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BEST PROXIMITY POINTS OF α-β-ψ-PROXIMAL CONTRACTIVE MAPPINGS IN COMPLETE METRIC SPACES ENDOWED WITH GRAPHS

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Abstract. In this work, we present the notion of an α-β-ψ-G-proximal contraction in metric spaces endowed with graph. We investigate the existence and uniqueness of best proximity points for this modified contractive mapping. The results obtained extended and generalized some fixed and best proximity points results in literature. Examples are given to validate the main results.

1. Introduction

The importance of fixed point theory emerges from the fact that it furnishes a unified approach and constitutes an important tool in solving equations which are not necessarily linear. A large number of problems can be formulated as nonlinear equations of the form $T(x) = x$, where T is a self-mapping in some framework. Nevertheless, an equation of the type $T(x) = x$ does not necessarily possess a solution if T happens to be a nonself-mapping. In this case, one seeks an appropriate solution that is optimal in the sense that

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$d(x, T(x))$ is minimum. That is, we resolve a problem of finding an element x such that x is in best proximity to $T(x)$ in some sense.

Best proximity point theorem analyzes the condition under which the optimization problem, namely, $\inf_{x\in A} d(x, Tx)$, has a solution. Te point x is called the best proximity point of $T : A \to B$, if $d(x, Tx) = d(A, B)$, where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$. Note that the best proximity point reduces to a fixed point if T is a self-mapping. A best proximity point problem is a problem of achieving the minimum distance between two sets through a function defined on one of the sets to the other. The very popular best approximation theorem is due to Fan [1]. If A is a nonempty compact subset of a Hausdorff locally convex topological vector space B and $T : A \to B$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(A, Tx)$. Fans results are not without shortcomings; the best approximation theorem only ensures the existence of approximate solutions, without necessarily yielding an optimal solution. But the best proximity point theorem provides sufficient conditions that ensure the existence of approximate solutions which are also optimal. Afterwards many authors such as Eldred and Veeramani [2] have derived extensions of Fans Theorem and the best approximation theorems in many directions. Significant best proximity point results are in [3, 4, 5, 6, 7, 8] and other references therein.

Our purpose here is to establish best proximity point theorems in complete metric spaces endowed with graph. We recall the following notation and definitions. Let (X, d) be a metric space and let A and B be nonempty subsets of X.

$$A_0 := \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$$
$$B_0 := \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.$$

In 2011, Basha [9] gave the following definition of a proximal contraction for non-self mappings in a metric space:

Definition 1.1. ([9]) Let $T : A \to B$ be a non-self mapping. Then T is called a *proximal contraction* if there exists $k \in [0, 1)$ and for every $u, v, x, y \in A$,

$$\left.\begin{array}{l} d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B) \end{array}\right\} \quad \Rightarrow \quad d(u, v) \leq kd(x, y).$$

Next, we recall some mappings and notions regarding a graph.

Let (X, d) be a metric space and $\Delta := \{(x, x) : x \in X\}$ be the diagonal of $X \times X$. Let G be a directed graph such that the set $V(G)$ of its vertices coincides with X and $\Delta \subset E(G)$, where $E(G)$ is the set of edges of the graph. Assume also that G has no parallel edges, and thus one can identify G with the pair $(V(G), E(G))$. We say a metric space (X, d) is endowed with a graph G, if G is a directed graph such that $V(G) = X$ and $\Delta \subseteq E(G)$.

We denote by Ψ a family of functions $\psi : [0, \infty) \to [0, \infty)$ such that for each $\psi \in \Psi$ and $t > 0$,

(i) ψ is non-decreasing,

(ii) $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$, where n is the n-th iterate of ψ.

Remark 1.2. If $\psi \in \Psi$, then $\psi^n(t) \to 0$ as $n \to \infty$, for all $t \geq 0$ and $\psi(t) < t$, for all $t > 0$.

Definition 1.3. ([10]) Let $(X, \preceq)$ be a partially ordered space with metric d. We say that $f : X \to X$ is an α-β-ψ-contractive mapping if there exist three functions $\alpha, \beta : X \times X \to [0, \infty), \psi \in \Psi$ such that

$$\alpha(x, y)d(f(x), f(y)) \leq \beta(x, y)\psi\big(d(x, y)\big),$$

for all $x, y \in X$ with $x \preceq y$.

Definition 1.4. ([10]) Let $f : X \to X$, $\alpha, \beta : X \times X \to [0, \infty)$ and $C_\alpha > 0$, $C_\beta \geq 0$. We say that f is an α-β-admissible mapping, if for all $x, y \in X$ with $x \preceq y$ hold

(i) $\alpha(x, y) \geq C_\alpha \Rightarrow \alpha(fx, fy) \geq C_\alpha$,

(ii) $\beta(x, y) \leq C_\beta \Rightarrow \beta(fx, fy) \leq C_\beta$,

(iii) $0 \leq \frac{C_\beta}{C_\alpha} \leq 1$.

In 2015, Asgari and Badehian [10], proved fixed point theorems for α-β-ψ-contractive mappings in partially ordered space with complete metric.

