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WEAK AND STRONG CONVERGENCE THEOREMS FOR THREE STEP ITERATION OF ASYMPTOTICALLY FIRMLY TYPE NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we establish weak and strong convergence theorems using three step iteration schemes which is defined by Xu and Noor, for an asymptotically firmly type nonexpansive mappings in Banach spaces.

1. Introduction

Let X be a normed linear space and E be a closed subset of X. Let $T : E \to E$ be a mapping which has at least one fixed point. In the second half of the twentieth century iterative procedures play a major role in the approximation of fixed points. Banach [1] in 1922 used one step iteration scheme $\{x_n\}$ (that is, Picard iteration)

$$x_0 \in E, x_{n+1} = Tx_n, n \geq 0, \tag{1.1}$$

to approximate fixed point for the contraction mappings in a complete metric space.

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Later, Mann [6] defined new one step iteration scheme $\{x_n\}$ in matrix form in the year 1953. In 1976, Rhoades [9] generalized the Mann iteration scheme as

$$x_0 \in E, x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, n \geq 0, \tag{1.2}$$

where $\{\alpha_n\} \subset [0, 1]$ and he proved that this iteration scheme $\{x_n\}$ converges to a fixed point of a continuous non-decreasing mappings. The iteration (1.2) is called a Krasnoselskii-Mann iteration.

In 1974, Ishikawa [5] introduced two step iteration scheme as

$$\begin{cases} x_0 \in E; \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n; \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n, \quad n \geq 0, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and he approximate the fixed point for Lipschitizian pseudocontractive mappings.

In 1991, Schu [10] defined the modification of Mann iteration as

$$x_0 \in E, x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n(x_n), \quad n \geq 0, \tag{1.4}$$

and the modification of Ishikawa iteration as

$$\begin{cases} x_0 \in E; \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n; \\ y_n = (1 - \beta_n)x_n + \beta_n T^n x_n, \quad n \geq 0. \end{cases} \tag{1.5}$$

He proved the strong convergence of fixed points for asymptotically nonexpansive mappings by using (1.4). He also proved strong convergence of fixed points for asymptotically pseudocontractive mappings by using (1.5).

Glowinski and Tallec [3] used three step iterative schemes to find the approximate solutions of the elasto viscoplasticity problem, eigen value problem and liquid crystal theory. Also they showed that this iterative schemes perform better numerically. Haubruge et al.[4] studied the three step iteration schemes of Glowinski and Tallec and they introduced new splitting type algorithm for solving variational inequalities, separable convex programming, and minimization of a sum of convex functions.

This paper is organized as follows. In Section II we recall three step iteration schemes and the definition of asymptotically firmly type nonexpansive mappings with an example. We extend the Lemma 1.2 in [10] to the three step iteration schemes, and Section III contains weak and strong convergence theorem for asymptotically firmly type nonexpansive mappings in Banach spaces using three step iteration schemes.

2. Three step iteration schemes

In 2002, Xu and Noor [12] introduced the following three step iterative schemes. For an initial guess $x_0 \in E$, compute the sequences

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n; \\ y_n = (1-\beta_n)x_n + \beta_n T^n z_n; \\ z_n = (1-\gamma_n)x_n + \gamma_n T^n x_n, \quad n \geq 0, \end{cases} \tag{2.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences of real numbers in [0,1], and they proved strong convergence of fixed points for asymptotically nonexpansive mappings. The iteration (2.1) is called Noor's iteration schemes.

Here we recall the definition of asymptotically firmly type nonexpansive mapping.

Definition 2.1. ([7]) Let E be a nonempty subset of a normed linear space X. A mapping $T : E \to E$ is said to be asymptotically firmly type nonexpansive mapping if there exists a sequence $\{k_n\} \subset [1,\infty), k_n \to 1$ as $n \to \infty$ and $k \in (0,\infty)$ such that

$$\|T^n x - T^n y\|^2 \leq \phi_n(x,y),$$

where $\phi_n(x,y) := k_n\|x-y\|^2 - k\|(x-y)-(T^n x - T^n y)\|^2$, for all $x, y \in E$, $n \geq 1$.

