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CERTAIN INTEGRAL FORMULAS INVOLVING THE GENERALIZED K- BESSEL FUNCTIONS

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Abstract. Motivated by Nisar et al. [10], by using MacRobert integral formula, we establish some integral formulas involving the generalized k-Bessel function $w^{\gamma,\lambda}_{k,\nu,c,d}(z)$ of the first kind, which are expressed in terms of the generalized Wright (hypergeometric) functions. Some interesting special cases of our main results are also considered.

1. Introduction and preliminaries

A remakably large number of integral formulas involving a variety of special functions have been developed (see, e.g., [1, 3, 5, 8]). Also, many integral formulas involving the Bessel function $J_\nu(z)$ in (1.2) have been presented (see, e.g., [3, 4, 18]).

Recently, Kahn and Nisar [10] have presented an integral formula involving Wright generalized Bessel function (or generalized Bessel - Maitland function) $J^{\gamma,\lambda}_{k,\nu}(z)$ defined by Singh et al. [16]. Motivated by this work [10], we aim at presenting two definite integral formulas involving generalized k-Bessel function of the first kind, which are expressed in terms of the generalized (Wright)

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hypergeometric functions. For our purpose, we begin by recalling the generalized Bessel function $w_\nu(z)$ of the first kind (see, e.g., [2, p. 10, Eq. (1.15)])

$$w_\nu(z) = \sum_{k=0}^{\infty} \frac{(-c)^k \left(\frac{z}{2}\right)^{\nu+2k}}{k!\,\Gamma(\nu + k + \frac{b'+1}{2})}, \tag{1.1}$$

where $z \in \mathbb{C} \setminus \{0\}$ and $b', c, \nu \in \mathbb{C}$ with $\Re(\nu) > -1$. Here and in the following, let $\mathbb{C}$, $\mathbb{R}^+$ and $\mathbb{N}$ be the sets of complex numbers, positive real numbers, and positive integers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Setting $b' = c = 1$, (1.1) reduces to the Bessel function $J_\nu(z)$ (see, e.g., [18, p. 100])

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{\nu+2k}}{k!\,\Gamma(\nu + k + 1)}, \tag{1.2}$$

where $z \in \mathbb{C} \setminus \{0\}$ and $\nu \in \mathbb{C}$ with $\Re(\nu) > -1$.

The Bessel-Maitland function (or the Wright-generalized Bessel function) is defined by (see [12])

$$J_\nu^\mu(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{\Gamma(\nu + \mu k + 1)}, \tag{1.3}$$

where $z \in \mathbb{C} \setminus \{0\}$ and $\nu, \mu \in \mathbb{C}$ with $\Re(\mu) > 0$.

An interesting generalization of the Bessel-Maitland function is defined by (see [9])

$$J_{\nu,\sigma}^\mu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{\nu+2\sigma+2k}}{\Gamma(\sigma + k + 1)\,\Gamma(\nu + \sigma + \nu k + 1)}, \tag{1.4}$$

where $z, \nu, \sigma \in \mathbb{C}$ and $\mu > 0$. Setting $\sigma = 0$, (1.4) reduces to $J_\nu(z)$. Further, another generalization of the generalized Bessel-Maitland function $J_{\nu,q}^{\mu,\gamma}(z)$ is defined by (see [16])

$$J_{\nu,q}^{\mu,\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk}}{\Gamma(\nu + \mu k + 1)} \frac{(-z)^k}{k!}, \tag{1.5}$$

where $\mu, \nu, \gamma \in \mathbb{C}$; $\Re(\nu) > 0$, $\Re(\mu) > 0$, $\Re(\gamma) > 0$, $q \in (0,1) \cup \mathbb{N}$ and $(\gamma)_{qk} = \frac{\Gamma(\gamma+qk)}{\Gamma(\gamma)}$ denote the generalized Pochhammer symbol.

The k-Wright function is defined by (see [14, 15])

$$w_{k,\beta}^{\gamma,\alpha}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\alpha n + \beta)} \frac{(z)^n}{(n!)^2} \tag{1.6}$$

$$(k \in \mathbb{R}^+;\ \alpha, \gamma, \beta \in \mathbb{C};\ \Re(\alpha) > -1 \text{ and } \Re(\beta) > 0),$$

$(\gamma)_{n,k}$ are given by [6]

$$(\gamma)_{n,k} := \begin{cases} \dfrac{\Gamma_k(\gamma + nk)}{\Gamma_k(\gamma)} & (k \in \mathbb{R}; \gamma \in \mathbb{C} \setminus \{0, -k, -2k, \dots\}) \\ \gamma(\gamma + k)\dots(\gamma + (n-1)k) & (n \in \mathbb{N}; \gamma \in \mathbb{C}) \end{cases} \tag{1.7}$$

and

$$\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right). \tag{1.8}$$

Letting $k = 1$, $\Gamma_k(z)$ reduces to the familiar gamma function $\Gamma(z)$.