Theorem 1.5. ([10]) *Let $(X, \preceq)$ be a partially ordered space with complete metric d. Let $f : X \to X$ be a non-decreasing, α-β-ψ-contractive mapping satisfying the following conditions:*

(i) *f is continuous,*

(ii) *f is α-β-admissible,*

(iii) *there exists $x_0 \in X$ such that $x_0 \preceq fx_0$,*

(iv) *there exist $C_\alpha > 0, C_\beta \geq 0$ such that $\alpha(fx_0, x_0) \geq C_\alpha$, $\beta(fx_0, x_0) \leq C_\beta$.*

Then, f has a fixed point.

The next Section, we introduce a notion of α-β-G-proximal admissible mappings and α-β-ψ-G-proximal contractive mappings that we consider to prove our main results, we draw some corollaries and provide examples in support of our main results.

2. Preliminaries

Definition 2.1. Let A and B be nonempty subsets of a metric space (X,d) endowed with a graph G. A mapping $T : A \to B$ is said to be

(i) *proximally G-edge-preserving* if for each $x, y, u, v \in A$ with $(x, y) \in E(G)$:

$$\left.\begin{array}{l} d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{array}\right\} \quad \Rightarrow \quad (u,v) \in E(G).$$

(ii) *α-β-G-proximal admissible* if there exist functions $\alpha, \beta : A \times A \to [0,\infty)$, and $C_\alpha > 0, C_\beta \geq 0$ two constants such that for all $x, y, u, v \in A$ with $(x, y) \in E(G)$:

(a) $\alpha(x,y) \geq C_\alpha,\ d(u,Tx) = d(v,Ty) = d(A,B) \quad \Rightarrow \quad \alpha(u,v) \geq C_\alpha$;

(b) $\beta(x,y) \leq C_\beta,\ d(u,Tx) = d(v,Ty) = d(A,B) \quad \Rightarrow \quad \beta(u,v) \leq C_\beta$;

(c) $0 \leq \frac{C_\beta}{C_\alpha} \leq 1$.

If (a), (b) and (c) hold for $\alpha(x,y) = 1 = \beta(x,y)$, for all $x, y \in A$, then we say that T is *G-proximal admissible.*

(iii) *G-proximal contraction* if there exist a constant $k \in [0,1)$ such that for all $x, y, u, v \in A$ with $(x,y) \in E(G)$:

$$\left.\begin{array}{l} d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{array}\right\} \quad \Rightarrow \quad d(u,v) \leq kd(x,y);$$

(iv) *α-β-ψ-G-proximal contraction* if there exist functions $\alpha, \beta : A \times A \to [0,\infty), \psi \in \Psi$ such that for all $x, y, u, v \in A$ with $(x, y) \in E(G)$:

$$\left.\begin{array}{l} d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{array}\right\} \quad \Rightarrow \quad \alpha(x,y)d(u,v) \leq \beta(x,y)\psi\big(d(x,y)\big).$$

3. Main results

In this section, we will prove best proximity point theorems for an α-β-ψ-G-proximal contraction in a complete metric space endowed with a graph.

Theorem 3.1. *Let A and B be nonempty closed subsets of a complete metric space (X,d) endowed with a graph G with $A_0 \neq \emptyset$. Let $T : A \to B$ be a non-self mapping which satisfies the following properties:*

(i) *T is continuous;*

(ii) *T is proximally G-edge-preserving, α-β-G-proximal admissible and α-β-ψ-G-proximal contraction such that $T(A_0) \subseteq B_0$;*

(iii) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A,B) \quad \text{and} \quad (x_0, x_1) \in E(G).$$

(iv) *there exist $C_\alpha > 0, C_\beta \geq 0$ such that $\alpha(x_0,x_1) \geq C_\alpha, \beta(x_0,x_1) \leq C_\beta$.*

Then T has a best proximity point in A, that is, there exists an element $w \in A$ such that $d(w, Tw) = d(A, B)$. Further, the sequence $\{x_n\}$, defined by

$$d(x_n, Tx_{n-1}) = d(A, B), \quad \forall\, n \in \mathbb{N},$$

converges to the element w.

Proof. From the condition (iii), there exist $x_0, x_1 \in A_0$ such that

$$d(x_1, Tx_0) = d(A, B) \quad \text{and} \quad (x_0, x_1) \in E(G). \tag{3.1}$$

Since $T(A_0) \subseteq B_0$, we have $Tx_1 \in B_0$ and hence there exits $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A, B). \tag{3.2}$$

By the proximally G-edge preserving of T and using both (3.1) and (3.2), we get $(x_1, x_2) \in E(G)$. By continuing this process, we can form the sequence $\{x_n\}$ in A_0 such that

$$d(x_n, Tx_{n-1}) = d(A, B) \;\; \text{with} \;\; (x_{n-1}, x_n) \in E(G), \;\; \forall\, n \in \mathbb{N}. \tag{3.3}$$

Next, we will show that T has a best proximity point in A. Suppose that there exists $n_0 \in \mathbb{N}$, such that $x_{n_0} = x_{n_0+1}$. By using (3.3), we obtain that $d(x_{n_0}, Tx_{n_0}) = d(x_{n_0+1}, Tx_{n_0}) = d(A, B)$ and so x_{n_0} is a best proximity point of T. Now, we assume with out loss of generality that any two consecutive elements of $\{x_n\}$ are distinct.

