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ON A m-ORDER NONLINEAR INTEGRODIFFERENTIAL EQUATION IN N VARIABLES

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Abstract. In this paper, by applying the fixed point theorems coupled with establishing suitable Banach spaces and a sufficient condition for relatively compact subsets, we study the existence and the compactness of the set of solutions for m-order nonlinear integrodifferential equation in N variables. In order to illustrate the results obtained here, we present two examples.

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1. Introduction

In this paper, we consider the following m-order nonlinear integrodifferential equation in N variables

$$u(x) = g(x) + \int_{\Omega} K(x, y; u(y), D_1 u(y), \cdots, D_1^m u(y))dy, \tag{1.1}$$

where $x = (x_1, \cdots, x_N) \in \Omega = [0,1]^N$ and $g : \Omega \to \mathbb{R}$, $K : \Omega \times \Omega \times \mathbb{R}^{m+1} \to \mathbb{R}$ are given functions. Denote by $D_1^i u = \frac{\partial^i u}{\partial x_1^i}$, the partial derivative of order $i = \overline{1,m}$ of a function $u(x)$ defined on Ω, with respect to the first variable.

It is well known that, integral and integrodifferential equations have attracted the interest of scientists not only because of their major role in the fields of functional analysis but also because of their important role in numerous applications, for example, mechanics, physics, population dynamics, economics and other fields of science, see Corduneanu [5], Deimling [6]. In general, existence results of integral and integrodifferential equations in one variable or N variables, have been obtained via the fundamental methods in which the fixed point theorems are often applied, see [1]-[20] and the references given therein.

In [17], based on the applications of the well-known Banach fixed point theorem coupled with Bielecki type norm and a certain integral inequality with explicit estimate, Pachpatte proved uniqueness and other properties of solutions of the following Fredholm type integrodifferential equation

$$x(t) = g(t) + \int_a^b f(t, s, x(s), x'(s), \cdots, x^{(n-1)}(s))ds, \ t \in [a, b],$$

where x, g, f are real valued functions and $n \geq 2$ is an integer. With the same methods, Pachpatte studied the existence, uniqueness and some basic properties of solutions of the Fredholm type integral equation in two variables as follows, see [18],

$$u(x, y) = f(x, y) + \int_0^a \int_0^b g\left(x, y, s, t; u(s, t), D_1 u(s, t), D_2 u(s, t)\right) dtds,$$

and those of certain Volterra integral and integrodifferential equations in two variables, see [19].

In [4], El-Borai et al. have proved the existence of a unique solution of a nonlinear integral equation of type Volterra-Hammerstein in n-dimensional of the form

$$\mu\phi(x, t) = f(x, t) + \lambda \int_0^t \int_{\Omega} F(t, \tau) K(x, y) \gamma\left(\tau, y, \phi(y, \tau)\right) dyd\tau,$$

where $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$; μ, λ are constants. After that, in [1], Abdou et al. investigated the following mixed nonlinear integral equation of the second kind in $n-$dimensional

$$\begin{aligned} \mu\phi(x,t) &= \lambda \int_\Omega k(x,y)\gamma\left(t,y,\phi(y,t)\right) dy \\ &+\lambda \int_0^t \int_\Omega G(t,\tau)k(x,y)\gamma\left(\tau,y,\phi(y,\tau)\right) dyd\tau \\ &+\lambda \int_0^t F(t,\tau)\phi(x,\tau)d\tau + f(x,t), \end{aligned}$$

where $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$. Also using the Banach fixed point theorem, the existence of a unique solution of these equations were proved.

Abdou et al. also considered the existence of integrable solution of nonlinear integral equation, of type Hammerstein–Volterra of the second kind, by using the technique of measure of weak noncompactness and Schauder fixed point theorem, see [2].

In [11], Lauran established sufficient conditions for the existence of solutions of the integral equation of Volterra type by using the concepts of nonexpansive operators, contraction principles and the Schaefer's fixed point theorem.

In [3], Aghajani et al. proved some results on the existence, uniqueness and estimation of the solutions of Fredholm type integro-differential equations in two variables, by using Perov's fixed point theorem.

Recently, in [8], [12]-[16], using tools of functional analysis and a fixed point theorem of Krasnosel'skii type, we have investigated solvability and asymptotically stable of nonlinear functional integral equations in one variable or two variables, or N variables.

Motivated by the above mentioned works, in this paper, we consider (1.1) and prove two existence theorems. First, applying the Banach theorem, we obtain the unique existence of a solution of (1.1) in Theorem 2.2. Next, applying the Schauder theorem, the existence of solutions of (1.1) will be given in Theorem 3.2. Furthermore, the compactness of solutions set is also proved. In order to illustrate the results obtained here, two examples are given.

2. The unique existence

We begin this section by constructing an appropriate Banach space for (1.1) as follows. By $X = C(\Omega; \mathbb{R})$, we denote the space of all continuous functions from Ω into $\mathbb{R}$ equipped with the standard norm

$$\|u\|_X = \sup_{x\in\Omega} |u(x)|\,, \ u \in X. \tag{2.1}$$

Put

$$X_m = \{u \in X = C(\Omega;\mathbb{R}) : D_1^i u \in X,\ i = \overline{1,m}\}. \tag{2.2}$$

We remark that $C^1(\Omega;\mathbb{R}) \setminus X_m \neq \phi$, $X_m \setminus C^1(\Omega;\mathbb{R}) \neq \phi$, $X_m \cap C^1(\Omega;\mathbb{R}) \neq \phi$, $X_m \neq C^k(\Omega;\mathbb{R})$ for all $k = 1, 2, \cdots$, $m = 2, 3, \cdots$.