Romero et al. [15] introduced the k-Bessel function of the first kind defined by

$$W_{k,\nu}^{\gamma,\lambda}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\lambda n + \nu + 1)} \frac{(-1)^n \left(\frac{z}{2}\right)^{\nu+2n}}{(n!)^2} \tag{1.9}$$

$$(k \in \mathbb{R}^+;\ \lambda, \gamma, \nu \in \mathbb{C};\ \Re(\lambda) > 0 \text{ and } \Re(\nu) > 0).$$

It is noted that the special case of (1.9) when $k = \lambda = \gamma = 1$ reduces to $J_\nu(z)$.

A more generalized form of k-Bessel function of the first kind is given as follows (see [13])

$$w_{k,\nu,b',c}^{\gamma,\alpha}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n(\gamma)_{n,k}}{\Gamma_k(\alpha n + \nu + \frac{b'+1}{2})} \frac{\left(\frac{z}{2}\right)^{\nu+2n}}{(n!)^2} \tag{1.10}$$

$$(k \in \mathbb{R}^+;\ \alpha, \gamma, \nu,\ b',\ c \in \mathbb{C};\ \Re(\nu) > 0).$$

From (1.6) and (1.10), we find

$$w_{k,\nu,1,1}^{\gamma,\alpha}(z) = \left(\frac{z}{2}\right)^{\nu} w_{k,\nu+1}^{\gamma,\alpha}\left(-\frac{z^2}{4}\right) \tag{1.11}$$

It is noted that $w_{1,\nu,b,c}^{1,1}(z)$ is the generalized Bessel function of the first kind $w_\nu(z)$, $w_{1,\nu,1,1}^{1,1}(z)$ is the $J_\nu(z)$ in (1.2), and also $w_{k,\nu,1,1}^{\gamma,\lambda}(z)$ is the k-Bessel function of the first kind $W_{k,\nu}^{\gamma,\lambda}(z)$ in (1.9) .

An interesting further generalization of the generalized hypergeometric series ${}_pF_q$ is due to Fox [7] and Wright [19, 20] who studied the asymptotic expansion of the generalized (Wright) hypergeometric function defined by

$${}_p\Psi_q\left[\begin{array}{l} (\alpha_1, A_1), \dots, (\alpha_p, A_p); \\ (\beta_1, B_1), \dots, (\beta_q, B_q); \end{array} z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(\alpha_j + A_j n)}{\prod_{j=1}^{q}\Gamma(\beta_j + B_j n)} \frac{z^n}{n!}, \tag{1.12}$$

where the coefficients$A_1, \ldots, A_p \in \mathbb{R}^+$ and $B_1, \ldots, B_q \in \mathbb{R}^+$ are positive real numbers such that

$$1 + \sum_{j=0}^{q} B_j - \sum_{j=0}^{p} A_j \geq 0. \tag{1.13}$$

A special case of (1.12) is

$$ {}_p\Psi_q \left[\begin{array}{c} (\alpha_1, 1), \ldots, (\alpha_p, 1); \\ (\beta_1, 1), \ldots, (\beta_q, 1); \end{array} z \right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j)}{\prod_{j=1}^{q} \Gamma(\beta_j)} {}_pF_q \left[\begin{array}{c} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \end{array} z \right], \tag{1.13}$$

where ${}_pF_q$ is the generalized hypergeometric series (see, e.g., [17, Section 1.5])

$$\begin{aligned} {}_pF_q \left[\begin{array}{c} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \end{array} z \right] &= {}_pF_q \left[\alpha_1, \ldots \alpha_p; \beta_1, \ldots, \beta_q; z \right] \\ &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!}, \end{aligned} \tag{1.14}$$

where $(\lambda)_\nu$ denotes the Pochhammer symbol defined (for λ, $\nu \in \mathbb{C}$), in terms of the familiar Gamma function Γ, by (see, e.g., [17, p. 2 and p. 4-6]

$$(\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}). \end{cases} \tag{1.15}$$