Example 2.2. Let B denote the unit ball in the Hilbert space l^2 and let $T : B \to B$ be defined as: $T(x,y) = (xy, \frac{y}{2})$. Let $a = (x_1, y_1)$ and $b = (x_2, y_2)$ in B. Then

$$\begin{aligned}
\| T^n a - T^n b \|^2 &= \left\| \left(\frac{x_1 {y_1}^n}{\prod_{i=0}^{n-1} 2^i} - \frac{x_2 {y_2}^n}{\prod_{i=0}^{n-1} 2^i}, \frac{y_1}{2^n} - \frac{y_2}{2^n} \right) \right\|^2. \\
&= \frac{(x_1 {y_1}^n - x_2 {y_2}^n)^2}{\prod_{i=0}^{n-1} 4^i} + \frac{(y_1-y_2)^2}{4^n}. \\
&= \left(\frac{x_1 {y_1}^n - x_2 {y_2}^n}{x_1 - x_2} \right)^2 \frac{(x_1-x_2)^2}{\prod_{i=0}^{n-1} 4^i} + \frac{(y_1-y_2)^2}{4^n}. \\
&= \frac{c_n(x_1-x_2)^2}{4^n} + \frac{(y_1-y_2)^2}{4^n}. \\
&\leq \frac{\max\{1, c_n\}}{4^n}\|a-b\|^2,
\end{aligned}$$

where

$$c_n = \frac{4^n}{\prod_{i=0}^{n-1} 4^i} \left(\frac{x_1 {y_1}^n - x_2 {y_2}^n}{x_1 - x_2} \right)^2, \ x_1 \neq x_2.$$

Since

$$\| (a-b) - (T^n a - T^n b) \|^2 \leq \| a - b \|^2 + \| T^n a - T^n b \|^2,$$

it implies that

$$\| (a-b) - (T^n a - T^n b) \|^2 - \| a - b \|^2 \leq \frac{max\{1, c_n\}}{4^n} \|a-b\|^2.$$

Now

$$\begin{aligned} \| T^n a - T^n b \|^2 &\leq \frac{2\max\{1, c_n\}}{4^n}\|a-b\|^2 - \frac{\max\{1, c_n\}}{4^n}\|a-b\|^2 \\ &\leq (1 + \frac{2\max\{1, c_n\}}{4^n})\|a-b\|^2 - \| (a-b) - (T^n a - T^n b) \|^2 \\ &\leq k_n \| a - b \|^2 - k \| (a-b) - (T^n a - T^n b) \|^2, \end{aligned}$$

where $k_n = 1 + \dfrac{2\max\{1, c_n\}}{4^n}$ and $k \in (0, 1]$.

For $x_1 = x_2$,

$$\begin{aligned} \| T^n a - T^n b \|^2 &= \left\| \left(\frac{x_1 {y_1}^n}{\prod_{i=0}^{n-1} 2^i} - \frac{x_1 {y_2}^n}{\prod_{i=0}^{n-1} 2^i} \frac{y_1}{2^n} - \frac{y_2}{2^n} \right) \right\|^2 \\ &= \frac{{x_1}^2 ({y_1}^n - {y_2}^n)^2}{\prod_{i=0}^{n-1} 4^i} + \frac{(y_1 - y_2)^2}{4^n} \\ &= \frac{{x_1}^2 \left((y_1 - y_2)({y_1}^{n-1} + {y_1}^{n-2} y_2 + ... + {y_2}^{n-1}) \right)^2}{\prod_{i=0}^{n-1} 4^i} + \frac{(y_1 - y_2)^2}{4^n} \\ &= \frac{d_n (y_1 - y_2)^2}{4^n} + \frac{(y_1 - y_2)^2}{4^n} \\ &\leq \frac{\max\{1, d_n\}}{4^n} \|a-b\|^2, \end{aligned}$$

where $d_n = \dfrac{4^n {x_1}^2}{\prod_{i=0}^{n-1} 4^i} \left({y_1}^{n-1} + {y_1}^{n-2} y_2 + ... + {y_2}^{n-1} \right)^2$. Clearly $d_n \to 0$ as $n \to \infty$, since $d_n \leq \dfrac{4^n n^2}{\prod_{i=0}^{n-1} 4^i}$.

