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THE EDGE METRIC DIMENSION OF CAYLEY GRAPHS $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ AND ITS BARYCENTRIC SUBDIVISIONS

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Abstract. The main objective of this study is to determine the edge metric dimension(EMD) of the Cayley graphs $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ and its barycentric subdivision. Infact, it is proved that the Cayley graphs and its subdivisions have constant EMD and its edge metric generator(EMG) set contains only three vertices to resolve all the edges of Cayley graphs $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ and its barycentric subdivisions. In particular EMD remains invariant under the barycentric subdivisions of $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$. On the contrary, in [4] it was proved that the metric dimension of the Cayley graphs $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ does not remain invariant under its barycentric subdivisions.

1. Introduction

Let $G = (V, E)$ be a connected graph. Then the distance between the vertex x and the edge $e = uv \in E$ is defined as $d_G(e, x) = \min\{d_G(u, x), d_G(v, x)\}$. A vertex $x \in V$ *distinguishes* two edges $e, f \in E$ if $d_G(x, e) \neq d_G(x, f)$. Let $\emptyset \neq S \subset V$ be an edge metric generator (EMG) for G if for any e_1 and e_2 in $E(G)$, there is at least one vertex $v \in S$ such that $d_G(e_1, v)$ and $d_G(e_2, v)$ are distinct. Then an EMG with the minimum size is referred to as an edge

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metric basis (EMB) for G, and its size is said to be an edge metric dimension (EMD), which is denoted by $dim_e(G)$. This theory was first dealt by Kelenc, Tratnik and Yero in [14].

The idea of metric dimension was first mentioned by Slater in 1975 to address the issue of exclusively identifying the position of an intruder in a network [16]. Let $\emptyset \neq S \subset V$ be a metric generator for G, if for any two vertices of G, say x and y, there is at least one vertex $v \in S$ such that $d(x,v) \neq d(y,v)$. A metric generator of minimum size is called a *metric basis* for G and its size is the *metric dimension* of G, which is denoted by $dim(G)$.

Harary and Melter went on to explore the concepts further. This graph parameter is useful in the fields of robotics, chemical and computer sciences, for further applications readers can refer to [1, 2, 3]. The families of graphs with constant metric dimension have been characterized by many different authors, one can see [4-11]. There are some other variants of the standard metric dimension that have been studied in recent years. For more details refer to [2, 13].

The new parameter EMD has been recently introduced, and the authors in [14] determined its value for various graphs. Since the determination of the EMD is NP hard, so one has to consider particular classes of graphs to find the EMD. In [19], the author characterized the graphs for which $dim_e(G) = n-1$. In 2018, Peterin and Yero computed the EMD of some graph products in terms of the graphs of the products. In particular, they give the results for the join, lexicographic and the corona products of the graphs[15]. More recently, a characterization of graphs with maximum EMD has been given in [20]. It was proved in [14] that the metric dimension of the wheel graph is strictly less than the EMD, i.e. $dim(W_{n,1}) < dim_e(W_{n,1})$. The authors asked to find some classes of the graph G for which $dim(G) < dim_e(G)$, $dim(G) > dim_e(G)$ or $dim(G) = dim_e(G)$. The metric dimension of the wheel related families have been determined in [17, 18].

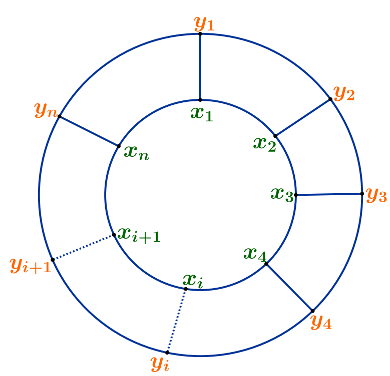
The barycentric subdivision of a graph G is the subdivision in which one new vertex is inserted in the interior of each edge[6]. The barycentric subdivision of any graph is a loopless bipartite graph. A planar graph can be characterized with the help of the subdivision process. A graph G is planar iff every subdivision of G is planar.