Indeed,

(i) with $u(x) = \left|x_1 - \frac{1}{2}\right| \left(x_1 - \frac{1}{2}\right) \left|x_2 - \frac{1}{2}\right| \left(x_2 - \frac{1}{2}\right) \cdots \left|x_N - \frac{1}{2}\right| \left(x_N - \frac{1}{2}\right)$, we have $u \in C^1(\Omega;\mathbb{R})$, but $u \notin X_m$. Hence $C^1(\Omega;\mathbb{R}) \setminus X_m \neq \phi$;

(ii) with $v(x) = x_1^{m+1}\left|x_2 - \frac{1}{2}\right| + e^{x_3+\cdots+x_N}$, we have $v \in X_m$, but $v \notin C^1(\Omega;\mathbb{R})$. So $X_m \setminus C^1(\Omega;\mathbb{R}) \neq \phi$;

(iii) $X_m \cap C^1(\Omega;\mathbb{R}) \neq \phi$ holds, by $w \equiv 0 \in X_m \cap C^1(\Omega;\mathbb{R})$;

(iv) $X_m \neq C^k(\Omega;\mathbb{R})$ for all $k = 1, 2, \cdots$, hold, because $X_m \setminus C^1(\Omega;\mathbb{R}) \neq \phi$.

Lemma 2.1. *X_m is a Banach space with the norm defined by*

$$\|u\|_{X_m} = \|u\|_X + \sum_{i=1}^m \left\|D_1^i u\right\|_X = \sum_{i=0}^m \left\|D_1^i u\right\|_X,\ u \in X_m. \tag{2.3}$$

Proof. Let $\{u_p\} \subset X_m$ be a Cauchy sequence in X_m. Then

$$\|u_p - u_q\|_{X_m} = \|u_p - u_q\|_X + \sum_{i=1}^m \left\|D_1^i u_p - D_1^i u_q\right\|_X \to 0, \text{ as } p,\ q \to \infty.$$

It implies that $\{u_p\}$ and $\{D_1^i u_p\}$ are also the Cauchy sequences in X. Since X is complete, $\{u_p\}$ converges to u and $\{D_1^i u_p\}$ converges to $v^{(i)}$ in X, i.e.,

$$\|u_p - u\|_X \to 0,\ \left\|D_1^i u_p - v^{(i)}\right\|_X \to 0, \text{ as } p \to \infty,\ i = \overline{1,m}. \tag{2.4}$$

We have to prove $D_1^i u = v^{(i)}$, $i = \overline{1,m}$. For $i = 1$, we have

$$u_p(x_1, x') - u_p(0, x') = \int_0^{x_1} D_1 u_p(s, x')ds,\ \forall (x_1, x') \in \Omega. \tag{2.5}$$

By $\|u_p - u\|_X \to 0$, we get

$$u_p(x_1, x') - u_p(0, x') \to u(x_1, x') - u(0, x'),\ \forall (x_1, x') \in \Omega. \tag{2.6}$$

On the other hand, it follows from $\left\|D_1 u_p - v^{(1)}\right\|_X \to 0$ that

$$\int_0^{x_1} D_1 u_p(s, x')ds \to \int_0^{x_1} v^{(1)}(s, x')ds,\ \forall (x_1, x') \in \Omega, \tag{2.7}$$

since

$$\begin{aligned} &\left|\int_0^{x_1} D_1 u_p(s, x')ds - \int_0^{x_1} v^{(1)}(s, x')ds\right| \\ &\le \int_0^{x_1} \left|D_1 u_p(s, x') - v^{(1)}(s, x')\right| ds \\ &\le \left\|D_1 u_p - v^{(1)}\right\|_X \ \to\ 0. \end{aligned}$$

Combining (2.5)-(2.7) yields

$$u(x_1, x') - u(0, x') = \int_0^{x_1} v^{(1)}(s, x')ds, \ \forall (x_1, x') \in \Omega. \qquad (2.8)$$

It implies that $D_1 u = v^{(1)} \in X$. Let $D_1^i u = v^{(i)}$, $i = 1, \cdots, r < m$. We shall show that $D_1^{r+1} u = v^{(r+1)}$. We have

$$D_1^r u_p(x_1, x') - D_1^r u_p(0, x') = \int_0^{x_1} D_1^{r+1} u_p(s, x')ds, \ \forall (x_1, x') \in \Omega. \qquad (2.9)$$

Because of $\|D_1^r u_p - D_1^r u\|_X \to 0$ and $\left\|D_1^{r+1} u_p - v^{(r+1)}\right\|_X \to 0$, we obtain

$$D_1^r u(x_1, x') - D_1^r u(0, x') = \int_0^{x_1} v^{(r+1)}(s, x')ds, \ \forall (x_1, x') \in \Omega. \qquad (2.10)$$

Then $D_1^{r+1} u = v^{(r+1)} \in X$. By induction, we deduce that $D_1^i u = v^{(i)}$, $i = \overline{1, m}$. Therefore $u \in X_m$ and $u_p \to u$ in X_m. Lemma 2.1 is proved. □

Now, we make the following assumptions.

(A_1) $g \in X_m$;

(A_2) $K \in C(\Omega \times \Omega \times \mathbb{R}^{m+1}; \mathbb{R})$,
such that $D_1 K, D_1^2 K, \cdots, D_1^m K \in C(\Omega \times \Omega \times \mathbb{R}^{m+1}; \mathbb{R})$,

and there exist nonnegative functions $k_0, k_1, \cdots, k_m : \Omega \times \Omega \to \mathbb{R}$ satisfying

(i) $\beta = \sum_{i=0}^m \sup_{x \in \Omega} \int_\Omega k_i(x, y)dy < 1$,

(ii) $\left|D_1^i K(x, y; u_0, \cdots, u_m) - D_1^i K(x, y; \bar{u}_0, \cdots, \bar{u}_m)\right|$

$\leq k_i(x, y) \sum_{j=0}^m |u_j - \bar{u}_j|$, $\forall (x, y) \in \Omega \times \Omega$, $\forall (u_0, \cdots, u_m), (\bar{u}_0, \cdots, \bar{u}_m)$

$\in \mathbb{R}^{m+1}$, $i = \overline{0, m}$.