As T is α-β-G-proximal admissible, condition (iv) and (3.3), the following holds:

$$\left.\begin{array}{l} \alpha(x_0, x_1) \geq C_\alpha, \;\; \beta(x_0, x_1) \leq C_\beta \\ (x_0, x_1) \in E(G) \\ d(x_1, Tx_0) = d(A, B) \\ d(x_2, Tx_1) = d(A, B) \end{array}\right\} \Rightarrow \;\; \alpha(x_1, x_2) \geq C_\alpha, \;\; \beta(x_1, x_2) \leq C_\beta. \tag{3.4}$$

Since T is an α-β-ψ-G-proximal contraction and by (3.4), we have

$$\begin{aligned} C_\alpha d(x_1, x_2) &\leq \alpha(x_0, x_1) d(x_1, x_2) \\ &\leq \beta(x_0, x_1)\psi\big(d(x_0, x_1)\big) \\ &\leq C_\beta \psi\big(d(x_0, x_1)\big), \end{aligned}$$

therefore

$$d(x_1, x_2) \;\leq\; \frac{C_\beta}{C_\alpha}\psi\big(d(x_0, x_1)\big) \;\leq\; \psi\big(d(x_0, x_1)\big). \tag{3.5}$$

Again, since T is an α-β-G-proximal admissible, (3.3) and (3.4), we have

$$\left.\begin{array}{l} \alpha(x_1,x_2)\geq C_\alpha,\ \beta(x_1,x_2)\leq C_\beta \\ (x_1,x_2)\in E(G) \\ d(x_2,Tx_1)=d(A,B) \\ d(x_3,Tx_2)=d(A,B) \end{array}\right\} \Rightarrow\ \alpha(x_2,x_3)\geq C_\alpha,\ \beta(x_2,x_3)\leq C_\beta. \tag{3.6}$$

By the fact that T is an α-β-ψ-G-proximal contraction, we have

$$\begin{aligned} C_\alpha d(x_2,x_3) &\leq \alpha(x_1,x_2)d(x_2,x_3) \\ &\leq \beta(x_1,x_2)\psi\big(d(x_1,x_2)\big) \\ &\leq C_\beta\psi\big(d(x_1,x_2)\big), \end{aligned}$$

from (3.5), we have

$$d(x_2,x_3)\leq \frac{C_\beta}{C_\alpha}\psi\big(d(x_1,x_2)\big)\leq \psi\big(d(x_1,x_2)\big)\leq \psi^2\big(d(x_0,x_1)\big).$$

On continuing this process, we obtain

$$\left.\begin{array}{l} \alpha(x_{n-1},x_n)\geq C_\alpha, \\ \beta(x_{n-1},x_n)\leq C_\beta \\ (x_{n-1},x_n)\in E(G) \\ d(x_n,Tx_{n-1})=d(A,B) \\ d(x_{n+1},Tx_n)=d(A,B) \end{array}\right\} \Rightarrow\ \alpha(x_n,x_{n+1})\geq C_\alpha,\ \beta(x_n,x_{n+1})\leq C_\beta,$$

for $n=1,2,3,\dots$ and

$$d(x_n,x_{n+1})\ \leq\ \psi^n\big(d(x_0,x_1)\big).$$

Since $\psi\in\Psi$, we have $\psi^n\big(d(x_0,x_1)\big)\to 0$ as $n\to 1$.

Now, we show that $\{x_n\}$ is a Cauchy sequence. We fix $\varepsilon>0$ and choose $n_0\in\mathbb{N}$ such that $\sum_{n=n_0}^{\infty}\psi^n\big(d(x_0,x_1)\big)<\varepsilon$. Let $m,n\in\mathbb{N}$ with $m>n>n_0$. Therefore by applying triangle inequality, we have

$$\begin{aligned} d(x_n,x_m) &\leq d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\cdots+d(x_{m-1},x_m) \\ &\leq \psi^n\big(d(x_0,x_1)\big)+\psi^{n+1}\big(d(x_0,x_1)\big)+\cdots+\psi^{m-1}\big(d(x_0,x_1)\big) \\ &= \sum_{n=n_0}^{m-1}\psi^n\big(d(x_0,x_1)\big) \\ &\leq \sum_{n=n_0}^{\infty}\psi^n\big(d(x_0,x_1)\big)<\varepsilon \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in A. Since A is a closed subset of a complete metric space and hence it is complete, so there exists $w\in A$ such that $x_n\to w$.

By the continuing of T, we have $Tx_n \to Tw$ as $n \to \infty$. As the metric function is continuous, we obtain

$$d(x_{n+1}, Tx_n) \to d(w, Tw) \quad \text{as } n \to \infty.$$

This implies that $w \in A$ is a best proximity point of T.

Indeed, the sequence $\{x_n\}$ defined by

$$d(x_{n+1}, Tx_n) = d(A, B), \quad n \in \mathbb{N},$$

converges to an element w. The proof is completed. □

If we drop the continuity assumption from Theorem 3.1, we obtain the following result.