For our present investigation, we also need to recall the following MacRobert's integral formula [10]:

$$\begin{aligned} \int_0^1 x^{\alpha-1}\ (1-x)^{\beta-1}\ [ax + b(1-x)]^{-\alpha-\beta}\ dx &= \frac{\Gamma(\alpha)\,\Gamma(\beta)}{a^\alpha b^\beta \Gamma(\alpha + \beta)} \\ &= \frac{B(\alpha, \beta)}{a^\alpha b^\beta}, \end{aligned} \tag{1.16}$$

provided $\Re(\alpha) > 0$ and $\Re(\beta) > 0$.

2. Main results

We establish two integral formulas, which are expressed in terms of the generalized Wright hypergeometric function and the generalized hypergeometric function, by inserting the generalized k-Bessel function of the first kind (1.10) with suitable argument into the integrand of (1.16).

Theorem 2.1. *The following integral formula holds: For* $k \in \mathbb{R}^+$, γ, ν, α', b', $c \in \mathbb{C}$ *with* $\Re(\alpha') > 0$, $\Re(\nu) > 0$, $\Re(\alpha) > 0$, *and* $\Re(\beta) > 0$,

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta}\, w^{\gamma,\alpha'}_{k,\nu,b',c}\left(\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right)dx$$
$$= \frac{1}{a^{\alpha}b^{\beta}\, k^{\frac{2\nu+b'+1}{2k}}\,\Gamma(\frac{\gamma}{k})}$$
$$\times {}_3\Psi_3\left[\begin{array}{c} (\frac{\gamma}{k},1),\,(\alpha+\nu,2),\,(\beta+\nu,2); \\ (1,1),\,(\frac{2\nu+b'+1}{2k},\frac{\alpha'}{k}),\,(\alpha+\beta+2\nu,4); \end{array} -ck^{1-\frac{\alpha'}{k}}\right]. \tag{2.1}$$

Proof. Let $\mathcal{L}$ denote the left hand side of (2.1). By applying (1.10) to the integrand of (2.1) and then interchanging the order of integral sign and summation, which is verified uniform convergence of the involved series under the given condition, we get

$$\mathcal{L} = \sum_{n=0}^{\infty}\frac{(-c)^n(\gamma)_{n,k}(ab)^{\nu+2n}}{\Gamma_k(\alpha' n+\nu+\frac{b'+1}{2})(n!)^2}$$
$$\times\int_0^1 x^{\alpha+\nu+2n-1}(1-x)^{\beta+\nu+2n-1}[ax+b(1-x)]^{-\alpha-\beta-2\nu-4n}\,dx. \tag{2.2}$$

Applying the integral formula (1.16) to the integral in (2.2), we obtain the following expression:

$$\mathcal{L} = \sum_{n=0}^{\infty}\frac{(-c)^n(\gamma)_{n,k}}{\Gamma_k(\alpha' n+\nu+\frac{b'+1}{2})(n!)^2}\frac{\Gamma(\alpha+\nu+2n)\Gamma(\beta+\nu+2n)}{\Gamma(\alpha+\beta+2\nu+4n)}, \tag{2.3}$$

which, upon using (1.7) and (1.8), leads to the right-hand side of (2.1). This completes the proof. □

Next, we consider other expression (2.1). We establish an integral formula for the generalized k-Bessel function of the first kind, which is expressed in terms of the generalized hypergeometric function ${}_pF_q$.

Theorem 2.2. *The following integral formula holds: For $k\in\mathbb{R}^+$, γ, ν, α', b', $c\in\mathbb{C}$ and $\frac{\alpha'}{k}\in\mathbb{N}$ with $\Re(\alpha')>0$, $\Re(\nu)>0$, $\Re(\alpha)>0$, and $\Re(\beta)>0$,*

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta}\, w^{\gamma,\alpha'}_{k,\nu,b',c}\left(\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right)dx$$
$$= \frac{B(\alpha+\nu,\beta+\nu)}{a^{\alpha}b^{\beta}\, k^{\frac{2\nu+b'+1}{2k}}\,\Gamma(\frac{2\nu+b'+1}{2k})}$$
$$\times {}_5F_{(\alpha'/k)+5}\left[\begin{array}{c} \Delta(2;\alpha+\nu),\,\Delta(2;\beta+\nu),\,\frac{\gamma}{k}; \\ \Delta\left(\frac{\alpha'}{k};\frac{2\nu+b'+1}{2k}\right),\,\Delta(4;\alpha+\beta+2\nu),\,1; \end{array} -ck^{1-\frac{\alpha'}{k}}\right], \tag{2.4}$$

where $\Delta(m;l)$ *abbreviates the array of* m *parameters* $\frac{l}{m}, \frac{l+1}{m}, \cdots, \frac{l+m-1}{m}$, $m \in \mathbb{N}$.