Theorem 2.2. *Let the functions g, K in* (1.1) *satisfy the assumptions* (A_1), (A_2). *Then the equation* (1.1) *has a unique solution in* X_m.

Proof. For every $u \in X_m$, we put

$$(Au)(x) = g(x) + \int_\Omega K(x, y; u(y), D_1 u(y), \cdots, D_1^m u(y))dy, \ x \in \Omega. \qquad (2.11)$$

A simple verification shows that $Au \in X_m$ for all $u \in X_m$. It is obvious that $A : X_m \to X_m$ is a contraction map, if we show that

$$\|Au - A\bar{u}\|_{X_m} \leq \beta \|u - \bar{u}\|_{X_m}, \ \forall u, \bar{u} \in X_m. \qquad (2.12)$$

For every u, $\bar{u} \in X_m$, for all $x \in \Omega$, using (A_2) and (ii) with $i = 0$, (2.11) implies

$$\begin{aligned}|(Au)(x)-(A\bar{u})(x)| &\leq \int_\Omega |K(x,y;u(y),D_1u(y),\cdots,D_1^m u(y)) \\ &\qquad -K(x,y;\bar{u}(y),D_1\bar{u}(y),\cdots,D_1^m\bar{u}(y))|\,dy \\ &\leq \int_\Omega k_0(x,y)\sum\nolimits_{j=0}^m \left|D_1^j u(y)-D_1^j\bar{u}(y)\right| dy \\ &\leq \left(\sup_{x\in\Omega}\int_\Omega k_0(x,y)dy\right)\|u-\bar{u}\|_{X_m}.\end{aligned}$$

Thus we have

$$\|Au-A\bar{u}\|_X \leq \left(\sup_{x\in\Omega}\int_\Omega k_0(x,y)dy\right)\|u-\bar{u}\|_{X_m}. \tag{2.13}$$

Similarly, by

$$D_1^i(Au)(x)=D_1^i g(x)+\int_\Omega D_1^i K(x,y;u(y),D_1u(y),\cdots,D_1^m u(y))dy, x\in\Omega,$$

using (A_2) and (ii) with $i=\overline{1,m}$, we get

$$\begin{aligned}\left|D_1^i(Au)(x)-D_1^i(A\bar{u})(x)\right| &\leq \int_\Omega \left|D_1^i K(x,y;u(y),D_1u(y),\cdots,D_1^m u(y))\right. \\ &\qquad \left.-D_1^i K(x,y;\bar{u}(y),D_1\bar{u}(y),\cdots,D_1^m\bar{u}(y))\right| dy \\ &\leq \int_\Omega k_i(x,y)\sum\nolimits_{j=0}^m \left|D_1^j u(y)-D_1^j\bar{u}(y)\right| dy \\ &\leq \left(\sup_{x\in\Omega}\int_\Omega k_i(x,y)dy\right)\|u-\bar{u}\|_{X_m}.\end{aligned}$$

Hence we have

$$\left\|D_1^i(Au)-D_1^i(A\bar{u})\right\|_X \leq \left(\sup_{x\in\Omega}\int_\Omega k_i(x,y)dy\right)\|u-\bar{u}\|_{X_m}. \tag{2.14}$$

From (2.13) and (2.14), (2.12) holds. Applying the Banach fixed point theorem, Theorem 2.2 is proved. □

3. The compactness of solutions set

In this section, by applying the Schauder fixed point theorem, we prove that the existence of solutions of (1.1) in X_m and the compactness of solutions set can be obtained by making the following assumptions:

(A_1) $g\in X_m$;

$(\bar{A}_2)$ $K \in C(\Omega \times \Omega \times \mathbb{R}^{m+1};\mathbb{R})$ such that
$D_1K,\ D_1^2K, \cdots, D_1^mK \in C(\Omega \times \Omega \times \mathbb{R}^{m+1};\mathbb{R})$,

and there exist nonnegative functions $\bar{k}_0, \bar{k}_1, \cdots, \bar{k}_m : \Omega \times \Omega \to \mathbb{R}$ satisfying

(i) $\bar{\beta} = \sum_{i=0}^{m} \sup\limits_{x\in\Omega} \int_\Omega \bar{k}_i(x,y)dy < 1$,

(ii) $\left|D_1^iK(x,y;u_0,\cdots,u_m)\right| \leq \bar{k}_i(x,y)\left(1+\sum_{j=0}^{m}|u_j|\right)$, $\forall(x,y) \in \Omega\times\Omega$, $\forall(u_0,\cdots,u_m) \in \mathbb{R}^{m+1}$, $i=\overline{0,m}$.

For the above purpose, we need a sufficient condition for relatively compact subsets of X_m as follows.

Lemma 3.1. *Let $\mathcal{F} \subset X_m$. Then $\mathcal{F}$ is relatively compact in X_m if and only if the following conditions are satisfied*

(i) $\exists M > 0 : \|u\|_{X_m} \leq M,\ \forall u \in \mathcal{F}$;

(ii) $\forall\varepsilon > 0,\ \exists\delta > 0 : \forall x,\ \bar{x} \in \Omega,\ |x-\bar{x}| < \delta \Longrightarrow \sup\limits_{u\in\mathcal{F}} [u(x)-u(\bar{x})]_\bigstar < \varepsilon$,

$$\text{where } [u(x)-u(\bar{x})]_\bigstar = |u(x)-u(\bar{x})| + \sum_{i=1}^{m}\left|D_1^iu(x)-D_1^iu(\bar{x})\right|. \tag{3.1}$$

