th

[3]Faculty of Science, Kasetsart University
Bangkok 10900, Thailand
e-mail: fscilkk@ku.ac.th

[4]School of Science, University of Phayao
Phayao 56000, Thailand
e-mail: jack4463@hotmail.com

Abstract. In this article, we introduce a new modified SP-iteration algorithm for solving fixed point problems of continuous functions on an arbitrary interval. Convergence theorems are established for the new iteration algorithm. Further, we test numerical experiments of the proposed algorithm to compare with Mann, Ishikawa, Noor, CP and SP- iterations.

1. Introduction

Let C be a closed interval on the real line and let $f : C \to C$ be a continuous function. A point $p \in C$ is called a *fixed point* of f if $f(p) = p$.

[0]Received February 21, 2019. Revised September 20, 2019.
[0]2010 Mathematics Subject Classification: 54E70, 47H25.
[0]Keywords: Continuous function, convergence theorem, fixed point.
[0]Corresponding author: C. Chairatsiripongg(jack4463@hotmail.com).

In the original way to estimate a fixed point of a nonlinear mapping, Mann [4] introduced Mann iteration, which generates a sequence $\{x_n\}$ as follows :

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n f(x_n) \tag{1.1}$$

for all $n \geq 1$, where $\{\alpha_n\} \in [0,1]$. In 1991, Borwein and Borwein [1] proved the convergence theorem for a continuous function on the closed and bounded interval in the real line by using iteration (1.1).

Another classical iteration process was introduced by Ishikawa [3] which is formulated as follows:

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n f(x_n), \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n f(y_n) \end{aligned} \tag{1.2}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. Such an iterative algorithm is called Ishikawa iteration. In 2006, Qing and Qihou [8] proved the convergence theorem of the sequence generated by iteration (1.2) for a continuous function on the closed interval in the real line (see also [7]).

In 2000, Noor [5] defined the following iterative scheme by $x_1 \in C$ and

$$\begin{aligned} z_n &= (1-\mu_n)x_n + \mu_n f(x_n), \\ y_n &= (1-\beta_n)x_n + \beta_n f(z_n), \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n f(y_n) \end{aligned} \tag{1.3}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$, such an iterative algorithm is known as Noor iteration.

Next Phuengrattana and Suantai [6] introduced and studied the SP-iteration as follows: $x_1 \in C$ and

$$\begin{aligned} z_n &= (1-\mu_n)x_n + \mu_n f(x_n), \\ y_n &= (1-\beta_n)z_n + \beta_n f(z_n), \\ x_{n+1} &= (1-\alpha_n)y_n + \alpha_n f(y_n) \end{aligned} \tag{1.4}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\mu_n\}$ are sequences in [0,1].

Recently, Cholamjiak and Pholasa [2] introduced the CP-iteration as follows: $x_1 \in C$ and

$$\begin{aligned} z_n &= (1-\mu_n)x_n + \mu_n f(x_n), \\ y_n &= (1-\tau_n-\beta_n)x_n + \tau_n z_n + \beta_n f(z_n), \\ x_{n+1} &= (1-\gamma_n-\alpha_n)z_n + \gamma_n y_n + \alpha_n f(y_n) \end{aligned} \tag{1.5}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\mu_n\}$, $\{\tau_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$. They proved a convergence theorem of the iteration (1.5) for continuous function on an arbitrary interval in the real line.

In this paper, motivated by the previous ones, we suggest a new modified SP-iteration algorithm for solving a fixed point problem of continuous function on an arbitrary interval in the real line. Such an iterative algorithm is called MSP-iteration. We also present numerical examples and the comparison to iteration of Mann, Ishikawa, Noor, CP and SP iterations.

2. Main result

In this section, we give some crucial lemmas.