Let $S \subset G$ a nonempty subset of a semi group of G. Then the Cayley graph $\Gamma(G,S)$ of G with respect to the set S is a graph with vertex set G, and two vertices x, y are adjacent if $sx = y$ for some $s \in S$. These graphs corresponding to groups play important role in both group theory and graph theory. For any group G the corresponding graph $\Gamma(G,S)$ is symmetric or undirected iff $S = S^{-1}$. The Cayley graph $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$, $n \geq 3$ is a cubic graph. In particular, the Cayley graph $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$,$n \geq 3$ consists of an outer n-cycle $y_1, y_2, \ldots, y_n$, an inner n-cycle $x_1, x_2, \ldots, x_n$,and a set of n spokes $x_iy_i, i = 1, 2, \ldots, n$. We have number of vertices, edge and face sets of the Cayley graph $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ as $|V(\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)| = 2n, |E(\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)| = 3n$ and $|F(\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)| = n+2$. The metric dimension of Cayley graph $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ and its barycentric subdivision have been determined in [1] and [4] respectively. In the same paper, the author noted that under the subdivisions, the metric dimension does not remain same. There are several characterization of families of graph with constant metric dimension, where the metric dimension does not depend on the number of the vertices of the corresponding graphs, for more details, one can see the references [4-12].

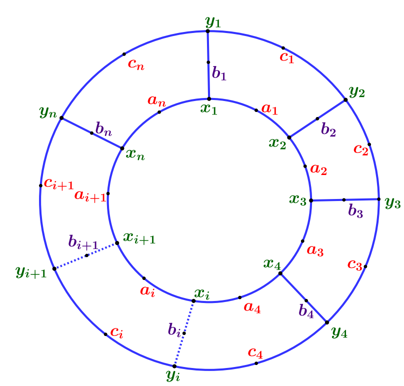
The main objective of this paper is to study the EMD of the Cayley graphs $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ and its barycentric subdivisions. We prove that the Cayley graphs and its subdivisions of have constant EMD and only three vertices are sufficient to resolve all the edges of the graph $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ and its barycentric subdivision.

2. The edge metric dimension of Cayley graphs $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$

In this section the EMD of the graph $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ has been determined and its always greater or equal to the metric dimension of $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$. The following figure shows the Cayley graph $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ and corresponding barycentric subdivision of the Cayley graph $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$.



1(a) Cayley graph



1(b) Barycentric subdivision

Theorem 2.1. *Let* $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ *be the Cayley graphs of the group* $\mathbb{Z}_n \oplus \mathbb{Z}_2$*,* $n \geq 5$*. Then*

$$dim_e(\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)) = 3.$$

Proof. Let us consider the vertex set of the Cayley graph $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ as $V(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2)) = \{x_1, x_2, \ldots, x_n\} \cup \{y_1, y_2, \ldots, y_n\}$. Then the edge set will be a union of three types of edges given by $E(\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)) = \{x_i y_i, x_i x_{i+1}, y_i y_{i+1} : i = 1, 2, \ldots, n\}$, where the indices are taken under mod n. Let $S = \{y_1, y_2, y_{k+1}\}$ be the set of vertices of the Cayley graphs $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$. We claim that S is an EMG for the Cayley graphs $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$. The distance vector depends on the type of n. So we consider two cases:

Even Case. If n is even, i.e. $n = 2l$, with $l = 3, 4, 5, \ldots$. Let e be an edge of the Cayley graphs $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$. Consider the following cases depending on the type of edge e.

1. If $e = x_i y_i$ for some $i \in \{1, 2, 3, \ldots, n\}$, then
$$d(x_i y_i; S) = \begin{cases} (0, 1, l), & i = 1; \\ (i-1, i-2, l-i+1), & 2 \le i \le l+1; \\ (n-i+1, n-i+2, i-l-1), & l+2 \le i \le n. \end{cases}$$
2. If $e = x_i x_{i+1}$ for some $i \in \{1, 2, 3, \ldots, n-1\}$, then
$$d(x_i x_{i+1}; S) = \begin{cases} (0, 0, l-1), & i = 1; \\ (i-1, i-2, l-i), & 2 \le i \le l; \\ (l-1, l-1, 0), & i = l; \\ (n-i, n-i+1, i-l-1), & l+2 \le i \le n. \end{cases}$$
3. If $e = y_i y_{i+1}$ for some $i \in \{1, 2, 3, \ldots, n-1\}$, then
$$d(y_i y_{i+1}; S) = \begin{cases} (1, 1, l), & i = 1; \\ (i, i-1, l-i+1), & 2 \le i \le l; \\ (l, l, 1), & i = l; \\ (n-i+1, n-i+2, i-l), & l+2 \le i \le n. \end{cases}$$

Thus the distance vector representation for any two distinct edges are different, hence $dim_e(\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)) \le 3$.