Proof. (a) Let $\mathcal{F}$ be relatively compact in X_m. Then $\mathcal{F}$ is bounded, so we have (i). It remains to show that (ii) holds. For every $\varepsilon > 0$, considering a collection of open balls in X_m, with center at $u \in \mathcal{F}$ and radius $\frac{\varepsilon}{3}$, as follows

$$B(u,\tfrac{\varepsilon}{3}) = \{\bar{u} \in X_m : \|u-\bar{u}\|_{X_m} < \tfrac{\varepsilon}{3}\},\ u \in \mathcal{F}.$$

It is not difficult to verify that $\overline{\mathcal{F}} \subset \bigcup_{u\in\mathcal{F}} B(u,\frac{\varepsilon}{3})$. Since $\overline{\mathcal{F}}$ compact in X_m, the open cover $\{B(u,\frac{\varepsilon}{3}),\ u\in\mathcal{F}\}$ of $\overline{\mathcal{F}}$ contains a finite subcover, it means that there are $u_1,\cdots,u_q \in \mathcal{F}$ such that $\overline{\mathcal{F}} \subset \bigcup_{j=1}^{q} B(u_j,\frac{\varepsilon}{3})$.

The functions $u_j,\ D_1^iu_j,\ i=\overline{1,m},\ j=\overline{1,q}$ are uniformly continuous on Ω, so there exists $\delta > 0$ such that

$$\forall x,\bar{x} \in \Omega,\ \ |x-\bar{x}| < \delta \Longrightarrow [u_j(x)-u_j(\bar{x})]_\bigstar < \frac{\varepsilon}{3},\ \forall j=\overline{1,q}.$$

For all $u \in \mathcal{F}$, note that $u \in B(u_{j_0},\frac{\varepsilon}{3})$ for some $j_0 = \overline{1,q}$. Thus, for all x, $\bar{x} \in \Omega$, if $|x-\bar{x}| < \delta$ then we get

$$\begin{aligned}
[u(x)-u(\bar{x})]_\bigstar &\leq [u(x)-u_{j_0}(x)]_\bigstar + [u_{j_0}(x)-u_{j_0}(\bar{x})]_\bigstar + [u_{j_0}(\bar{x})-u(\bar{x})]_\bigstar \\
&\leq 2\,\|u-u_{j_0}\|_{X_m} + [u_{j_0}(x)-u_{j_0}(\bar{x})]_\bigstar \\
&< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

This implies that (ii) holds.

(b) Conversely, let (3.1) hold. Then we have to prove that $\mathcal{F}$ is relatively compact in X_m. Let $\{u_p\}$ be a sequence in $\mathcal{F}$, we have to show that there exists a convergent subsequence of $\{u_p\}$.

By (3.1), $\mathcal{F}_1 = \{u_p : p \in \mathbb{N}\}$ and $\mathcal{F}_2^i = \{D_1^i u_p : p \in \mathbb{N}\}$ are uniformly bounded and equicontinuous in X. Hence an application of the Ascoli-Arzela theorem to $\mathcal{F}_1$ implies that it is relatively compact in X, so there exists a subsequence $\{u_{p_k}\}$ of $\{u_p\}$ and $u \in X$ such that

$$\|u_{p_k} - u\|_X \to 0, \text{ as } k \to \infty.$$

Remark that $\{D_1^i u_{p_k} : k \in \mathbb{N}\} \subset \mathcal{F}_2^i$ is also uniformly bounded and equicontinuous in X, so it is also relatively compact in X. We deduce the existence of a subsequence of $\{D_1^i u_{p_k}\}$, denoted by the same symbol, and $v^{(i)} \in X$, such that

$$\left\| D_1^i u_{p_k} - v^{(i)} \right\|_X \to 0, \text{ as } k \to \infty.$$

By the fact that

$$u_{p_k}(x_1, x') - u_{p_k}(0, x') = \int_0^{x_1} D_1 u_{p_k}(s, x')ds, \ \forall (x_1, x') \in \Omega,$$

furthermore $\|u_{p_k} - u\|_X \to 0$ and $\left\|D_1 u_{p_k} - v^{(1)}\right\|_X \to 0$, we obtain

$$u(x_1, x') - u(0, x') = \int_0^{x_1} v^{(1)}(s, x')ds, \ \forall (x_1, x') \in \Omega.$$

It implies that $D_1 u = v^{(1)} \in X$.

Let $D_1^i u = v^{(i)}$, $i = 1, \cdots, r < m$. We shall show that $D_1^{r+1} u = v^{(r+1)}$.

We have

$$D_1^r u_p(x_1, x') - D_1^r u_p(0, x') = \int_0^{x_1} D_1^{r+1} u_p(s, x')ds, \ \forall (x_1, x') \in \Omega. \tag{3.2}$$

From $\|D_1^r u_p - D_1^r u\|_X \to 0$ and $\left\|D_1^{r+1} u_p - v^{(r+1)}\right\|_X \to 0$, we obtain

$$D_1^r u(x_1, x') - D_1^r u(0, x') = \int_0^{x_1} v^{(r+1)}(s, x')ds, \ \forall (x_1, x') \in \Omega. \tag{3.3}$$

Then $D_1^{r+1} u = v^{(r+1)} \in X$. By induction, we deduce that $D_1^i u = v^{(i)}$, $i = \overline{1, m}$. Therefore $u \in X_m$ and $u_{p_k} \to u$ in X_m. This completes the proof. □

Theorem 3.2. *Let the functions g, K in* (1.1) *satisfy the assumptions* (A_1), $(\bar{A}_2)$. *Then the equation* (1.1) *has a solution in* X_m. *Furthermore, the set of solutions of this equation is compact.*