Now

$$\begin{aligned}\| T^n a - T^n b \|^2 &\leq \frac{2\max\{1,d_n\}}{4^n}\|a-b\|^2 - \frac{\max\{1,d_n\}}{4^n}\|a-b\|^2 \\ &\leq (1+\frac{2\max\{1,d_n\}}{4^n})\|a-b\|^2 - \| (a-b)-(T^n a - T^n b) \|^2 \\ &\leq k_n \| a-b \|^2 - k \| (a-b)-(T^n a - T^n b) \|^2,\end{aligned}$$

where $k_n = 1 + \frac{2\max\{1,d_n\}}{4^n}$ and $k \in (0,1]$. Hence T is an asymptotically firmly type nonexpansive mapping.

Definition 2.3. ([8]) A Banach space X is said to satisfy Opial's condition if for any sequence $x_n \in X, x_n \rightharpoonup x$ implies

$$\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|,$$

for all $y \in X$, with $x \neq y$.

Definition 2.4. ([2]) A mapping T is called demiclosed at zero if $x_n \rightharpoonup x$ and $Tx_n \to 0$, then $Tx = 0$.

Lemma 2.5. ([11]) *Let X be a normed linear space. Then for all $x, y \in X$ and $t \in [0,1]$, then*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2.$$

Lemma 2.6. ([2]) *Suppose that $\{u_n\}$ and $\{v_n\}$ are two sequences of non-negative numbers such that $u_{n+1} \leq u_n + v_n$ for all $n \geq 1$. If $\sum_{n=1}^{\infty} v_n$ converges, then $\lim_{n\to\infty} u_n$ exists.*

Lemma 2.7. ([10]) *Let E be a nonempty convex subset of a normed linear space X and $T : E \to E$ be a uniformly L-Lipschitzian mapping. For $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $x_0 \in E$, define $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n(y_n)$, and $y_n = (1-\beta_n)x_n + \beta_n T^n(x_n), n \geq 0$. Then*

$$\| x_n - T(x_n) \| \leq c_n + c_{n-1}L(1+3L+2L^2),$$

where $c_n = \| x_n - T^n(x_n) \|$, for all $n \in \mathbb{N}$.

We extend the above Lemma 2.7 into three step iteration (Noor's Iteration) schemes.

Lemma 2.8. *Let E be a nonempty convex subset of a normed linear space X and $T : E \to E$ be a uniformly L-Lipschitzian mapping. For $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset$*

$[0,1]$ *and* $x_0 \in E$, *define* $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, y_n = (1-\beta_n)x_n + \beta_n T^n z_n$ *and* $z_n = (1-\gamma_n)x_n + \gamma_n T^n x_n, n \geq 0$. *Then*

$$\| x_n - T(x_n) \| \leq c_n + c_{n-1}L(2 + 2L + 2L^2 + L^3),$$

where $c_n = \| x_n - T^n(x_n) \|$, *for all* $n \in \mathbb{N}$.

Proof. Let $x_1 \in E$ and for $n \in N$, $d_n = \|x_n - T^n y_n\|, e_n = \|x_n - y_n\|, f_n = \|y_{n-1} - x_n\|, g_n = \|x_{n+1} - x_n\|, h_n = \|x_n - T^n z_n\|, l_n = \|x_n - z_n\|, m_n = \|x_{n-1} - T^{n-1}x_n\|$ and $k_n = \|x_n - T^{n-1}x_n\|$, respectively. Then we have