Lemma 2.1. *Let C be a closed interval on the real line (can be unbounded) and let $f : C \to C$ be a continuous function. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0,1]$. Choose an arbitrary initial guess $x_1 \in C$. Assume $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ have been constructed. Compute x_{n+1} via the formula*

$$\begin{aligned} z_n &= (1-\mu_n)x_n + \mu_n f(x_n), \\ y_n &= (1-\tau_n-\beta_n)z_n + \tau_n f(x_n) + \beta_n f(z_n), \\ x_{n+1} &= (1-\gamma_n-\alpha_n)y_n + \gamma_n f(z_n) + \alpha_n f(y_n),\ n \geq 1, \end{aligned} \tag{2.1}$$

where $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \beta_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} \tau_n < \infty$. *If $x_n \to a$, then a is a fixed point of f.*

Proof. Let $x_n \to a$, and suppose $a \neq f(a)$. Then $\{x_n\}$ is bounded. So, $\{f(x_n)\}$ is bounded by the continuity of f. So are $\{y_n\}, \{z_n\}, \{f(y_n)\}$ and $\{f(z_n)\}$. Moreover, $z_n \to a$ since $x_n \to a$ and $\mu_n \to 0$. We also have $y_n \to a$ since $z_n \to a$, $\tau_n \to 0$ and $\beta_n \to 0$. From (2.1), we obtain

$$\begin{aligned} x_{n+1} &= (1-\gamma_n)y_n - \alpha_n y_n + \gamma_n f(z_n) + \alpha_n f(y_n) \\ &= (1-\gamma_n)y_n + \gamma_n f(z_n) + \alpha_n(f(y_n) - y_n) \\ &= (1-\gamma_n)y_n(1-\tau_n-\beta_n)z_n + (1-\gamma_n)\tau_n f(x_n) + (1-\gamma_n)\beta_n f(z_n) \\ &\quad + \alpha_n(f(y_n) - y_n) \\ &= (1-\gamma_n)z_n - (1-\gamma_n)\tau_n z_n - (1-\gamma_n)\beta_n z_n + (1-\gamma_n)\tau_n f(x_n) \\ &\quad + (1-\gamma_n)\beta_n f(z_n) + \gamma_n f(z_n) + \alpha_n(f(y_n) - y_n) \\ &= z_n - \gamma_n z_n - (1-\gamma_n)\tau_n z_n - (1-\gamma_n)\beta_n z_n + (1-\gamma_n)\tau_n f(x_n) \\ &\quad + (1-\gamma_n)\beta_n f(z_n) + \gamma_n f(z_n) + \alpha_n(f(y_n) - y_n) \end{aligned}$$

$$
\begin{aligned}
&= z_n - (1-\gamma_n)\tau_n z_n + (1-\gamma_n)\tau_n f(x_n) + \gamma_n(f(z_n) - z_n) \\
&\quad + (1-\gamma_n)\beta_n(f(z_n) - z_n) + \alpha_n(f(y_n) - y_n) \\
&= (1-(1-\gamma_n)\tau_n)(1-\mu_n)x_n + (1-(1-\gamma_n)\tau_n)\mu_n f(x_n) \\
&\quad + (1-\gamma_n)\tau_n f(x_n) + \gamma_n(f(z_n) - z_n) + (1-\gamma_n)\beta_n(f(z_n) - z_n) \\
&\quad + \alpha_n(f(y_n) - y_n) \\
&= (1-(1-\gamma_n)\tau_n)x_n - (1-(1-\gamma_n)\tau_n)\mu_n x_n + (1-(1-\gamma_n)\tau_n)\mu_n f(x_n) \\
&\quad + (1-\gamma_n)\tau_n f(x_n) + \gamma_n(f(z_n) - z_n) + (1-\gamma_n)\beta_n(f(z_n) - z_n) \\
&\quad + \alpha_n(f(y_n) - y_n) \\
&= x_n - (1-\gamma_n)\tau_n x_n - (1-(1-\gamma_n)\tau_n)\mu_n x_n + (1-(1-\gamma_n)\tau_n)\mu_n f(x_n) \\
&\quad + (1-\gamma_n)\tau_n f(x_n) + \gamma_n(f(z_n) - z_n) + (1-\gamma_n)\beta_n(f(z_n) - z_n) \\
&\quad + \alpha_n(f(y_n) - y_n) \\
&= x_n + (1-\gamma_n)\tau_n(f(x_n) - x_n) + (1-(1-\gamma_n)\tau_n)\mu_n(f(x_n) - x_n) \\
&\quad + \gamma_n(f(z_n) - z_n) + (1-\gamma_n)\beta_n(f(z_n) - z_n) + \alpha_n(f(y_n) - y_n) \\
&= x_n + [(1-\gamma_n)\tau_n + (1-(1-\gamma_n)\tau_n)\mu_n](f(x_n) - x_n) \\
&\quad + [\gamma_n + (1-\gamma_n)\beta_n](f(z_n) - z_n) + \alpha_n(f(y_n) - y_n) \\
&= x_n + (\tau_n - \gamma_n\tau_n + \mu_n - \mu_n\tau_n + \gamma_n\tau_n\mu_n)(f(x_n) - x_n) + \alpha_n(f(y_n) - y_n) \\
&\quad + (\gamma_n + \beta_n - \gamma_n\beta_n)(f(z_n) - z_n).
\end{aligned} \tag{2.2}
$$