Let S be an EMG for the graph $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ with size r. We proved that $r \ge 3$. Now, we need to prove that $r \le 3$. Assume on the contrary that there is an EMG $S \subset V(\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2))$ such that $r \le 2$. Then we have three cases to consider:

1. If $S = \{x_i, x_j\}$ such that $1 \le i\ ,\ j \le l+1$ and $i < j$, then observe that
   - $d(x_j x_{j+1}|S) = d(x_j y_j|S) = (j-i, 0)$ for $j - i < l$,
   - $d(x_{j-1} x_j|S) = d(x_j x_{j+1}|S) = (l-1, 0)$ for $j - i = l$.
2. If $S = \{y_i, y_j\}$, then the result would be similar to the above case 1.
3. If $S = \{x_i, y_j\}$, then we have
   - $d(x_n y_n|S) = d(x_2 y_2|S) = (1, 1)$ for $i = j = 1$,
   - $d(x_{i-1} y_{i-1}|S) = d(x_{i+1} y_{i+1}|S) = (1, 1)$ for $i = j \ne 1$,
   - $d(x_i x_{i+1}|S) = d(x_i y_i|S) = (0, j-i)$ for $i < j$,
   - $d(y_j y_{j+1}|S) = d(x_j y_j|S) = (i-j, 0)$ for $i > j$,

which is a contradiction in all three cases, hence, S is not an EMG. Thus $|S| \geq 3$ and $dim_e(\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)) = 3$.

Odd Case. If n is odd, i.e. $n = 2l - 1$, with $l = 3, 4, ...$ Let e be an edge of the Cayley graphs $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$. Consider the following cases on the type of the edge e.

1. If $e = x_i y_i$ for some $i \in \{1, 2, 3, \ldots, n\}$, then
$$d(x_i y_i; S) = \begin{cases} (0, 1, l-1), & i = 1; \\ (i-1, i-2, l-i+1), & 2 \leq i \leq l; \\ (l-1, l-1, 0), & i = l; \\ (n-i+1, n-i+2, i-l-1), & l+2 \leq i \leq n. \end{cases}$$
2. If $e = x_i x_{i+1}$ for some $i \in \{1, 2, 3, \ldots, n-1\}$, then
$$d(x_i x_{i+1}; S) = \begin{cases} (0, 0, l-1), & i = 1; \\ (i-1, i-2, l-i), & 2 \leq i \leq l; \\ (n-i, n-i+1, i-l-1), & l+1 \leq i \leq n. \end{cases}$$
3. If $e = y_i y_{i+1}$ for some $i \in \{1, 2, 3, \ldots, n-1\}$, then
$$d(y_i y_{i+1}; S) = \begin{cases} (1, 1, l), & i = 1; \\ (i, i-1, l-i+1), & 2 \leq i \leq l; \\ (n-i+1, n-i+2, i-l), & l+1 \leq i \leq n. \end{cases}$$

Thus the distance vector representation for any two distinct edges are different, hence $dim_e(\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)) \leq 3$.

Let S be an EMG for the graph $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ with size r. We proved that $r \geq 3$. Now, we need to prove that $r \leq 3$. Assume on the contrary that there is an EMG $S \subset V(\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2))$ such that $r \leq 2$. Then we have three cases to consider:

1. If $S = \{x_i, x_j\}$ such that $1 \leq i$, $j \leq l+1$ and $i < j$, then observe that
   - $d(x_j x_{j+1}|S) = d(x_j y_j|S) = (j-i, 0)$ for $j - i < l$,
   - $d(x_{j-1} x_j|S) = d(x_j y_j|S) = (l-1, 0)$ for $j - i = l$,

which is a contradiction, hence, S is not an EMG.