$$\begin{aligned}
d_n &= \|x_n - T^n y_n\| \leq \|x_n - T^n x_n\| + \|T^n x_n - T^n y_n\| \\
&\leq c_n + L\|x_n - y_n\| = c_n + Le_n \leq (1 + L + L^2)c_n, \\
e_n &= \|x_n - y_n\| = \|x_n - \beta_n T^n z_n - (1-\beta_n)x_n\| = \beta_n \|x_n - T^n z_n\| \\
&= \beta_n h_n \leq h_n = (1+L)c_n, \\
f_n &= \|y_{n-1} - x_n\| \leq \|y_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\| \leq e_{n-1} + g_{n-1} \\
&\leq (1+L)c_{n-1} + (1 + L + L^2)c_{n-1} = (2 + 2L + L^2)c_{n-1}, \\
g_n &= \|x_{n+1} - x_n\| = \|\alpha_n T^n y_n + (1-\alpha_n)x_n - x_n\| = \alpha_n \|T^n y_n - x_n\| \\
&= \alpha_n d_n \leq d_n \leq (1 + L + L^2)c_n, \\
h_n &= \|x_n - T^n z_n\| \leq \|x_n - T^n x_n\| + \|T^n x_n - T^n z_n\| \\
&\leq c_n + L\|x_n - z_n\| = c_n + Ll_n \leq (1+L)c_n, \\
l_n &= \|x_n - z_n\| = \|x_n - \gamma_n T^n x_n - (1-\gamma_n)x_n\| \\
&= \gamma_n c_n \leq c_n, \\
m_n &= \|x_{n-1} - T^{n-1}x_n\| \\
&\leq \|x_{n-1} - T^{n-1}x_{n-1}\| + \|T^{n-1}x_{n-1} - T^{n-1}x_n\| \\
&\leq c_{n-1} + Lg_{n-1} \leq (1 + L + L^2 + L^3)c_{n-1}, \\
k_n &= \|x_n - T^{n-1}x_n\| \\
&\leq \|x_n - x_{n-1}\| + \|x_{n-1} - T^{n-1}x_n\|, \\
&= g_{n-1} + m_n \\
&\leq (1 + L + L^2)c_{n-1} + (1 + L + L^2 + L^3)c_{n-1} \\
&= (2 + 2L + 2L^2 + L^3)c_{n-1}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\|x_n - T(x_n)\| &\leq \|x_n - T^n x_n\| + \|T^n x_n - Tx_n\| \\
&\leq c_n + L\|T^{n-1}x_n - x_n\| \\
&\leq c_n + L(2 + 2L + 2L^2 + L^3)c_{n-1}.
\end{aligned}$$

This completes the proof. □

3. Convergence results

Theorem 3.1. *Let X be a real reflexive space that satisfies Opial's condition. Let E be a non-empty closed convex subset of X and $T : E \to E$ be an asymptotically firmly type nonexpansive mapping with the sequence $\{k_n\} \subset [1, \infty), \Sigma(k_n - 1) < \infty$ and assume $Fix(T) := \{x : Tx = x\} \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0, 1]$ with*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$,
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
(iii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.

Then the sequence of iteration defined in (2.1) *converges weakly to some fixed point of T if (I-T) is demiclosed at zero.*

Proof. Let $p \in Fix(T)$. Then we have

$$\phi_n(x_n, p) = k_n\|x_n - p\|^2 - k\|x_n - T^n x_n\|^2,$$

$$\begin{aligned}
\phi_n(z_n, p) &= k_n\|z_n - p\|^2 - k\|z_n - T^n z_n\|^2 \\
&= k_n\|\gamma_n T^n x_n + (1-\gamma_n)x_n - p\|^2 - k\|z_n - T^n z_n\|^2 \\
&\leq k_n\{(1-\gamma_n)\|x_n - p\|^2 + \gamma_n\phi_n(x_n, p)\} - k\|z_n - T^n z_n\|^2
\end{aligned}$$

and

$$\begin{aligned}
\phi_n(y_n, p) &= k_n\|y_n - p\|^2 - k\|y_n - T^n y_n\|^2 \\
&= k_n\|\beta_n T^n z_n + (1-\beta_n)x_n - p\|^2 - k\|y_n - T^n y_n\|^2 \\
&\leq k_n\{(1-\beta_n)\|x_n - p\|^2 + \beta_n\phi_n(z_n, p)\} - k\|y_n - T^n y_n\|^2.
\end{aligned}$$