Let $p_k = f(x_k) - x_k, q_k = f(y_k) - y_k$ and $r_k = f(z_k) - z_k$. Then, we see that

$$
\begin{aligned}
\lim_{k\to\infty} p_k &= \lim_{n\to\infty}(f(x_k) - x_k) = f(a) - a \neq 0, \\
\lim_{k\to\infty} q_k &= \lim_{n\to\infty}(f(y_k) - y_k) = f(a) - a \neq 0, \\
\lim_{k\to\infty} r_k &= \lim_{n\to\infty}(f(z_k) - z_k) = f(a) - a \neq 0.
\end{aligned} \tag{2.3}
$$

So, from (2.2) we obtain

$$
\begin{aligned}
x_n &= x_1 + \sum_{k=1}^{n}(\tau_k - \gamma_k\tau_k + \mu_k - \mu_k\tau_k + \gamma_k\tau_k\mu_k)(f(x_k) - x_k) \\
&\quad + \sum_{k=1}^{n}\alpha_k(f(y_k) - y_k) + \sum_{k=1}^{n}(\gamma_k + \beta_k - \gamma_k\beta_k)(f(z_k) - z_k) \\
&= x_1 + \sum_{k=1}^{n}(\tau_k - \gamma_k\tau_k + \mu_k - \mu_k\tau_k + \gamma_k\tau_k\mu_k)p_k + \sum_{k=1}^{n}\alpha_k q_k \\
&\quad + \sum_{k=1}^{n}(\gamma_k + \beta_k - \gamma_k\beta_k)r_k.
\end{aligned}
$$

Since $\lim_{k\to\infty} p_k \neq 0$ and $\sum_{k=1}^{\infty}\mu_k < \infty$, $\sum_{k=1}^{\infty}\gamma_k < \infty$ and $\sum_{k=1}^{\infty}\tau_k < \infty$, it is easy to see that

$$\sum_{k=1}^{\infty}(\tau_k - \gamma_k\tau_k + \mu_k - \mu_k\tau_k + \gamma_k\tau_k\mu_k)p_k < \infty.$$

Similarly, we have $\sum_{k=1}^{\infty}(\gamma_k + \beta_k - \gamma_k\beta_k)r_k < \infty$ since $\lim_{k\to\infty} r_k \neq 0$, $\sum_{k=1}^{\infty}\beta_k < \infty$ and $\sum_{k=1}^{\infty}\gamma_k < \infty$. This shows that $\{x_n\}$ is a divergent sequence since $\lim_{k\to\infty} q_k \neq 0$ and $\sum_{k=1}^{\infty}\alpha_k = \infty$. This is a contradiction to the convergence of $\{x_n\}$. Hence $f(a) = a$ and a is fixed point of f. □

Lemma 2.2. *Let C be a closed interval on the real line (can be unbounded) and let $f : C \to C$ be a continuous function. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0,1]$. Let $\{x_n\}$ be a sequence generated iteratively by $x_1 \in C$ and*

$$\begin{aligned} z_n &= (1-\mu_n)x_n + \mu_n f(x_n),\\ y_n &= (1-\tau_n-\beta_n)z_n + \tau_n f(x_n) + \beta_n f(z_n),\\ x_{n+1} &= (1-\gamma_n-\alpha_n)y_n + \gamma_n f(z_n) + \alpha_n f(y_n),\ n \geq 1, \end{aligned}$$

where $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\beta_n < \infty$, $\sum_{n=1}^{\infty}\mu_n < \infty$, $\sum_{n=1}^{\infty}\gamma_n < \infty$ and $\sum_{n=1}^{\infty}\tau_n < \infty$. If $\{x_n\}$ is bounded, then $\{x_n\}$ is convergent.