Theorem 3.2. *Let A and B be nonempty closed subsets of a complete metric space (X, d) endowed with a graph G, with a nonempty and closed set A_0. Let $T : A \to B$ be a non-self mapping which satisfies the following properties:*

- (i) *T is proximally G-edge-preserving, α-β-G-proximal admissible and α-β-ψ-G-proximal contraction such that $T(A_0) \subseteq B_0$;*
- (ii) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A, B) \quad \textit{and} \quad (x_0, x_1) \in E(G).$$

- (iii) *there exist $C_\alpha > 0, C_\beta \geq 0$ such that $\alpha(x_0, x_1) \geq C_\alpha, \beta(x_0, x_1) \leq C_\beta$.*
- (iv) *If $\{x_n\}$ is a sequence in A such that $(x_n, x_{n+1}) \in E(G)$, $\alpha(x_n, x_{n+1}) \geq C_\alpha$, $\beta(x_n, x_{n+1}) \leq C_\beta$ for all n and $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$, $\alpha(x_{n_k}, x) \geq C_\alpha$, $\beta(x_{n_k}, x) \leq C_\beta$ for all k.*

Then there exists an element $z \in A$ such that $d(z, Tz) = d(A, B)$. Further, the sequence $\{x_n\}$, defined by

$$d(x_{n+1}, Tx_n) = d(A, B), \quad \forall\, n \in \mathbb{N},$$

converges to the element z.

Proof. Following the proof of Theorem 3.1, there exists a sequence $\{x_n\}$ in A_0 satisfying

$$\begin{gathered} d(x_n, Tx_{n-1}) = d(A, B) \text{ with } (x_{n-1}, x_n) \in E(G), \\ \alpha(x_{n-1}, x_n) \geq C_\alpha, \beta(x_{n-1}, x_n) \leq C_\beta, \end{gathered} \tag{3.7}$$

for all $n \in \mathbb{N}$, and $x_n \to w$. Since A_0 is closed, we get $w \in A_0$. By (i), we have $T(A_0) \subseteq B_0$, so $Tw \in B_0$. Then there exists $z \in A$ such that

$$d(z, Tw) = d(A, B). \tag{3.8}$$

By (iv), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, w) \in E(G)$, $\alpha(x_{n_k}, w) \geq C_\alpha, \beta(x_{n_k}, w) \leq C_\beta$, for all $k \in \mathbb{N}$. Indeed, by the fact that T is an α-β-ψ-G-proximal contraction, we get

$$\begin{aligned} C_\alpha d(x_{n_k+1}, z) &\leq \alpha(x_{n_k}, w) d(x_{n_k+1}, z) \\ &\leq \beta(x_{n_k}, w)\psi\big(d(x_{n_k}, w)\big) \\ &\leq C_\beta \psi\big(d(x_{n_k}, w)\big) \end{aligned}$$

and therefore

$$d(x_{n_k+1}, z) \ \leq \frac{C_\beta}{C_\alpha}\psi\big(d(x_{n_k}, w)\big).$$

Since $\psi \in \Psi$, we get

$$d(x_{n_k+1}, z) < d(x_{n_k}, w).$$

If $n \to \infty$, we obtain $z = w$. Therefore there exists $z \in A$ such that $d(z, Tz) = d(A, B)$. The proof is completed. □

The following corollaries are obtained directly from Theorems 3.1 and 3.2.

Corollary 3.3. *Let A and B be nonempty closed subsets of a complete metric space (X, d) endowed with a graph G with $A_0 \neq \emptyset$. Let $T : A \to B$ be proximally G-edge-preserving, α-β-G-proximal admissible and α-β-ψ-G-proximal contraction such that $T(A_0) \subseteq B_0$. Assume that there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A, B) \quad \text{and} \quad (x_0, x_1) \in E(G)$$

and there exist $C_\alpha > 0, C_\beta \geq 0$ such that $\alpha(x_0, x_1) \geq C_\alpha, \beta(x_0, x_1) \leq C_\beta$. Suppose that either

(i) *T is continuous or*

(ii) *If $\{x_n\}$ is a sequence in A such that $(x_n, x_{n+1}) \in E(G)$, $\alpha(x_n, x_{n+1}) \geq C_\alpha$, $\beta(x_n, x_{n+1}) \leq C_\beta$ for all n and $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$, $\alpha(x_{n_k}, x) \geq C_\alpha$, $\beta(x_{n_k}, x) \leq C_\beta$ for all k and A_0 is closed.*

Then there exists an element $z \in A$ such that $d(z, Tz) = d(A, B)$. Further, the sequence $\{x_n\}$, defined by

$$d(x_{n+1}, Tx_n) = d(A, B), \ \ \forall\, n \in \mathbb{N},$$

converges to the element z.