Now

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n T^n y_n + (1-\alpha_n)x_n - p\|^2 \\
&\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\phi_n(y_n, p) \\
&\leq (1-\alpha_n)\|x_n - p\|^2 \\
&\quad + \alpha_n[k_n\{(1-\beta_n)\|x_n - p\|^2 + \beta_n\phi_n(z_n, p)\} - k\|y_n - T^n y_n\|^2] \\
&\leq \{1 - \alpha_n + \alpha_n k_n(1-\beta_n)\}\|x_n - p\|^2 + \alpha_n k_n\beta_n\phi_n(z_n, p) \\
&\quad - k\alpha_n\|y_n - T^n y_n\|^2 \\
&\leq \{1 - \alpha_n + \alpha_n k_n(1-\beta_n)\}\|x_n - p\|^2 \\
&\quad + \alpha_n k_n\beta_n[k_n\{(1-\gamma_n)\|x_n - p\|^2 + \gamma_n\phi_n(x_n, p)\} \\
&\quad - k\|z_n - T^n z_n\|^2] - k\alpha_n\|y_n - T^n y_n\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \{1-\alpha_n+\alpha_n k_n(1-\beta_n)\}\|x_n-p\|^2+\alpha_n {k_n}^2\beta_n(1-\gamma_n)\|x_n-p\|^2\\
&\quad+\alpha_n {k_n}^2\beta_n\gamma_n\phi_n(x_n,p)-k\alpha_n k_n\beta_n\|z_n-T^n z_n\|^2\\
&\quad-k\alpha_n\|y_n-T^n y_n\|^2\\
&\leq \{1-\alpha_n+\alpha_n k_n(1-\beta_n)+\alpha_n {k_n}^2\beta_n(1-\gamma_n)\}\|x_n-p\|^2\\
&\quad+\alpha_n {k_n}^2\beta_n\gamma_n\{k_n\|x_n-p\|^2-k\|x_n-T^n x_n\|^2\} \qquad (3.1)\\
&\quad-k\alpha_n k_n\beta_n\|z_n-T^n z_n\|^2-k\alpha_n\|y_n-T^n y_n\|^2\\
&\leq \{1-\alpha_n+\alpha_n k_n(1-\beta_n)+\alpha_n {k_n}^2\beta_n(1-\gamma_n)\\
&\quad+\alpha_n {k_n}^3\beta_n\gamma_n\}\|x_n-p\|^2-k\alpha_n {k_n}^2\beta_n\gamma_n\|x_n-T^n x_n\|^2\\
&\quad-k\alpha_n k_n\beta_n\|z_n-T^n z_n\|^2-k\alpha_n\|y_n-T^n y_n\|^2.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\|x_{n+1}-p\|^2 \leq \{1+\alpha_n(k_n-1)(\beta_n\gamma_n {k_n}^2+\beta_n k_n+1)\}\|x_n-p\|^2&\\
-k\alpha_n {k_n}^2\beta_n\gamma_n\|x_n-T^n x_n\|^2& \qquad (3.2)\\
-k\alpha_n k_n\beta_n\|z_n-T^n z_n\|^2-k\alpha_n\|y_n-T^n y_n\|^2).&
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \{1+(k_n-1)({k_n}^2+k_n+1\}\|x_n-p\|^2\\
&\leq \{1+({k_n}^3-1)\}\,\|x_n-p\|^2\\
&\leq \{1+\lambda_n\}\,\|x_n-p\|^2 \quad (\lambda_n:={k_n}^3-1) \qquad (3.3)\\
&\leq \{1+\lambda_n\}\{1+\lambda_{n-1}\}\cdots\{1+\lambda_1\}\,\|x_1-p\|^2\\
&\leq e^{\sum_{i=1}^n\lambda_i}\,\|x_1-p\|^2.
\end{aligned}
$$

Therefore, we have

$$
\|x_{n+1}-p\| \leq e^{\sum_{i=1}^n({k_i}^3-1)}\,\|x_1-p\|.
$$

This implies that $\{\|x_n-p\|\}$ is non-increasing and so it is bounded. Also from (3.3), we have

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \|x_n-p\|^2+({k_n}^3-1)\,\|x_n-p\|^2\\
&\leq \|x_n-p\|^2+\mu_n,
\end{aligned}
$$

where $\mu_n=({k_n}^3-1)\,\|x_n-p\|^2$. Since $\sum {k_n}^3-1<\infty$, $\sum_{i=1}^n\mu_n<\infty$ and by Lemma 2.6, we can get that

$$
\lim_{n\to\infty}\|x_n-p\| \ \textit{exists.}
$$

From (3.2), we deduce the following:

$$
k\alpha_n {k_n}^2\beta_n\gamma_n\|x_n-T^n x_n\|^2 \leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+M(k_n-1), \qquad (3.4)
$$

$$k\alpha_n\|y_n - T^n y_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + M(k_n - 1) \tag{3.5}$$

and

$$k\alpha_n k_n \beta_n \|z_n - T^n z_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + M(k_n - 1). \tag{3.6}$$