Proof. Suppose $\{x_n\}$ is not convergent. Let $a = \liminf_n x_n$ and $b = \limsup_n x_n$. Then $a < b$. We first show that if $a < m < b$, then $f(m) = m$. Suppose $f(m) \neq m$. Without loss of generality, we suppose $f(m) - m > 0$. Since f is continuous, there exists δ with $0 < \delta < b - a$ such that for $|x - m| \leq \delta$,

$$f(x) - x > 0.$$

Since $\{x_n\}$ in bounded and f is continuous, $\{f(x_n)\}$ is bounded. Hence $\{z_n\}, \{y_n\}, \{f(z_n)\}$ and $\{f(y_n)\}$ are all bounded. Using

$$\begin{aligned} x_{n+1} - x_n &= (1-\gamma_n-\alpha_n)(y_n - x_n) + \gamma_n(f(z_n) - x_n) + \alpha_n(f(y_n) - x_n),\\ y_n - x_n &= \tau_n(f(x_n) - z_n) + \beta_n(f(z_n) - z_n) + (z_n - x_n),\\ z_n - x_n &= \mu_n(f(x_n) - x_n), \end{aligned}$$

we can easily show that $|z_n - x_n| \to 0$, $|y_n - x_n| \to 0$ and $|x_{n+1} - x_n| \to 0$. Thus, there exists a positive integer N such that for all $n > N$,

$$|x_{n+1} - x_n| < \frac{\delta}{2},\ |y_n - x_n| < \frac{\delta}{2},\ |z_n - x_n| < \frac{\delta}{2}. \tag{2.4}$$

Since $b = \limsup_n x_n > m$, there exists $k_1 > N$ such that $x_{n_{k_1}} > m$. Let $n_{k_1} = k$. Then $x_k > m$. For x_k, there exist the following two cases

(*i*) $x_k > m + \frac{\delta}{2}$, then $x_{k+1} > x_k - \frac{\delta}{2} \geq m$ using (2.4). So, we have $x_{k+1} > m$.

(*ii*) $m < x_k < m + \frac{\delta}{2}$, then $m - \frac{\delta}{2} < y_k < m + \delta$ and $m - \frac{\delta}{2} < z_k < m + \delta$ by (2.4). So, we obtain $|x_k - m| < \frac{\delta}{2} < \delta, |y_k - m| < \delta$ and $|z_k - m| < \delta$.

Hence

$$f(x_k) - x_k > 0, f(y_k) - y_k > 0 \textit{ and } f(z_k) - z_k > 0. \tag{2.5}$$

From (2.2) and (2.5), we have

$$\begin{aligned} x_{k+1} &= x_k + (\tau_k - \gamma_k \tau_k + \mu_k - \mu_k \tau_k + \gamma_k \tau_k \mu_k)(f(x_k) - x_k) \\ &\quad + \alpha_k (f(y_k) - y_k) + (\gamma_k + \beta_k - \gamma_k \beta_k)(f(z_k) - z_k) \\ &> x_k. \end{aligned}$$

Thus $x_{k+1} > x_k > m$. From (i) and (ii), we have $x_{k+1} > m$. Similarly, we get that $x_{k+2} > m$, $x_{k+3} > m, \ldots$. Thus we have $x_n > m$ for all $n > k = n_{k_1}$. So $a = \lim_{k\to\infty} x_{n_k} \geq m$, which is a contradiction with $a < m$. Thus $f(m) = m$.

We next consider the following two cases.

(1) There exists x_M such that $a < x_M > b$. Then $f(x_M) = x_M$. It follows that

$$z_M = (1 - \mu_M)x_M + \mu_M f(x_M) = x_M$$

and

$$\begin{aligned} y_M &= (1 - \tau_M - \beta_M)z_M + \tau_M f(x_M) + \beta_M f(z_M) \\ &= (1 - \tau_M - \beta_M)x_M + \tau_M f(x_M) + \beta_M f(x_M) \\ &= x_M. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} x_{M+1} &= (1 - \gamma_M - \alpha_M)y_M + \gamma_M f(z_M) + \alpha_M f(y_M) \\ &= (1 - \gamma_M - \alpha_M)x_M + \gamma_M f(x_M) + \alpha_M f(x_M) \\ &= x_M. \end{aligned}$$

Similarly, we obtain $x_M = x_{M+1} = x_{M+2} = \ldots$. So, we conclude that $x_n \to x_M$. Since there exists $x_{n_k} \to a$, $x_M = a$. This shows that $x_n \to a$, which is a contradiction.