Lemma 3.4. *In addition to the hypotheses of Theorem 3.1 (Theorem 3.2), if x is a best proximity point of T with $(x, u) \in E(G), \alpha(x, u) \geq C_\alpha$ and $\beta(x, u) \leq C_\beta$ for some $u \in A_0$, then there exists a sequence $\{u_n\} \subseteq A_0$ such that $d(u_n, Tu_{n-1}) = d(A, B)$, $(x, u_n) \in E(G)$, for all $n \in \mathbb{N}$ and $u_n \to x$ as $n \to \infty$.*

Proof. Let x be a best proximity point of T, *i.e.*,

$$d(x, Tx) = d(A, B). \tag{3.9}$$

Let $u \in A_0$ such that $(x, u) \in E(G)$. We set $u_0 = u$. Since $T(A_0) \subseteq B_0$ and $u = u_0 \in A_0$, we have $Tu_0 \in B_0$. Hence there exists $u_1 \in A$ such that

$$d(u_1, Tu_0) = d(A, B). \tag{3.10}$$

By the definition of A_0 and B_0, we have $u_1 \in A_0$. Since T is proximally G-edge-preserving on A_0, from $(x, u_0) \in E(G)$, (3.9) and (3.10), we have $(x, u_1) \in E(G)$.

On continuing this process we can construct a sequence $\{u_n\}$ in A_0 such that

$$d(u_n, Tu_{n-1}) = d(A, B), \tag{3.11}$$

satisfying

$$(x, u_n) \in E(G), \ \ \forall\, n \in \mathbb{N}. \tag{3.12}$$

By assumption, from (3.9) and (3.10) we have

$$\left.\begin{array}{l} \alpha(x, u_0) \geq C_\alpha, \ \beta(x, u_0) \leq C_\beta \\ (x, u_0) \in E(G) \\ d(x, Tx) = d(A, B) \\ d(u_1, Tu_0) = d(A, B) \end{array}\right\} \Rightarrow \ \alpha(x, u_1) \geq C_\alpha, \ \beta(x, u_1) \leq C_\beta. \tag{3.13}$$

Since T is an α-β-ψ-G-proximal contraction we have

$$\begin{aligned} C_\alpha d(x, u_1) &\leq \alpha(x, u_0) d(x, u_1) \\ &\leq \beta(x, u_0) \psi\big(d(x, u_0)\big) \\ &\leq C_\beta \psi\big(d(x, u_0)\big), \end{aligned}$$

and it follows that

$$d(x, u_1) \leq \frac{C_\beta}{C_\alpha} \psi\big(d(x, u_0)\big) \leq \psi\big(d(x, u_0)\big) = \psi\big(d(x, u)\big).$$

From (3.9), (3.11), (3.12) and (3.13) we have

$$\left.\begin{array}{l} \alpha(x, u_1) \geq C_\alpha, \ \beta(x, u_1) \leq C_\beta \\ (x, u_1) \in E(G) \\ d(x, Tx) = d(A, B) \\ d(u_2, Tu_1) = d(A, B) \end{array}\right\} \Rightarrow \ \alpha(x, u_2) \geq C_\alpha, \ \beta(x, u_2) \leq C_\beta, \tag{3.14}$$

T is an α-β-ψ-G-proximal contraction we have

$$\begin{aligned} C_\alpha d(x, u_2) &\leq \alpha(x, u_1) d(x, u_2) \\ &\leq \beta(x, u_1) \psi\big(d(x, u_1)\big) \\ &\leq C_\beta \psi\big(d(x, u_1)\big), \end{aligned}$$

and it follows that

$$d(x,u_2) \leq \frac{C_\beta}{C_\alpha}\psi\big(d(x,u_1)\big) \leq \psi\big(d(x,u_1)\big) \leq \psi^2\big(d(x,u)\big).$$

On continuing this process, we obtain

$$d(x,u_n) \leq \psi^n\big(d(x,u)\big) \to 0 \quad \text{as} \quad n \to \infty.$$

i.e., $u_n \to x$ as $n \to \infty$. □

Theorem 3.5. *Suppose that all the hypotheses of Theorem* 3.1 *(Theorem* 3.2*) are satisfied. We Assume the following hypothesis.*
$(\star)$ *There exists* $u \in A_0$ *such that for every* $x, y \in A_0$ *with* $(x,u), (y,u) \in E(G)$,

$$\begin{aligned} \alpha(x,u) \geq C_\alpha \ \textit{and} \ (x,u) \leq \beta \\ \alpha(y,u) \geq C_\alpha \ \textit{and} \ (y,u) \leq \beta. \end{aligned} \tag{3.15}$$

Then T *has a unique best proximity point in* A_0.

Proof. By the proof of Theorem 3.1 (Theorem 3.2), the set of best proximity points of T is nonempty. Let x, y be two best proximity points of T in A_0. By our assumption, we have there exists $u \in A_0$ such that $(x,u), (y,u) \in E(G)$, $\alpha(x,u) \geq C_\alpha$ and $\alpha(x,u) \leq C_\beta$, $\alpha(y,u) \geq C_\alpha$ and $\alpha(y,u) \leq C_\beta$. Now by applying Lemma 3.4, it follows that there exists a sequence $\{u_n\} \subseteq A_0$ such that $u_n \to x$ and $u_n \to y$ as $n \to \infty$. Hence by the uniqueness of limits we have $x = y$. □

Corollary 3.6. *Let* A *and* B *be nonempty closed subsets of a complete metric space* (X,d) *endowed with a graph* G *with* $A_0 \neq \emptyset$. *Let* $T : A \to B$ *be a non-self mapping which satisfies the following properties:*

(i) T *is a* G*-proximal contraction;*
(ii) T *is continuous;*
(iii) T *is proximally* G*-edge-preserving,* G*-proximal admissible and* G*-proximal contraction such that* $T(A_0) \subseteq B_0$*;*
(iv) *there exist* $x_0, x_1 \in A_0$ *such that* $(x_0,x_1) \in E(G)$ *and* $d(x_1,Tx_0) = d(A,B)$.