2. If $S = \{y_i, y_j\}$, then the result would be similar to above case 1.

3. If $S = \{x_i, y_j\}$, then we have
   - $d(x_n y_n|S) = d(x_2 y_2|S) = (1, 1)$ for $i = j = 1$,
   - $d(x_{i-1} y_{i-1}|S) = d(x_{i+1} y_{i+1}|S) = (1, 1)$ for $i = j \neq 1$,
   - $d(x_i x_{i+1}|S) = d(x_i y_i|S) = (0, j-i)$ for $i < j$,
   - $d(y_j y_{j+1}|S) = d(x_j y_j|S) = (i-j, 0)$ for $i > j$,

which is a contradiction, thus, S is not an EMG.

Therefore, $r \geq 3$ and $dim_e(\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)) = 3$. □

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3. The edge metric dimension of barycentric subdivision of Cayley graphs $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$

Let $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ be the Cayley graph. Then its barycentric subdivision is obtained by adding a new vertex on every edge of $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$. In particular, we will add vertices a_i and c_i on the edges of the inner cycle and outer cycle, respectively, and add vertices b_i in between the vertices x_i and y_i of the graph $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$. It is clear that the resulting barycentric subdivisions graph $B\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ has $5n$ vertices and $6n$ edges.

For the sake of simplicity, the cycle induced by $\{x_i, a_i : 1 \le i \le n\}$, will be called the inner cycle, the cycle induced by $\{y_i, c_i : 1 \le i \le n\}$, will be called the outer cycle and set of vertices $\{b_i : 1 \le i \le n\}$, will be called the set of interior vertices of the barycentric subdivision graph $B\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$. The figure 1(b) depicts the the barycentric subdivision graph $B\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ of the Cayley Graphs $\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$.

Theorem 3.1. *If $n \ge 4$, then $dim_e(B\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)) = 3$.*

Proof. The distance vector depends on the type of n. So we consider two cases:

Even Case. If n is even, i.e. $n = 2l$, with $l = 3, 4, \ldots$. Let us consider the subset $S = \{x_1, x_2, x_{l+1}\}$ of the vertices of the barycentric subdivision graph $B\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$. We claim that S is an EMG for the graph $B\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$. Let e be an edge of the barycentric subdivisions graph $B\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$. Consider the following cases on the type of the edge e.

1. If $e = x_i b_i$ for some $i \in \{1, 2, 3, \ldots, 2l\}$, then

$$d(x_ib_i; S) = \begin{cases} (0, 2, 2l), & i = 1; \\ (2, 0, 2l-2), & i = 2; \\ (2i-2, 2i-4, 2l-2i+2), & 3 \le i \le l+1; \\ (4l-2i+2, 4l-2i+4, 2i-2l-2), & l+2 \le i \le 2l. \end{cases}$$

2. If $e = y_i b_i$ for some $i \in \{1, 2, 3, \ldots, 2l\}$, then

$$d(y_ib_i; S) = \begin{cases} (1, 3, 2l+1), & i = 1; \\ (3, 1, 2l-1), & i = 2; \\ (2i-1, 2i-3, 2l-2i+3), & 3 \le i \le l+1; \\ (4l-2i+3, 4l-2i+5, 2i-2l-1), & l+2 \le i \le 2l. \end{cases}$$

3. If $e = y_i c_i$ for some $i \in \{1, 2, 3, \ldots, 2l\}$, then

$$d(y_ic_i;S)=\begin{cases}(2,3,2l+1), & i=1;\\ (2i,2i-2,2l-2i+3), & 2\le i\le l;\\ (2l+1,2l,2), & i=l+1;\\ (4l-2i+3,4l-2i+5,2i-2l), & l+2\le i\le 2l.\end{cases}$$

4. If $e=x_ia_i$ for some $i\in\{1,2,3,\ldots,2l\}$, then

$$d(x_ia_i;S)=\begin{cases}(0,1,2l-1), & i=1;\\ (2i-2,2i-4,2l-2i+1), & 2\le i\le l;\\ (2l-1,2l-2,0), & i=l+1;\\ (4l-2i+1,4l-2i+3,2i-2l-2), & l+2\le i\le 2l.\end{cases}$$

5. If $e=c_iy_{i+1}$ for some $i\in\{1,2,3,\ldots,2l\}$, then

$$d(c_iy_{i+1};S)=\begin{cases}(3,2,2l), & i=1;\\ (2i+1,2i-1,2l-2i+2), & 2\le i\le l;\\ (2l,2l+1,3), & i=l+1;\\ (4l-2i+2,4l-2i+4,4l-2i+1), & l+2\le i\le 2l.\end{cases}$$