Proof. Using

$$(\lambda)_{mn} = m^{mn} \prod_{j=1}^{m} \left(\frac{\lambda+j-1}{m}\right)_n$$

and

$$\Gamma(\lambda+mn) = \Gamma(\lambda)\,(\lambda)_{mn}$$

in (2.3) and summing up the given series with the help of (1.16), we find that, when the lasting resulting summation is expressed in terms of ${}_pF_q$ in (1.14), we get the expression (2.4). □

3. Special cases

We have the following special cases:

(1) Setting $\alpha' = k = \gamma = 1$ in (2.1), we obtain

$$\begin{aligned}&\int_0^1 x^{\alpha-1}\,(1-x)^{\beta-1}\,[ax+b(1-x)]^{-\alpha-\beta}\,w_\nu\left(\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right)dx\\ &= \frac{1}{a^\alpha b^\beta}\,{}_2\Psi_2\left[\begin{matrix}(\alpha+\nu,2),\,(\beta+\nu,2);\\ \left(\frac{2\nu+b'+1}{2},1\right),\,(\alpha+\beta+2\nu,4);\end{matrix}\;-c\right],\end{aligned} \tag{3.1}$$

where $\nu,\,\alpha,\,\beta,\,c \in \mathbb{C}$ with $\Re(\nu)>0$, $\Re(\alpha)>0$, and $\Re(\beta)>0$.

(2) Setting $\alpha' = k = \gamma = 1$ in (2.4), we get

$$\begin{aligned}&\int_0^1 x^{\alpha-1}\,(1-x)^{\beta-1}\,[ax+b(1-x)]^{-\alpha-\beta}\,w_\nu\left(\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right)dx\\ &= \frac{B\,(\alpha+\nu,\beta+\nu)}{a^\alpha b^\beta\,\Gamma\left(\frac{2\nu+b'+1}{2}\right)}\,{}_4F_5\left[\begin{matrix}\Delta(2;\alpha+\nu),\,\Delta(2;\beta+\nu);\\ \Delta\left(1;\frac{2\nu+b'+1}{2}\right),\,\Delta(4;\alpha+\beta+2\nu);\end{matrix}\;-c\right]\end{aligned} \tag{3.2}$$

where $\nu,\,\alpha,\,\beta,\,c \in \mathbb{C}$ with $\Re(\nu)>0$, $\Re(\alpha)>0$, and $\Re(\beta)>0$.

(3) Setting $\alpha' = k = \gamma = b' = c = 1$ in (2.1), we have

$$\begin{aligned}&\int_0^1 x^{\alpha-1}\,(1-x)^{\beta-1}\,[ax+b(1-x)]^{-\alpha-\beta}\,J_\nu\left(\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right)dx\\ &= \frac{1}{a^\alpha b^\beta}\,{}_2\Psi_2\left[\begin{matrix}(\alpha+\nu,2),\,(\beta+\nu,2);\\ (\nu+1,1),\,(\alpha+\beta+2\nu,4);\end{matrix}\;-1\right],\end{aligned} \tag{3.3}$$

where ν, α, $\beta \in \mathbb{C}$ with $\Re(\nu) > 0$, $\Re(\alpha) > 0$, and $\Re(\beta) > 0$.

(4) Setting $\alpha' = k = \gamma = b' = c = 1$ in (2.4), we find

$$\begin{aligned} &\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,[ax+b(1-x)]^{-\alpha-\beta}\,J_\nu\left(\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right)dx \\ &= \frac{B(\alpha+\nu,\beta+\nu)}{a^\alpha b^\beta\,\Gamma(\nu+1)}\,{}_4F_5\left[\begin{array}{c} \Delta(2;\alpha+\nu),\,\Delta(2;\beta+\nu); \\ \Delta(1;\nu+1),\,\Delta(4;\alpha+\beta+2\nu); \end{array} -1\right], \end{aligned} \tag{3.4}$$

where ν, α, β, $c \in \mathbb{C}$ with $\Re(\nu) > 0$, $\Re(\alpha) > 0$, and $\Re(\beta) > 0$.