Then T *has a best proximity point in* A_0.

Proof. Follows by choosing $\psi(t) = kt, t \geq 0$, and $\alpha(x,y) = \beta(x,y) = 1$, for all x, y in A, with $C_\alpha = C_\beta = 1$ in Theorem 3.1. □

If the continuity assumption is removed from Corollary 3.6, we have the following result.

Corollary 3.7. *Let A and B be nonempty closed subsets of a complete metric space (X,d) endowed with a graph G with $A_0 \neq \emptyset$. Let $T : A \to B$ be a non-self mapping which satisfies the following properties:*

(i) *T is a G-proximal contraction;*

(ii) *If $\{x_n\}$ is a sequence in A such that $(x_n, x_{n+1}) \in E(G)$ for all n and $x_n \to x$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all k and A_0 is closed;*

(iii) *T is a proximally G-edge-preserving, G-proximal admissible and G-proximal contraction such that $T(A_0) \subseteq B_0$;*

(iv) *there exist $x_0, x_1 \in A_0$ such that*

$$d(x_1, Tx_0) = d(A,B) \quad \textit{and} \quad (x_0, x_1) \in E(G).$$

Then T has a best proximity point in A_0

Example 3.8. Let $X = \mathbb{R}^2$, with an Euclidean metric d. Let G be a directed graph with $V(G) = X$ and $E(G) = \{\big((x,y),(u,v)\big) : x \geq u \text{ and } y \geq v\}$, for all $(x,y),(u,v) \in X$.

Let $A = \{-1\} \times [0,\infty) = A_0, B = \{1\} \times [0,\infty) = B_0$. Clearly $d(A,B) = 2$. We define $T : A \to B$ by

$$T(-1,x) = \begin{cases} (1, \frac{x}{6}) & \text{if } x \in [0,1], \\ (1, \frac{7}{6}x - 1) & \text{if } x > 1. \end{cases}$$

Clearly T is continuous and $T(A_0) \subseteq B_0$. Next, we show that T is proximally G-edge-preserving. Let $(-1,x),(-1,y),(-1,u)$ and $(-1,v) \in A$ with $\big((-1,x),(-1,y)\big) \in E(G)$ such that

$$d\big((-1,u), T(-1,x)\big) = d\big((-1,v), T(-1,y)\big) = d(A,B) = 2. \tag{3.16}$$

We now show that $\big((-1,u),(-1,v)\big) \in E(G)$.

Case I: $x, y \in [0,1]$ with $x \geq y$, from (3.16), we obtain $u = \frac{x}{6} \geq \frac{y}{6} = v$ then $\big((-1,u),(-1,v)\big) \in E(G)$.

Case II: $y \in [0,1]$ and $x > 1$, from (3.16), we obtain $u = \frac{7x}{6} - 1 > \frac{x}{6} > \frac{y}{6} = v$ then $\big((-1,u),(-1,v)\big) \in E(G)$.

Case III: $x \geq y > 1$, from (3.16), we obtain $u = \frac{7x}{6} - 1 \geq \frac{7y}{6} - 1 = v$ then $\big((-1,u),(-1,v)\big) \in E(G)$.

From all the above cases, we conclude that T is proximally G-edge-preserving.

Now, we define functions $\alpha, \beta : A \times A \to [0,\infty)$ by

$$\alpha\big((-1,x),(-1,y)\big) = \begin{cases} \frac{5}{6} & \text{if } x, y \in [0,1] \text{ with } \big((-1,x),(-1,y)\big) \in E(G), \\ 0 & \text{otherwise} \end{cases}$$

and

$$\beta\big((-1,x),(-1,y)\big)=\begin{cases}\frac{1}{2} & \text{if } x,y\in[0,1] \text{ with } \big((-1,x),(-1,y)\big)\in E(G),\\ 0 & \text{otherwise.}\end{cases}$$

Let $\psi(t)=\frac{t}{3}$ for all $t\geq 0$. We now show that T is α-β-ψ-G-proximal contractive mapping, let $(-1,x),(-1,y),(-1,u),(-1,v)\in A$ with $\big((-1,x),(-1,y)\big)\in E(G)$ such that

$$d\big((-1,u),T(-1,x)\big)=d\big((-1,v),T(-1,y)\big)=d(A,B)=2. \tag{3.17}$$

Consider the case $x,y\in[0,1]$ with $x\geq y$. Then $\alpha\big((-1,x),(-1,y)\big)=\frac{5}{6}$ and $\beta\big((-1,x),(-1,y)\big)=\frac{1}{2}$. From (3.17), we obtain $u=\frac{x}{6}$ and $v=\frac{y}{6}\in[0,\frac{1}{6}]$. Therefore

$$\begin{aligned}&\alpha\big((-1,x),(-1,y)\big)d\big((-1,u),(-1,v)\big)=\tfrac{5}{6}\big|u-v\big|\\ &\leq\big|u-v\big|=\tfrac{1}{2}\left(\tfrac{\left|x-y\right|}{3}\right)\\ &=\beta\big((-1,x),(-1,y)\big)\psi\Big(\big((-1,x),(-1,y)\big)\Big).\end{aligned}$$