Proof. Considering the operator A as in (2.11). It is not hard to verify $A : X_m \to X_m$. For $\rho > 0$, considering a closed ball in X_m as follows

$$B_\rho = \{u \in X_m : \|u\|_{X_m} \le \rho\}.$$

We can show that there exists $\rho > 0$ such that $A : B_\rho \to B_\rho$. Indeed, for every $u \in B_\rho$, for all $x \in \Omega$, we have

$$\begin{aligned} |(Au)(x)| &\le |g(x)| + \int_\Omega |K(x,y;u(y),D_1u(y),\cdots,D_1^m u(y))|\,dy \\ &\le \|g\|_X + \int_\Omega \bar{k}_0(x,y)\left(1+\sum\nolimits_{i=0}^m \left|D_1^i u(y)\right|\right)dy \\ &\le \|g\|_X + \int_\Omega \bar{k}_0(x,y)\left(1+\|u\|_{X_m}\right)dy \\ &\le \|g\|_X + (1+\rho)\left(\sup_{x\in\Omega}\int_\Omega \bar{k}_0(x,y)dy\right), \end{aligned}$$

it implies that

$$\|Au\|_X \le \|g\|_X + (1+\rho)\left(\sup_{x\in\Omega}\int_\Omega \bar{k}_0(x,y)dy\right). \tag{3.4}$$

Similarly, we have

$$\begin{aligned} \left|D_1^i(Au)(x)\right| &\le \left|D_1^i g(x)\right| + \int_\Omega \left|D_1^i K(x,y;u(y),D_1u(y),\cdots,D_1^m u(y))\right|dy \\ &\le \left\|D_1^i g\right\|_X + (1+\rho)\left(\sup_{x\in\Omega}\int_\Omega \bar{k}_i(x,y)dy\right), \end{aligned}$$

therefore

$$\left\|D_1^i(Au)\right\|_X \le \left\|D_1^i g\right\|_X + (1+\rho)\left(\sup_{x\in\Omega}\int_\Omega \bar{k}_i(x,y)dy\right). \tag{3.5}$$

This yields

$$\|Au\|_{X_m} \le \|g\|_{X_m} + (1+\rho)\sum\nolimits_{i=0}^m \sup_{x\in\Omega}\int_\Omega \bar{k}_i(x,y)dy \le \|g\|_{X_m} + (1+\rho)\,\bar{\beta}. \tag{3.6}$$

Choosing $\rho \ge \|g\|_{X_m} + (1+\rho)\,\bar{\beta}$, i.e. $\rho \ge \frac{\|g\|_{X_m}+\bar{\beta}}{1-\bar{\beta}}$. Then $A : B_\rho \to B_\rho$.

Now we show that the operator A satisfies two conditions as below.

(i) $A : B_\rho \to B_\rho$ is continuous.
(ii) $\mathcal{F} = A(B_\rho)$ is relatively compact in X_m.

To prove (i), let $\{u_p\} \subset B_\rho$, $\|u_p - u\|_{X_m} \to 0$, as $p \to \infty$, we have to prove that

$$\|Au_p - Au\|_X \to 0 \quad \text{and} \quad \sum\nolimits_{i=1}^m \left\|D_1^i(Au_p) - D_1^i(Au)\right\|_X \to 0, \tag{3.7}$$

as $p \to \infty$.

Note that

$$|(Au_p)(x) - (Au)(x)| \le \int_\Omega |K(x,y;u_p(y), D_1u_p(y), \cdots, D_1^m u_p(y)) \quad (3.8)$$
$$-K(x,y;u(y), D_1u(y), \cdots, D_1^m u(y))|\,dy.$$

Give $\varepsilon > 0$. Since the function K is uniformly continuous on $\Omega \times \Omega \times [-\rho,\rho]^{m+1}$, there exists $\delta > 0$ such that for all $(u_0, \cdots, u_m)$, $(\bar{u}_0, \cdots, \bar{u}_m) \in [-\rho,\rho]^{m+1}$,

$$\sum_{i=0}^{m} |u_i - \bar{u}_i| < \delta.$$

Hence

$$|K(x,y;u_0,\cdots,u_m) - K(x,y;\bar{u}_0,\cdots,\bar{u}_m)| < \varepsilon,$$

for all $(x,y) \in \Omega \times \Omega$.

By $\|u_p - u\|_X + \sum_{i=1}^m \|D_1^i u_p - D_1^i u\|_X \to 0$, there is $p_0 \in \mathbb{N}$ such that for all $p \in \mathbb{N}$ with $p \ge p_0$,

$$\|u_p - u\|_X + \sum_{i=1}^{m} \|D_1^i u_p - D_1^i u\|_X < \delta.$$

It follows that for all $p \in \mathbb{N}$, with $p \ge p_0$,

$$|K(x,y;u_p(y), D_1u_p(y), ..., D_1^m u_p(y)) - K(x,y;u(y), D_1u(y), ..., D_1^m u(y))| < \varepsilon,$$

for all $(x,y) \in \Omega \times \Omega$. So we have

$$|(Au_p)(x) - (Au)(x)| < \varepsilon, \quad \forall x \in \Omega, \quad \forall p \ge p_0,$$

it means that

$$\|Au_p - Au\|_X < \varepsilon, \quad \forall p \ge p_0, \tag{3.9}$$

i.e., $\|Au_p - Au\|_X \to 0$, as $p \to \infty$.

By the same way, we get $\|D_1^i(Au_m) - D_1^i(Au)\|_X \to 0$, as $p \to \infty$, for all $i = \overline{1,m}$.

To prove (ii), we use Lemma 3.1. Condition (3.1) (i) holds because of $\mathcal{F} = A(B_\rho) \subset B_\rho$. It remains to show (3.1) (ii). We have

$$\begin{aligned}&[(Au)(x) - (Au)(\bar{x})]_\bigstar \\ &\le [g(x) - g(\bar{x})]_\bigstar + \int_\Omega [K(x,y;u(y), D_1u(y), \cdots, D_1^m u(y)) \\ &\quad -K(\bar{x},y;u(y), D_1u(y), \cdots, D_1^m u(y))]_\bigstar\,dy,\end{aligned} \tag{3.10}$$

for all $x, \bar{x} \in \Omega$, and $u \in B_\rho$.