6. If $e=a_ix_{i+1}$ for some $i\in\{1,2,3,\ldots,2l\}$, then

$$d(a_ix_{i+1};S)=\begin{cases}(1,0,2l-2), & i=1;\\ (2i-1,2i-3,2l-2i), & 2\le i\le l;\\ (2l-2,2l-1,1), & i=l+1;\\ (4l-2i,4l-2i+2,4l-2i-1), & l+2\le i\le 2l.\end{cases}$$

Thus the distance vector representation for any two distinct edges are different, hence $dim_e(B\Gamma(\mathbb{Z}_n\oplus\mathbb{Z}_2))\le 3$.

Let S be an EMG for the graph $B\Gamma(\mathbb{Z}_n\oplus\mathbb{Z}_2)$ with size r. We proved that $r\le 3$. Now, we need to prove that $r\ge 3$. Assume on the contrary that there is an EMG $S\subset V(B\Gamma(\mathbb{Z}_n\oplus\mathbb{Z}_2))$ such that $r\ge 2$. We restrict the range of n to l due to the symmetry of the graph. Then we have the following cases to consider:

Case 1. Let S be the set of vertices from the inner cycle such that $|S|=2$. Then we have following subcases:

Subcase 1a. Let $S=\{x_i,x_j\}$ s. t. $1\le i,j\le l$. If $i<j$. Then we have $d(a_{i-1}x_i,S)=(0,2j-2i)=d(b_ix_i,S)$ and if $j=l+1,i=1$, then $d(x_1a_n,S)=(0,2l-1)=d(x_1a_1,S)$, which is a contradiction.

Subcase 1b. If $S=\{a_i,a_j\}$ s. t. $1\le i,j\ge l$ and $i<j$, then we obtain that $d(a_{i-1}x_i,S)=(1,2j-2i+1)=d(b_ix_i,S)$ and if $j=l+1,i=1$, then $d(x_1a_1,S)=(0,2l-1)=d(x_2a_1,S)$, which is a contradiction.

Subcase 1c. Let $S=\{x_i,a_j\}$ s. t. $1\le i,j\le l$. If $i<j$, then we obtained $d(a_{i-1}x_i,S)=(0,2j-2i+1)=d(b_ix_i,S)$ and if $i>j$, then $d(a_{j-1}x_j,S)=(1,2i-2j)=d(b_{j-1}x_{j-1},S)$ and for $S=\{x_1,a_{l+1}\}$, we have $d(a_1x_1,S)=(0,2l-1)=d(b_1x_1,S)$ and similarly for $S=\{x_{l+1},a_1\}$, we have $d(a_{l+1}x_{l+1},S)=(0,2l-1)=d(b_{l+1}x_{l+1},S)$, which is a contradiction.

Case 2. Let $S = \{b_i, b_j\}$ s. t. $1 \leq i, j \leq l+1$. If $i < j$. Then it is obvious that $d(x_ib_i, S) = (0, 2j-2i+1) = d(b_iy_i, S)$, which is a contradiction.

Case 3. Let S be the set of vertices from the outer cycle such that $|S| = 2$. In particular $S = \{y_i, c_i\}$ for every $1 \leqslant i \leqslant n$. Then due to graph symmetry, this case is similar to Case 1.

Case 4. Let $S \subseteq \{x_i, a_l, b_j\}$, $\forall$ $\{1 \leqslant i, j, l \leqslant n\}$. Then we have the following two cases to consider depending on the types of the vertices of the set S.

Subcase 4a. If $S = \{x_i, b_j\}$ s. t. $1 \leqslant i, j \leqslant l$ and if $i < j$, then it is clear that $d(a_{i-1}x_i, S) = (0, 2j-2i+1) = d(b_ix_i, S)$ and if $i > j$, then $d(a_ix_j, S) = (0, 2i-2j+1) = d(b_ix_i, S)$ and for $i = j$ then $d(a_{i-1}x_i, S) = (0, 1) = d(a_ix_i, S)$, which is a contradiction.

Subcase 4b. If $S = \{a_i, b_j\}$ s. t. $1 \leqslant i, j \leqslant l$ and if $i < j$, then it is clear that $d(a_{i-1}x_i, S) = (1, 2j-2i+1) = d(b_ix_i, S)$ and if $i > j$, then $d(a_ix_i, S) = (1, 2i-2j+1) = d(b_ix_i, S)$ and for $i = j$ then $d(a_{i+1}x_{i+1}, S) = (1, 3) = d(b_{i+1}x_{i+1}, S)$, which is a contradiction.