For the other possible cases we have $\alpha\big((-1,x),(-1,y)\big)=\beta\big((-1,x),(-1,y)\big)=0$. Therefore

$$\begin{aligned}&\alpha\big((-1,x),(-1,y)\big)d\big((-1,u),(-1,v)\big)\\ &\leq\beta\big((-1,x),(-1,y)\big)\psi\Big(\big((-1,x),(-1,y)\big)\Big).\end{aligned}$$

Hence T is an α-β-ψ-G-proximal contractive mapping. We now show that T is an α-β-G-proximal admissible. For this purpose, we choose $C_\alpha=\frac{3}{4}$ and $C_\beta=\frac{1}{2}$ Clearly $0\leq\frac{C_\beta}{C_\alpha}\leq 1$. Let $(-1,x),(-1,y),(-1,u)$ and $(-1,v)\in A$ with $x,y\in[0,1]$ and $\big((-1,x),(-1,y)\big)\in E(G)$ such that

(i) $\alpha\big((-1,x),(-1,y)\big)=\frac{5}{6}\geq\frac{3}{4}=C_\alpha$ and

$$d\big((-1,u),T(-1,x)\big)=d\big((-1,v),T(-1,y)\big)=d(A,B)=2,$$

(ii) $\beta\big((-1,x),(-1,y)\big)=\frac{1}{2}\leq\frac{1}{2}=C_\beta$ and

$$d\big((-1,u),T(-1,x)\big)=d\big((-1,v),T(-1,y)\big)=d(A,B)=2.$$

From (i) and (ii), we obtain $u=\frac{x}{6}$ and $v=\frac{y}{6}$ in $[0,1]$. Since $x\geq y$, it follows that $u\geq v$ or $\big((-1,u),(-1,v)\big)\in E(G)$. Therefore

$$\alpha\big((-1,u),(-1,v)\big)=\frac{5}{6}\geq\frac{3}{4}=C_\alpha \text{ and } \beta\big((-1,u),(-1,v)\big)=\frac{1}{2}\leq\frac{1}{2}=C_\beta.$$

Hence T is an α-β-G-proximal admissible. We choose $x_0 = (-1,1), x_1 = (-1,\frac{1}{6})$ in A_0. Then $\big(x_0,x_1\big) \in E(G)$ and $d\big((-1,\frac{1}{4}),T(-1,1)\big) = 2 = d(A,B)$. Also,

$$\alpha\big((-1,1),(-1,\tfrac{1}{6})\big) = \tfrac{5}{6} \geq \tfrac{3}{4} = C_\alpha \text{ and } \beta\big((-1,1),(-1,\tfrac{1}{6})\big) = \tfrac{1}{2} \leq \tfrac{1}{2} = C_\beta.$$

Hence all the hypotheses of Theorem 3.1 are satisfied and $(-1,0)$ and $(-1,2)$ are two best proximity points of T. Here we observe that Condition $(\star)$ of Theorem 3.5 fails to hold: if $u = (-1,x_0) \in A_0$ with $x_0 > 0$, we choose $x = (-1,0), y = (-1,2)$ so that $\big(x,u\big) \notin E(G)$; if $u = (-1,0)$ then $\big(x,u\big) \in E(G)$ and $\big(y,u\big) \in E(G)$, $\alpha(x,u) = \frac{5}{6} > \frac{3}{4} = C_\alpha$, $\beta(x,u) = \frac{1}{2} = C_\beta$ and $\beta(y,u) = 0 < \frac{1}{2} = C_\beta$. But $\alpha(y,u) = 0 \ngeq \frac{3}{4} = C_\alpha$. Hence Condition $(\star)$ of Theorem 3.5 fails to hold.

The following example is in support of Theorem 3.5 in which T is not continuous.

Example 3.9. Let $X = [0,5] \times [0,5]$ with an Euclidean metric d. Let G be a directed graph with $V(G) = X$ and $E(G) = \{\big((x,y),(u,v)\big) : x \geq u \text{ and } y \geq v\}$, for all $(x,y),(u,v) \in X$.

Let $A = \{0\} \times [0,5] = A_0, B = \{1\} \times [0,5] = B_0$. We define $T : A \to B$ by

$$T(0,x) = \begin{cases} (1,\frac{x^2}{1+2x}) & \text{if } x \in [0,1], \\ (1,\frac{2}{3}x) & \text{if } x \in (1,5]. \end{cases}$$

Clearly $d(A,B) = 1$, T is not continuous, T is proximally G-edge-preserving, $T(A_0) \subseteq B_0$ and A_0 is closed.