(2) For all $n, x_n \leq a$ or $x_n \geq b$. Since $b - a > 0$ and $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$, there exists $\bar{N}$ such that $|x_{n+1} - x_n| < \frac{(b-a)}{2}$ for $n > \bar{N}$. So, it is seen that $x_n \leq a$ for $n > \bar{N}$, or it is always that $x_n \geq b$ for $n > \bar{N}$, then $b = \lim_{l\to\infty} x_{n_l} \leq a$, which is a contradiction with $a < b$. If $x_n \geq b$ for $n > \bar{N}$. It $x_n \leq a$ for $n > \bar{N}$ then $a = \lim_{k\to\infty} x_{n_k} \geq b$, which is a contradiction with $a < b$. Thus we conclude that $x_n \to a$. This completes the proof. □

Now, we are ready to prove the main results of this paper.

Theorem 2.3. *Let C be a closed interval on the real line (can be unbounded) and let $f : C \to C$ be a continuous function. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0, 1]$. Let $\{x_n\}$ be a sequence generated iteratively by $x_1 \in C$ and*

$$\begin{aligned} z_n &= (1 - \mu_n)x_n + \mu_n f(x_n) \\ y_n &= (1 - \tau_n - \beta_n)z_n + \tau_n f(x_n) + \beta_n f(z_n) \\ x_{n+1} &= (1 - \gamma_n - \alpha_n)y_n + \gamma_n f(z_n) + \alpha_n f(y_n),\ n \geq 1, \end{aligned}$$

where $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \beta_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \tau_n < \infty$. If $\{x_n\}$ is bounded, then $\{x_n\}$ is convergent to a fixed point of f.

Proof. Let $\{x_n\}$ be a bounded sequence. Then, by Lemma 2.2, $\{x_n\}$ is convergent. Hence, by Lemma 2.1, it converges to a fixed point of f. □

As a direct consequence of Theorem 2.3, we obtain the following result.

Theorem 2.4. *Let C be a closed interval on the real line (can be unbounded) and let $f : C \to C$ be a continuous function. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0, 1]$. Let $\{x_n\}$ be a sequence generated iteratively by $x_1 \in C$ and*

$$\begin{aligned} z_n &= (1 - \mu_n)x_n + \mu_n f(x_n) \\ y_n &= (1 - \tau_n - \beta_n)z_n + \tau_n f(x_n) + \beta_n f(z_n) \\ x_{n+1} &= (1 - \gamma_n - \alpha_n)y_n + \gamma_n f(z_n) + \alpha_n f(y_n),\ n \geq 1, \end{aligned}$$

where $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \beta_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \tau_n < \infty$. Then $\{x_n\}$ converges to a fixed point of f if and only if $\{x_n\}$ is bounded.

Corollary 2.5. *Let $f : [a, b] \to [a, b]$ be a continuous function. Let $\{\alpha_n\}, \{\beta_n\}$, $\{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0, 1]$. Let $\{x_n\}$ be a sequence generated iteratively by $x_1 \in [a, b]$ and*

$$\begin{aligned} z_n &= (1-\mu_n)x_n + \mu_n f(x_n) \\ y_n &= (1-\tau_n-\beta_n)z_n + \tau_n f(x_n) + \beta_n f(z_n) \\ x_{n+1} &= (1-\gamma_n-\alpha_n)y_n + \gamma_n f(z_n) + \alpha_n f(y_n), \ n \geq 1, \end{aligned}$$

where $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \beta_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \tau_n < \infty$. Then $\{x_n\}$ is convergent to a fixed point of f.

If we take $\alpha_n = \beta_n = 0$, then we obtain the following result.