Subcase 4c. If $i = 1, j = l+1$ s. t. $S = \{a_1, b_{l+1}\}$ then $d(c_ny_1, S) = d(c_1y_1, S) = (3, 2l)$. If $i = l+1, j = 1$, s. t. $S = \{a_{l+1}, b_1\}$ then $d(c_{l+1}y_{l+1}, S) = d(c_ly_{l+1}, S) = (3, 2l)$, which is a contradiction.

Case 5. Let $S \subseteq \{y_i, c_i, b_i\}$. Then due to graph symmetry, this case is similar to case 4.

Case 6. If $S \subseteq \{x_i, y_j, a_l, c_l\}$

Subcase 6a. If $S = \{x_i, y_j\}$ s. t. $1 \leqslant i, j \leqslant l$ and if $i < j$, then it is easy to see that $d(a_{i-1}x_i, S) = (0, 2j-2i+2) = d(b_ix_i, S)$ and if $i > j$, then $d(a_ix_i, S) = (0, 2i-2j+2) = d(b_ix_i, S)$ and for $i = j$ then $d(a_{i-1}x_i, S) = (0, 2) = d(a_ix_i, S)$, which si a contradiction.

If $i = 1, j = l+1$ s. t. $S = \{x_1, y_{l+1}\}$ then $d(a_{2l}x_1, S) = d(a_1x_1, S) = (0, 2l+1)$ and if $i = l+1, j = 1$, s. t. $S = \{x_{l+1}, y_1\}$ then $d(a_{2l}x_1, S) = d(a_1x_1, S) = (2, 2l-1)$, which is a contradiction.

Subcase 6b. If $S = \{x_i, c_j\}$ s. t. $1 \leqslant i, j \leqslant l+1$ and if $i < j$, then it is easy to see that $d(a_ix_i, S) = (0, 2j-2i+2) = d(b_ix_i, S)$ and if $i > j$, then $d(a_{i-1}x_i, S) = (0, 2i-2j) = d(b_ix_i, S)$ and if $i = j$ then $d(c_{i-1}y_i, S) = (1, 1) = d(b_iy_i, S)$, which si a contradiction.

If $i = 1, j = l+1$ s. t. $S = \{x_1, c_{l+1}\}$ then $d(c_ly_{l+1}, S) = d(b_ly_l, S) = (1, 2l+1)$ and if $i = l+1, j = 1$, then $S = \{x_{l+1}, c_1\}$ then $d(c_{2l}y_1, S) = d(b_1y_1, S) = (1, 2l+1)$, which is a contradiction.

Subcase 6c. If $S = \{a_i, y_i\}$, then it is similar to 6b.

Subcase 6d. If $S = \{a_i, c_j\}$ s. t. $1 \leqslant i, j \leqslant l+1$ and if $i < j+2$ and $j \neq i, i+1$, then it is clear that $d(a_{i+1}x_{i+1}, S) = (1, 2j-2i+4) = d(b_{i+1}x_{i+1}, S)$ and if $i \geqslant j+2$ and $i \neq j, j+1$, then $d(c_{i+1}y_{i+1}, S) = (2, 2i-2j+1) = d(b_{i+1}y_{i+1}, S)$. For $i+1 = j$, $S = \{a_ic_{i+1}\}$ $d(c_iy_i, S) = (3, 2) = d(b_{i+2}x_{i+2}, S)$, which is a contradiction. If $i = j$, then $d(a_ix_i, S) = d(a_ix_{i+1}, S) = (0,3)$. If $S = \{a_{i+1}, c_i\}$, then $d(a_ix_i, S) = (2,3) = d(b_{i+2}y_{i+2}, S)$. If $S = \{a_1, c_{l+1}\}$, then $d(a_1x_2, S) = (0, 2l+1) = d(a_1x_2, S)$. If $S = \{a_{l+1}, c_1\}$, then $d(a_{l+1}x_{l+1}, S) = (0, 2l+1) = d(a_{l+1}x_{l+2}, S)$, which is a contradiction.