Let $\varepsilon > 0$. By the fact that $D_1^i K$, $i = \overline{0,m}$ are uniformly continuous on $\Omega \times \Omega \times [-\rho,\rho]^{m+1}$, there exists $\delta_1 > 0$ such that for all x, $\bar{x} \in \Omega$,

$$|x - \bar{x}| < \delta_1 \Longrightarrow [K(x,y;v_0,\cdots,v_m) - K(\bar{x},y;v_0,\cdots,v_m)]_\bigstar < \frac{\varepsilon}{2},$$

for all $(y, v_0, \cdots, v_m) \in \Omega \times [-\rho, \rho]^{m+1}$. Then, for all x, $\bar{x} \in \Omega$, $|x - \bar{x}| < \delta_1$,

$$[K(x, y; u(y), D_1u(y), ..., D_1^m u(y)) - K(\bar{x}, y; u(y), D_1u(y), ..., D_1^m u(y))]_\bigstar < \frac{\varepsilon}{2},$$

for all $(y, u) \in \Omega \times B_\rho$. Hence, for all x, $\bar{x} \in \Omega$, $|x - \bar{x}| < \delta_1$,

$$\begin{aligned}&\int_\Omega [K(x, y; u(y), D_1u(y), \cdots, D_1^m u(y)) \\ &\qquad - K(\bar{x}, y; u(y), D_1u(y), \cdots, D_1^m u(y))]_\bigstar dy \\ &< \frac{\varepsilon}{2}, \ \forall\, u \in B_\rho.\end{aligned}$$

Since $D_1^i g$, $i = \overline{0, m}$ are also uniformly continuous on Ω, there is $\delta_2 > 0$ such that

$$\forall\, x, \bar{x} \in \Omega, \ \ |x - \bar{x}| < \delta_2 \Longrightarrow [g(x) - g(\bar{x})]_\bigstar < \frac{\varepsilon}{2}.$$

Choose $\bar{\delta}_1 = \min\{\delta_1, \delta_2\}$, it yields

$$\forall\, x, \bar{x} \in \Omega, \ \ |x - \bar{x}| < \bar{\delta}_1 \Longrightarrow [(Au)(x) - (Au)(\bar{x})]_\bigstar < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \ \forall u \in B_\rho. \tag{3.11}$$

Lemma 3.1 implies that $\mathcal{F} = A(B_\rho)$ is relatively compact in X_m.

Applying the Schauder fixed point theorem, the existence of a solution is proved. Finally, we show that the set of solutions, $S = \{u \in B_\rho : u = Au\}$, is compact in X_m. From the compactness of the operator $A : B_\rho \to B_\rho$ and $S = A(S)$, we only prove that S is closed. Let $\{u_p\} \subset S$, $\|u_p - u\|_{X_m} \to 0$. The continuity of A leads to

$$\begin{aligned}\|u - Au\|_{X_m} &\leq \|u - u_p\|_{X_m} + \|u_p - Au\|_{X_m} \\ &= \|u - u_p\|_{X_m} + \|Au_p - Au\|_{X_m} \to 0,\end{aligned}$$

so $u = Au \in S$. This completes the proof. □

4. Examples

In this section, we present two examples to illustrate the results obtained in sections 2, 3.

Example 4.1. Consider (1.1), with the functions g, K as follows

$$\begin{cases} K(x, y; u_0, ..., u_m) = k(x)\Big[y_1^{\alpha_0} ... y_N^{\alpha_0} \sin\Big(\frac{\pi u_0}{2w_0(y)}\Big) + \sum_{i=1}^m y_1^{\alpha_i} ... y_N^{\alpha_i} \cos\Big(\frac{2\pi u_i}{D_1^i w_0(y)}\Big)\Big], \\ g(x) = w_0(x) - \sum_{j=0}^m \frac{1}{(1+\alpha_j)^N} k(x), \end{cases} \tag{4.1}$$

where

$$\begin{aligned} w_0(x) &= e^{x_1} + x_1^{\gamma_1} |x_2 - \alpha|^{\gamma_2} + \sum_{i=3}^{N} |x_i - \alpha|, \qquad (4.2) \\ k(x) &= x_1^{\tilde{\gamma}_1} |x_2 - \tilde{\alpha}|^{\tilde{\gamma}_2} + \sum_{i=3}^{N} |x_i - \tilde{\alpha}|, \end{aligned}$$

and α, γ_1, γ_2, $\tilde{\alpha}$, $\tilde{\gamma}_1$, $\tilde{\gamma}_2$, α_0, $\alpha_1, \cdots, \alpha_m$ are positive constants satisfying

$$\begin{cases} 0 < \alpha, \tilde{\alpha} < 1, \quad 0 < \gamma_2, \tilde{\gamma}_2 \le 1, \quad \gamma_1 > m, \quad \tilde{\gamma}_1 > m, \\ 2\pi \sum_{j=0}^{m} \frac{1}{(1+\alpha_j)^N} \Big[\Big(1 + \sum_{i=1}^{m} i! C_{\tilde{\gamma}_1}^i \Big) \max\{\tilde{\alpha}^{\tilde{\gamma}_2}, (1-\tilde{\alpha})^{\tilde{\gamma}_2}\} \\ \qquad\qquad + (N-2) \max\{\tilde{\alpha}, 1 - \tilde{\alpha}\} \Big] < 1, \end{cases} \qquad (4.3)$$

with $C_{\tilde{\gamma}_1}^i = \dfrac{\tilde{\gamma}_1 (\tilde{\gamma}_1 - 1) \cdots (\tilde{\gamma}_1 - i + 1)}{i!}$. Then we have

$$\begin{aligned} w_0(x) &= e^{x_1} + x_1^{\gamma_1} |x_2 - \alpha|^{\gamma_2} + \sum_{i=3}^{N} |x_i - \alpha|, \\ D_1^i w_0(x) &= e^{x_1} + i! C_{\gamma_1}^i x_1^{\gamma_1 - i} |x_2 - \alpha|^{\gamma_2}, \end{aligned}$$

so w_0, $D_1^i w_0 \in X$ and $w_0(x) \ge 1$, $D_1^i w_0(x) \ge 1$. Hence $K \in C(\Omega \times \Omega \times \mathbb{R}^{m+1}; \mathbb{R})$.