Corollary 2.6. *Let C be a closed interval on the real line (can be unbounded) and let $f : C \to C$ be continuous function. Let $\{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0, 1]$. Let $\{x_n\}$ be a sequence generated iteratively by $x_1 \in C$ and*

$$\begin{aligned} z_n &= (1-\mu_n)x_n + \mu_n f(x_n), \\ y_n &= (1-\tau_n)z_n + \tau_n f(x_n), \\ x_{n+1} &= (1-\gamma_n)y_n + \gamma_n f(z_n), \ n \geq 1, \end{aligned}$$

where $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \tau_n < \infty$.
Then $\{x_n\}$ is convergent to a fixed point of f if and only if $\{x_n\}$ is bounded.

If we take $\tau_n = \gamma_n = 0$, then we obtain the following result.

Corollary 2.7. *Let C be a closed interval such that on the real line (can be unbounded) and let $f : C \to C$ be continuous function. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\mu_n\}$. Let $\{x_n\}$ be a sequence generated iteratively by $x_1 \in C$ and*

$$\begin{aligned} z_n &= (1-\mu_n)x_n + \mu_n f(x_n), \\ y_n &= (1-\beta_n)z_n + \beta_n f(z_n), \\ x_{n+1} &= (1-\alpha_n)y_n + \alpha_n f(y_n), \ n \geq 1, \end{aligned}$$

where $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \beta_n < \infty$, and $\sum_{n=1}^{\infty} \mu_n < \infty$. Then $\{x_n\}$ is convergent to a fixed point of f if and only if $\{x_n\}$ is bounded.

Remark 2.8. Corollary 2.7 extends the main result obtained in [8] from the modified Ishikawa iteration to the modified Noor iteration.

3. Numerical example

In this section, we give numerical examples to demonstrate the convergence of the algorithm defined in this paper. For convenience, we call the iteration (2.1) the MSP-iteration.

Example 3.1. Let $f : [1, \infty) \to [1, \infty)$ be defined by $f(x) = \sqrt{0.9 \ln x + 1}$. Then f is a continuous function. Use the initial point $x_1 = 3$ and the control conditions $\alpha_n = \frac{1}{(n+1)^{0.5}}, \beta_n = \frac{1}{(n+1)^{2.5}}, \mu_n = \frac{1}{(n+1)^{1.5}}, \gamma_n = \frac{1}{(n+1)^{1.9}}$ and $\tau_n = \frac{1}{(n+1)^{1.8}}$. The stopping criterion is $|x_{n+1} - x_n| < 10^{-7}$.

n	Mann	Ishikawa	Noor	CP	SP	MSP iteration	
	x_n	x_n	x_n	x_n	x_n	x_n	$\lvert f(x_n) - x_n\rvert$
1	3	3	3	3	3	3	1.5897
5	1.2460	1.2375	1.2371	1.1692	1.1288	1.0510	0.0288
10	1.0741	1.0715	1.0714	1.0469	1.0348	1.0124	0.0068
15	1.0313	1.0302	1.0302	1.0190	1.0140	1.0047	0.0026
20	1.0154	1.0149	1.0149	1.0091	1.0066	1.0022	0.0012
25	1.0083	1.0080	1.0080	1.0048	1.0035	1.0011	0.0006
30	1.0048	1.0046	1.0046	1.0027	1.0020	1.0006	0.0003
35	1.0029	1.0028	1.0028	1.0016	1.0011	1.0003	0.0002
40	1.0018	1.0017	1.0017	1.0010	1.0007	1.0002	0.0001
Iterations No.	152	151	151	139	132	111	

Table 1. Comparison of the convergence rate of Mann, Ishikawa, Noor, CP, SP and MSP iterations for the function given in Example 3.1.

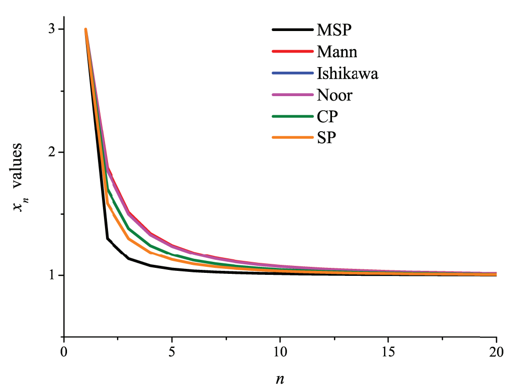

Figure 1. Convergence behavior of Mann, Ishikawa, Noor, CP, SP and MSP for the function given in Example 3.1.