Odd Case. If n is odd, i.e. $n = 2l+1$, with $l = 3, 4, \ldots$. Let us consider the subset $S = \{x_1, x_2, a_{l+1}\}$ of the vertices of the barycentric subdivision graph $B\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$. We claim that S is an EMG for the graph $B\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$. Let e be an edge of the barycentric subdivision graph $B\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)$. Consider the following cases on the type of the edge e.

1. If $e = x_ib_i$ for some $i \in \{1,2,3,\ldots,2l+1\}$, then
$$d(x_ib_i; S) = \begin{cases} (0,2,2l+1), i = 1; \\ (2i-2, 2i-4, 2l-2i+3),\ 2 \leqslant i \leqslant l+1; \\ (2l, 2l, 1), i = l+2; \\ (4l-2i+4, 4l-2i+6, 2i-2l-3), l+3 \leq i \leq 2l+1. \end{cases}$$
2. If $e = y_ib_i$ for some $i \in \{1,2,3,\ldots,2l+1\}$, then
$$d(x_ib_i; S) = \begin{cases} (1,3,2l+2), i = 1; \\ (2i-1, 2i-3, 2l-2i+4),\ 2 \leqslant i \leqslant l+1; \\ (2l+1, 2l+1, 2), i = l+2; \\ (4l-2i+5, 4l-2i+7, 2i-2l-2), l+3 \leq i \leq 2l+1. \end{cases}$$
3. If $e = y_ic_i$ for some $i \in \{1,2,3,\ldots,2l+1\}$, then
$$d(y_ic_i; S) = \begin{cases} (0,1,2l), i = 1; \\ (2i-2, 2i-4, 2l-2i+2),\ 2 \leq i \leq l+1; \\ (2l-1, 2l, 1), i = l+2; \\ (4l-2i+3, 4l-2i+5, 2i-2l-3), l+3 \leq i \leq 2l+1. \end{cases}$$
4. If $e = x_ia_i$ for some $i \in \{1,2,3,\ldots,2l+1\}$, then
$$d(x_ia_i; S) = \begin{cases} (0,1,2l), i = 1; \\ (2i-2, 2i-4, 2l-2i+2),\ 2 \leq i \leq l+1; \\ (2l-1, 2l, 1), i = l+2; \\ (4l-2i+3, 4l-2i+5, 2i-2l-3), l+3 \leq i \leq 2l+1. \end{cases}$$
5. If $e = c_iy_{i+1}$ for some $i \in \{1,2,3,\ldots,2l+1\}$, then
$$d(c_iy_{i+1}; S) = \begin{cases} (3,2,2l+1), i = 1; \\ (2i+1, 2i-1, 2l-2i+3),\ 2 \leq i \leq l; \\ (2l+2, 2l+1, 3), i = l+1; \\ (4l-2i+4, 4l-2i+6, 2i-2l), l+2 \leq i \leq 2l+1. \end{cases}$$
6. If $e = a_ix_{i+1}$ for some $i \in \{1,2,3,\ldots,2l\}$, then
$$d(a_ix_{i+1}; S) = \begin{cases} (1,0,2l-1), i = 1; \\ (2i-1, 2i-3, 2l-2i+1),\ 2 \leq i \leq l; \\ (2l, 2l-1, 0), i = l+1; \\ (4l-2i+2, 4l-2i+4, 2i-2l-2), l+2 \leq i \leq 2l+1. \end{cases}$$

Thus the distance vector representation for any two distinct edges are different, hence $dim_e(B\Gamma(\mathbb{Z}_n \oplus \mathbb{Z}_2)) \leq 3$. The converse is the same as for the even case, so we omit the proof. This completes the proof of theorem. □

4. Conclusion

To find the EMD of a graph is an NP-complete problem. In this paper, the EMD for Cayley graphs and its barycentric subdivisions have been determined. It is proved that these families of graphs have constant EMD and only three vertices are sufficient to resolve all the edges of the Cayley graphs and its barycentric subdivisions subdivisions(chosen appropriately). The EMD remains unchanged under the barycentric subdivisions of Cayley graphs. Therefore, it is natural to asked the following question.

Open Problem: If $B(G)$ denote the graph obtained after the barycentric subdivision of the graph G, then under what conditions on graphs does one has the following equality $dim_e(BG) = dim_e(G)$.

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