(5) Setting $b' = c = 1$ and $\alpha' = \lambda$ in (2.1), we get

$$\begin{aligned} &\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,[ax+b(1-x)]^{-\alpha-\beta}\,J_{k,\nu}^{\gamma,\lambda}\left(\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right)dx \\ &= \frac{1}{a^\alpha b^\beta\,k^{\frac{\nu+1}{k}}\,\Gamma(\frac{\gamma}{k})} \\ &\times {}_3\Psi_3\left[\begin{array}{c} (\frac{\gamma}{k},1),\,(\alpha+\nu,2),\,(\beta+\nu,2); \\ (1,1),\,(\frac{\nu+1}{k},\frac{\lambda}{k}),\,(\alpha+\beta+2\nu,4); \end{array} -k^{1-\frac{\lambda}{k}}\right], \end{aligned} \tag{3.5}$$

(6) Setting $b' = c = 1$ and $\alpha' = \lambda$ in (2.4), we have

$$\begin{aligned} &\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,[ax+b(1-x)]^{-\alpha-\beta}\,J_{k,\nu}^{\gamma,\lambda}\left(\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right)dx \\ &= \frac{B(\alpha+\nu,\beta+\nu)}{a^\alpha b^\beta\,k^{\frac{\nu+1}{k}}\,\Gamma(\frac{\nu+1}{k})} \\ &\times {}_5F_{(\lambda/k)+5}\left[\begin{array}{c} \Delta(2;\alpha+\nu),\,\Delta(2;\beta+\nu),\,\frac{\gamma}{k}; \\ \Delta\left(\frac{\lambda}{k};\frac{\nu+1}{k}\right),\,\Delta(4;\alpha+\beta+2\nu),\,1; \end{array} -k^{1-\frac{\lambda}{k}}\right], \end{aligned} \tag{3.6}$$

(7) Using (1.11) and setting $b' = c = 1$ and $\alpha' = \lambda$ in (2.1), we obtain

$$\begin{aligned} &\int_0^1 x^{\alpha+\nu-1}(1-x)^{\beta+\nu-1}\,[ax+b(1-x)]^{-\alpha-\beta-2\nu} \\ &\quad\times w_{k,\nu+1}^{\gamma,\lambda}\left(-\frac{[abx(1-x)]^2}{[ax+b(1-x)]^4}\right)dx \\ &= \frac{1}{a^{\alpha-\nu}b^{\beta-\nu}\,k^{\frac{\nu+1}{k}}\,\Gamma(\frac{\gamma}{k})} \\ &\quad\times {}_3\Psi_3\left[\begin{array}{c} (\frac{\gamma}{k},1),\,(\alpha+\nu,2),\,(\beta+\nu,2); \\ (1,1),\,(\frac{\nu+1}{k},\frac{\lambda}{k}),\,(\alpha+\beta+2\nu,4); \end{array} -k^{1-\frac{\lambda}{k}}\right]. \end{aligned} \tag{3.7}$$

(8) Setting $b' = c = 1$ and $\alpha' = \lambda$ in (2.4), we obtain

$$\begin{aligned} &\int_0^1 x^{\alpha+\nu-1}\,(1-x)^{\beta+\nu-1}\,\left[ax+b(1-x)\right]^{-\alpha-\beta-2\nu} \\ &\quad\times w_{k,\nu+1}^{\gamma,\lambda}\left(-\frac{[abx(1-x)]^2}{[ax+b(1-x)]^4}\right)dx \\ &= \frac{B\left(\alpha+\nu,\beta+\nu\right)}{a^{\alpha-\nu}b^{\beta-\nu}\,k^{\frac{\nu+1}{k}}\,\Gamma(\frac{\nu+1}{k})} \\ &\quad\times {}_5F_{(\lambda/k)+5}\left[\begin{array}{c} \Delta(2;\alpha+\nu),\,\Delta(2;\beta+\nu),\,\frac{\gamma}{k}; \\ \Delta\left(\frac{\lambda}{k};\frac{\nu+1}{k}\right),\,\Delta(4;\alpha+\beta+2\nu),\,1; \end{array} -k^{1-\frac{\lambda}{k}}\right], \end{aligned} \tag{3.8}$$

where $\frac{\lambda}{k} \in \mathbb{N}$.

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