Example 3.2. Let $f : [1, \infty) \to [1, \infty)$ be defined by $f(x) = 0.2\sqrt{x+1} + \sqrt{x}$. Then f is a continuous function. Use the initial point $x_1 = 5$ and the control conditions $\alpha_n = \frac{1}{(n+1)^{0.5}}, \beta_n = \frac{1}{(n+1)^2}, \mu_n = \frac{1}{(n+1)^{1.1}}, \gamma_n = \frac{1}{(n+1)^{1.7}}$ and $\tau_n = \frac{1}{(n+1)^{1.5}}$. The stopping criterion is $|x_{n+1} - x_n| < 10^{-7}$.

n	Mann	Ishikawa	Noor	CP	SP	MSP iteration	
	x_n	x_n	x_n	x_n	x_n	x_n	$\|f(x_n) - x_n\|$
1	5	5	5	5	5	5	2.2740
5	2.1774	2.1257	2.1197	1.9052	1.8012	1.6668	0.0491
10	1.7709	1.7533	1.7514	1.6623	1.6300	1.5940	0.0093
15	1.6613	1.6536	1.6528	1.6093	1.5962	1.5824	0.0031
20	1.6192	1.6153	1.6149	1.5916	1.5854	1.5791	0.0013
25	1.6000	1.5979	1.5976	1.5842	1.5810	1.5778	0.0006
30	1.5903	1.5890	1.5889	1.5808	1.5790	1.5773	0.0003
35	1.5850	1.5842	1.5841	1.5791	1.5780	1.5770	0.0001
40	1.5819	1.5814	1.5814	1.5781	1.5775	1.5769	0.0001
Iterations No.	177	175	174	142	131	106	

TABLE 2. Comparison of the convergence rate of Mann, Ishikawa, Noor, CP, SP and MSP iterations for the function given in Example 3.2.

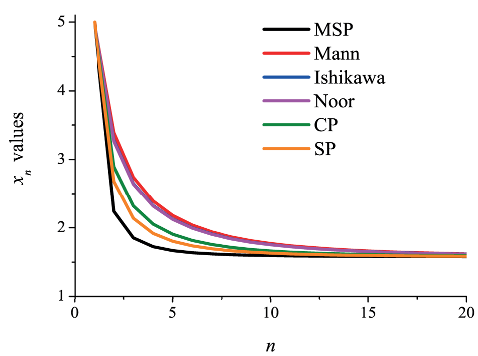

FIGURE 2. Convergence behavior of Mann, Ishikawa, Noor, CP, SP and MSP for the function given in Example 3.2.

Remark 3.3. From Table 1, Figure 1,Table 2 and Figure 2 ,we observe that the sequence generated by the MSP-iteration converges to a fixed point faster than that of Mann, Ishikawa, Noor, CP and SP iterations.

Acknowledgments: This study was supported by Unit of Excellence, University of Phayao, Phayao, Thailand (Grant No.UoE62001).

References

[1] D. Borwein and J. Borwein, *Fixed point iterations for real function,* J. Math. Anal. Appl., **157** (1991), 112–126.

[2] P. Cholamjiak and N. Pholasa , *Approximating fixed points for continuous functions on an arbitrary interval,* J.Ineq. and Appl., **2013**(214) (2013).

[3] S. Ishikawa, *Fixed points by a new iteration method,* Proc. Am. Math. Soc., **44** (1974), 147–150.

[4] W.R. Mann, *Mean value method in iteration,* Proc. Amer. Math. Soc., **4**(3) (1953), 506–510.

[5] M.A. Noor, *New approximation schemes for general variational inequalities,* J. Math. Anal. Appl., **251** (2000), 217–229.

[6] W. Phuengrattana and S. Suantai, *On the rate of convergence of Mann Ishikawa. Noor and SP iterations for continuous function on an arbitrary interval,* J. Comput. Appl. Math., **235** (2011), 3006–3014.

[7] Y. Qing, S.Y. Cho and X. Qin, *convergence of Ishikawa iteration with error terms on an arbitrary interval,* Commun. Korean Math. Soc., **26** (2011), 229-235.

[8] Y. Qing and L. Qihou, *The necessary and sufficient condition for the convergence of Ishikawa iteration on an arbitrary interval,* J. Math. Anal. Appl., **323** (2006), 1383–1386.