We can prove that (A_1), (A_2) hold. It is easy to see that (A_1) holds, since w_0, $k \in X_m$.

Assumption (A_2) holds, by the fact that:

First, we have $D_1^i k \in X$,

$$D_1^i K = D_1^i k(x) \left[y_1^{\alpha_0} \cdots y_N^{\alpha_0} \sin\left(\frac{\pi u}{2 w_0(y)} \right) + \sum_{i=1}^{m} y_1^{\alpha_i} \cdots y_N^{\alpha_i} \cos\left(\frac{2\pi u_i}{D_1^i w_0(y)} \right) \right],$$

so $D_1^i K \in C(\Omega \times \Omega \times \mathbb{R}^{m+1}; \mathbb{R})$;

$$\begin{aligned} & | \, K(x, y; u_0, \cdots, u_m) - K(x, y; \bar{u}_0, \cdots, \bar{u}_m) \, | \\ & \le k(x) \left[y_1^{\alpha_0} \cdots y_N^{\alpha_0} \frac{\pi |u_0 - \bar{u}_0|}{2 w_0(y)} + \sum_{i=1}^{m} y_1^{\alpha_i} \cdots y_N^{\alpha_i} \frac{2\pi |u_i - \bar{u}_i|}{D_1^i w_0(y)} \right] \\ & \le 2\pi k(x) \sum_{i=0}^{m} y_1^{\alpha_i} \cdots y_N^{\alpha_i} \sum_{j=0}^{m} |u_j - \bar{u}_j| \\ & \equiv k_0(x, y) \sum_{j=0}^{m} |u_j - \bar{u}_j|, \qquad (4.4) \end{aligned}$$

in which $k_0(x,y) = 2\pi k(x) \sum_{i=0}^{m} y_1^{\alpha_i} \cdots y_N^{\alpha_i}$. Similarly, we have

$$\left|D_1^i K(x,y;u_0,\cdots,u_m) - D_1^i K(x,y;\bar{u}_0,\cdots,\bar{u}_m)\right| \le k_i(x,y) \sum\nolimits_{j=0}^{m} |u_j - \bar{u}_j|, \tag{4.5}$$

with $k_i(x,y) = 2\pi \left|D_1^i k(x)\right| \sum_{i=0}^{m} y_1^{\alpha_i} \cdots y_N^{\alpha_i}$, $i = \overline{1,m}$.

Next, we have

$$\int_\Omega k_i(x,y)dy = 2\pi \left|D_1^i k(x)\right| \int_\Omega \sum_{j=0}^{m} y_1^{\alpha_j} ... y_N^{\alpha_j}\, dy = 2\pi \sum\nolimits_{j=0}^{m} \frac{1}{(1+\alpha_j)^N} \left|D_1^i k(x)\right|. \tag{4.6}$$

We also have the following lemma, its proof is easy so we omit.

Lemma 4.2. *Let positive constants* α, γ_2, γ_1 *satisfy* $0 < \alpha < 1$, $\gamma_1 > 0$, $0 < \gamma_2 \le 1$. *Then*

$$0 \le x_1^{\gamma_1} |x_2 - \alpha|^{\gamma_2} \le \max\{\alpha^{\gamma_2},\ (1-\alpha)^{\gamma_2}\},\ \ \forall x_1,\ x_2 \in [0,1].$$

Now, using Lemma 4.4, we obtain

$$\begin{aligned} 0 &\le k(x) = x_1^{\tilde{\gamma}_1} |x_2 - \tilde{\alpha}|^{\tilde{\gamma}_2} + \sum_{i=3}^{N} |x_i - \tilde{\alpha}| \qquad (4.7)\\ &\le \max\{\tilde{\alpha}^{\tilde{\gamma}_2}, (1-\tilde{\alpha})^{\tilde{\gamma}_2}\} + (N-2)\max\{\tilde{\alpha}, 1-\tilde{\alpha}\};\\ 0 &\le D_1 k(x) = \tilde{\gamma}_1 x_1^{\tilde{\gamma}_1 - 1} |x_2 - \tilde{\alpha}|^{\tilde{\gamma}_2} \le \tilde{\gamma}_1 \max\{\tilde{\alpha}^{\tilde{\gamma}_2}, (1-\tilde{\alpha})^{\tilde{\gamma}_2}\},\\ &\vdots\\ 0 &\le D_1^i k(x) = i! C_{\tilde{\gamma}_1}^i x_1^{\tilde{\gamma}_1 - i} |x_2 - \alpha|^{\tilde{\gamma}_2} \le i! C_{\tilde{\gamma}_1}^i \max\{\tilde{\alpha}^{\tilde{\gamma}_2}, (1-\tilde{\alpha})^{\tilde{\gamma}_2}\},\ i = \overline{1,m}, \end{aligned}$$

so

$$\begin{aligned} &\sup_{x\in\Omega} \int_\Omega k_0(x,y)dy\\ &\le 2\pi \sum_{j=0}^{m} \frac{1}{(1+\alpha_j)^N} \left[\max\{\tilde{\alpha}^{\tilde{\gamma}_2}, (1-\tilde{\alpha})^{\tilde{\gamma}_2}\} + (N-2)\max\{\tilde{\alpha}, 1-\tilde{\alpha}\}\right], \end{aligned}$$

$$\sup_{x\in\Omega} \int_\Omega k_i(x,y)dy \le 2\pi \sum_{j=0}^{m} \frac{1}{(1+\alpha_j)^N} i! C_{\tilde{\gamma}_1}^i \max\{\tilde{\alpha}^{\tilde{\gamma}_2}, (1-\tilde{\alpha})^{\tilde{\gamma}_2}\},\ i = \overline{1,m}.$$