Now, we define functions $\alpha,\beta : A \times A \to [0,\infty)$ by

$$\alpha\big((0,x),(0,y)\big) = \begin{cases} \frac{3}{4} & \text{if } x,y \in [0,1] \text{ with } \big((0,x),(0,y)\big) \in E(G), \\ 0 & \text{otherwise} \end{cases}$$

and

$$\beta\big((0,x),(0,y)\big) = \begin{cases} \frac{2}{5} & \text{if } x,y \in [0,1] \text{ with } \big((0,x),(0,y)\big) \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\psi(t) = \frac{7}{8}$ for all $t \geq 0$. We now show that T is α-β-ψ-G-proximal contractive mapping, let $(0,x),(0,y),(0,u)$ and $(0,v) \in A$ with $\big((0,x),(0,y)\big) \in E(G)$ such that

$$d\big((0,u),T(0,x)\big) = d\big((0,v),T(0,y)\big) = d(A,B) = 1. \tag{3.18}$$

Let us consider the case $x,y \in [0,1]$ with $\big((0,x),(0,y)\big) \in E(G)$. Then $\alpha\big((0,x),(0,y)\big) = \frac{3}{4}$ and $\beta\big((0,x),(0,y)\big) = \frac{2}{5}$. From (3.18), we obtain $u =$

$\frac{x^2}{1+2x} \in \big[0, \frac{1}{3}\big]$ and $v = \frac{y^2}{1+2y} \in \big[0, \frac{1}{3}\big]$. Therefore

$$\begin{aligned}\alpha\big((0,x),(0,y)\big)d\big((0,u),(0,v)\big) &= \tfrac{3}{4}\big|u-v\big| = \tfrac{3}{4}\left|\tfrac{x^2}{1+2x} - \tfrac{y^2}{1+2y}\right| \\ &= \tfrac{3}{4}\left(\tfrac{\big|x-y\big|(x+y+2xy)}{1+2(x+y+2xy)}\right) \\ &\le \tfrac{7}{20}\,|x-y| = \tfrac{2}{5}\left(\tfrac{7|x-y|}{8}\right) \\ &= \beta\big((0,x),(0,y)\big)\psi\Big(\big((0,x),(0,y)\big)\Big).\end{aligned}$$

For the other possible cases we have $\alpha\big((0,x),(0,y)\big) = \beta\big((0,x),(0,y)\big) = 0$. Therefore

$$\begin{aligned}&\alpha\big((0,x),(0,y)\big)d\big((0,u),(0,v)\big) \\ &\le \beta\big((0,x),(0,y)\big)\psi\Big(\big((0,x),(0,y)\big)\Big).\end{aligned}$$

Hence T is an α-β-ψ-G-proximal contractive mapping.

We now show that T is an α-β-G-proximal admissible. For this purpose, we choose $C_\alpha = \frac{2}{3}$ and $C_\beta = \frac{1}{2}$. Clearly $0 \le \frac{C_\beta}{C_\alpha} \le 1$. Let $(0,x),(0,y),(0,u)$ and $(0,v) \in A$ with $x \ge y$ such that

(a) $\alpha\big((0,x),(0,y)\big) = \frac{3}{4} \ge \frac{2}{3} = C_\alpha$ and

$$d\big((0,u),T(0,x)\big) = d\big((0,v),T(0,y)\big) = d(A,B) = 1,$$

(b) $\beta\big((0,x),(0,y)\big) = \frac{2}{5} \le \frac{1}{2} = C_\beta$ and

$$d\big((0,u),T(0,x)\big) = d\big((0,v),T(0,y)\big) = d(A,B) = 1.$$

From (a) and (b), we obtain $u = \frac{x^2}{1+2x}$ and $v = \frac{y^2}{1+2y}$ in $[0, \frac{1}{3}]$. Since $x \ge y$, it follows that $u \ge v$ or $\big((-1,u),(-1,v)\big) \in E(G)$. Therefore

$$\alpha\big((0,u),(0,v)\big) = \frac{3}{4} \ge \frac{2}{3} = C_\alpha \text{ and } \beta\big((0,u),(0,v)\big) = \frac{2}{5} \le \frac{1}{2} = C_\beta.$$

Hence T is an α-β-G-proximal admissible.

Now, we choose $x_0 = (0, \frac{1}{2})$, $x_1 = (0, \frac{1}{7})$, such that $d\big((0,\frac{1}{7}), T(0,\frac{1}{2})\big) = 1$, $\alpha\big((0,\frac{1}{2}),(0,\frac{1}{7})\big) = \frac{3}{4} \ge \frac{2}{3} = C_\alpha$, $\beta\big((0,\frac{1}{2}),(0,\frac{1}{7})\big) = \frac{2}{5} \le \frac{1}{2} = C_\beta$ and $\big(x_0, x_1\big) \in E(G)$. Finally, If $\{x_n\}$ is a sequence in A such that $x_n \ge x_{n+1}$, $\alpha(x_n, x_{n+1}) \ge C_\alpha$, $\beta(x_n, x_{n+1}) \le C_\beta$ for all n and $x_n \to x$ as $n \to \infty$, then by definition of α and β, $x_n \in [0,1]$. Thus $x \in [0,1]$ and there exists a subsequence $\{x_{n_k}\} \subseteq [0,1]$ such that $x_{n_k} \ge x$, $\alpha(x_{n_k}, x) \ge C_\alpha$, $\beta(x_{n_k}, x) \le C_\beta$ for all k. Hence all the hypotheses of Theorem 3.5 are satisfied and $(0,0)$ is the unique best proximity point of T.

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