Since $\liminf_{n\to\infty} \alpha_n > 0$, $\liminf_{n\to\infty} \beta_n > 0$ and $\liminf_{n\to\infty} \gamma_n > 0$, there exist some $a, b, c > 0$ and $n_0, n_1, n_2 \in N$ such that

$$\alpha_n > a \textit{ for all } n \ge n_0, \ \ \beta_n > b \textit{ for all } n \ge n_1 \ \ \textit{and} \ \ \gamma_n > c \textit{ for all } n \ge n_2.$$

Therefore, from (3.4)

$$k{k_n}^2 abc < k\alpha_n {k_n}^2 \beta_n \gamma_n \|x_n - T^n x_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + M(k_n - 1).$$

This implies that

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0.$$

Similarly from (3.5) and (3.6), we get

$$\lim_{n\to\infty} \|y_n - T^n y_n\| = 0 \ \textit{ and } \ \lim_{n\to\infty} \|z_n - T^n z_n\| = 0.$$

Since T is uniformly L-Lipschitzian, from Lemma 2.8, we have $\|x_n - T(x_n)\| \to 0$ as $n \to \infty$.

Since E is reflexive, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ weakly convergent to some point p. Then by the hypothesis that $I - T$ is demiclosed at zero, we have $(I - T)p = 0$. That is, $p = Tp$.

Let q be an another weak limit of $\{x_{n_k}\}$ and $p \ne q$. Then we can choose a subsequence $\{x_{n_j}\}$ that weakly converges to y. And also, we have $Tq = q$. Since $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in Fix(T)$, we have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - p\| &= \limsup_{k\to\infty} \|x_{n_k} - p\| \\
&< \limsup_{k\to\infty} \|x_{n_k} - q\| \\
&= \limsup_{k\to\infty} \|x_{n_j} - q\| \\
&< \limsup_{j\to\infty} \|x_{n_j} - p\| \\
&= \lim_{n\to\infty} \|x_n - p\|.
\end{aligned}$$

This is a contradiction, and hence $p = q$.

Further, $\|z_n - x_n\| = \gamma_n \|T^n x_n - x_n\| \to 0$ as $n \to \infty$, and $\|y_n - x_n\| = \beta_n \|T^n z_n - x_n\| \le c_n + L\|x_n - z_n\| \to 0$ as $n \to \infty$. That is, $\lim_{n\to\infty} y_n = p$ and $\lim_{n\to\infty} z_n = p$. This completes the proof. □

Lemma 3.2. *Let E be a non-empty closed convex subset of a Banach space X. Let $T : E \to E$ be continuous and the sequence $\{x_n\}$ be defined in E such that* $\lim_{n\to\infty} \| x_n - p \|$ *exists for each $p \in F(T)$. Suppose there exists a convergent* *subsequence* $\{x_{n_k}\}$ *such that* $\| x_{n_k} - T(x_{n_k}) \| \to 0$. *Then* $\{x_n\}$ *converges to a fixed point of* T.

Proof. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ that converges to some point $x \in E$. Since T is continuous and $\| x_{n_k} - T(x_{n_k}) \| \to 0$, we have $Tx = x$. Therefore $\{x_n\}$ converges to a fixed point x of T. □

Theorem 3.3. *Let* E *be a non-empty closed convex subset of a uniformly convex Banach space* X *and* $T : E \to E$ *be an asymptotically firmly type nonexpansive mapping with the sequence* $\{k_n\} \subset (1, \infty]$, $\sum(k_n - 1) < \infty$ *and* $Fix(T) \neq \emptyset$. *Let* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *be real sequences in* $[0,1]$ *with*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$,
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
(iii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.

Suppose T *is completely continuous. Then the sequence of iteration* $\{x_n\}$ *defined in* (2.1) *converges strongly to some fixed point of* T.

Proof. From the proof of Theorem 3.1, we have $\| x_n - T(x_n) \| \to 0$ as $n \to \infty$. Since T is completely continuous, we can find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\| x_{n_k} - T(x_{n_k}) \| \to 0$. Also from the proof of Theorem 3.1, we have $\lim_{n\to\infty} \| x_n - p \|$ exists for each $p \in Fix(T)$. Then by Lemma 3.2, we can get $\{x_n\}$ converges strongly to a fixed point p of T. □

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