Consequently

$$\begin{aligned}\beta &= \sum_{i=0}^{m}\sup_{x\in\Omega}\int_{\Omega}k_i(x,y)dy\\ &\leq 2\pi\sum_{j=0}^{m}\frac{1}{(1+\alpha_j)^N}\left[\left(1+\sum_{i=1}^{m}i!C_{\tilde{\gamma}_1}^{i}\right)\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\}\right.\\ &\qquad\left.+(N-2)\max\{\tilde{\alpha},1-\tilde{\alpha}\}\right]\\ &< 1. \end{aligned} \tag{4.8}$$

Therefore, Theorem 2.2 holds. Furthermore, $w_0 \in X_m$ is also a unique solution of (1.1).

Example 4.3. Considering (1.1), with the functions K, g defined by

$$\begin{cases} K(x,y;u_0,\cdots,u_m) = k(x)K_1(y;u_0,\cdots,u_m),\\ g(x) = w_0(x) - \sum_{j=0}^{m}\frac{1}{(1+\alpha_j)^N}k(x), \end{cases} \tag{4.9}$$

where

$$\begin{cases} K_1(y;u_0,\cdots,u_m) = \sum_{i=0}^{m} y_1^{\alpha_i}\cdots y_N^{\alpha_i}\left|\frac{u_i}{D_1^i w_0(y)}\right|^{\frac{1}{i+2}},\\ w_0(x) = e^{x_1} + x_1^{\gamma_1}\left|x_2-\alpha\right|^{\gamma_2} + \sum_{i=3}^{N}\left|x_i-\alpha\right|,\\ k(x) = x_1^{\tilde{\gamma}_1}\left|x_2-\tilde{\alpha}\right|^{\tilde{\gamma}_2} + \sum_{i=3}^{N}\left|x_i-\tilde{\alpha}\right|, \end{cases} \tag{4.10}$$

and α, γ_1, γ_2, $\tilde{\alpha}$, $\tilde{\gamma}_1$, $\tilde{\gamma}_2$, $\alpha_0, \alpha_1, \cdots, \alpha_m$ are positive constants satisfying

$$\begin{cases} 0<\alpha,\ \tilde{\alpha}<1, \quad 0<\gamma_2,\ \tilde{\gamma}_2\leq 1, \quad \gamma_1>m, \quad \tilde{\gamma}_1>m,\\ \sum\limits_{j=0}^{m}\frac{1}{(1+\alpha_j)^N}\left[\left(1+\sum\limits_{i=1}^{m}i!C_{\tilde{\gamma}_1}^{i}\right)\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\}\right.\\ \qquad\qquad \left.+(N-2)\max\{\tilde{\alpha},1-\tilde{\alpha}\}\right]<1. \end{cases} \tag{4.11}$$

We can prove that (A_1), $(\bar{A}_2)$ hold, because of the following.

First, w_0, $D_1^i w_0 \in X$ and $w_0(x) \geq 1$, $D_1^i w_0(x) \geq 1$. Then $K \in C(\Omega\times\Omega\times\mathbb{R}^{m+1};\mathbb{R})$. By $D_1^i k \in X$, $D_1^i K = D_1^i k(x)K_1(y,u_0,\cdots,u_m)$, so $D_1^i K \in C(\Omega\times\Omega\times\mathbb{R}^{m+1};\mathbb{R})$.

Applying the inequality $a \leq 1 + a^q$, $\forall a \geq 0$, $\forall q \geq 1$, we obtain

$$|K_1(y;u_0,...,u_m)| \leq \sum_{i=0}^{m} y_1^{\alpha_i}...y_N^{\alpha_i}\left(1+\frac{|u_i|}{D_1^i w_0(y)}\right) \leq \sum_{i=0}^{m} y_1^{\alpha_i}...y_N^{\alpha_i}\left(1+\sum_{j=0}^{m}|u_j|\right), \tag{4.12}$$

it gives

$$\left|D_1^i K(x,y;u_0,...,u_m)\right| = \left|D_1^i k(x)\right| |K_1(y;u_0,...,u_m)| \leq \bar{k}_i(x,y)\left(1+\sum_{j=0}^{m}|u_j|\right), \tag{4.13}$$

with $\bar{k}_i(x,y) = \left|D_1^i k(x)\right| \sum_{j=0}^m y_1^{\alpha_j} \cdots y_N^{\alpha_j}$, $i = \overline{0,m}$.

Next,

$$\int_\Omega \bar{k}_i(x,y)dy = \left|D_1^i k(x)\right| \int_\Omega \sum_{j=0}^m y_1^{\alpha_j} \cdots y_N^{\alpha_j} dy = \left|D_1^i k(x)\right| \sum_{j=0}^m \frac{1}{(1+\alpha_j)^N},$$

so

$$\begin{aligned} \bar{\beta} &= \sum_{i=0}^m \sup_{x\in\Omega} \int_\Omega \bar{k}_i(x,y)dy \\ &\leq \sum_{j=0}^m \frac{1}{(1+\alpha_j)^N}\left[\left(1+\sum_{i=1}^m i!C_{\tilde{\gamma}_1}^i\right)\max\{\tilde{\alpha}^{\tilde{\gamma}_2},(1-\tilde{\alpha})^{\tilde{\gamma}_2}\}\right. \\ &\qquad \left. + (N-2)\max\{\tilde{\alpha},1-\tilde{\alpha}\}\right] \\ &< 1. \end{aligned} \tag{4.14}$$

Theorem 3.2 is fulfilled. Furthermore, $w_0 \in X_m$ is also a solution of (1.1).

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