**Nonlinear Functional Analysis and Applications**
**Vol. 24, No. 4 (2019), pp. 813-826**
**ISSN: 1229-1595(print), 2466-0973(online)**

http://nfaa.kyungnam.ac.kr/journal-nfaa


# INEQUALITIES CONCERNING THE POLAR DERIVATIVES OF POLYNOMIALS WITH RESTRICTED ZEROS

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**Abstract.** In this paper some inequalities for the maximum modulus of the polar derivative for polynomials with restricted zeros are obtained by using the boundary Schwarz lemma of Osserman. Our results generalize and refine some well-known results concerning the polynomials due to Turán, Dubinin and others.

## 1. Introduction and preliminaries

Let $\mathcal{P}_n$ denote the class of all algebraic polynomials of the form:

$$P(z) = \sum_{j=o}^{n} a_j z^j, \ a_n \neq 0, \ n \geq 1.$$

[0]Received April 17. 2019. Revised October 3, 2019.
[0]2010 Mathematics Subject Classification: 26D10, 41A17, 30C15.
[0]Keywords: Polynomials, Schwarz lemma, inequalities in the complex domain.
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It was conjectured by Erdös and later verified by Lax [9] that if $P \in \mathcal{P}_n$ does not vanish in $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.1}$$

On the other hand Turán [15] showed that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.2}$$

Equality in (1.1) and (1.2) holds for $P(z) = az^n + b$, $|a| = |b| \neq 0$.

As an extension of (1.2), Govil [8] proved that if $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k, k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|. \tag{1.3}$$

The result is sharp as shown by the polynomial $P(z) = z^n + k^n$.

By involving the minimum modulus of $P(z)$ on $|z| = 1$, Aziz and Dawood [2] proved under the hypothesis of inequality (1.2) that

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \left\{ \max_{|z|=1} |P(z)| + \min_{|z|=1} |P(z)| \right\}. \tag{1.4}$$

Equality in (1.4) holds for $P(z) = az^n + b$, $|a| = |b| \neq 0$.

In literature, there exist several generalizations and extensions of (1.2), (1.3) and (1.4) (see [1]-[5], [11], [13], [14]). Dubinin [7] obtain a refinement of (1.2) by involving some of the coefficients of polynomial $P \in \mathcal{P}_n$ in the bound of inequality (1.2). More precisely, proved that if all the zeros of the polynomial $P \in \mathcal{P}_n$ lie in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{1}{2} \left( n + \frac{|a_n| - |a_0|}{|a_n| + |a_0|} \right) \max_{|z|=1} |P(z)|. \tag{1.5}$$

The polar derivative $D_\alpha P(z)$ of $P \in \mathcal{P}_n$ with respect to the point $\alpha \in \mathbb{C}$ is defined by

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n - 1$ and it generalizes the ordinary derivative $P'(z)$ of $P(z)$ in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z)$$

uniformly for $|z| \leq R$, $R > 0$.

Aziz [1], Aziz and Rather ([4], [5]) obtained several sharp estimates for maximum modulus of $D_\alpha P(z)$ on $|z| = 1$ and among other things they extended inequality (1.3) to the polar derivative of a polynomial by showing that if

$P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \le k, k \ge 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \ge k$

$$\max_{|z|=1} |D_\alpha P(z)| \ge \frac{n(|\alpha| - k)}{1 + k^n} \max_{|z|=1} |P(z)|. \tag{1.6}$$

In this paper, we are interested in estimating the lower bound for the maximum modulus of the polar derivative of $P(z)$ on $|z| = 1$ for $P \in \mathcal{P}_n$ not vanishing in the region $|z| > k$ where $k \ge 1$ and establish some refinements and generalizations of the inequalities (1.2), (1.3), (1.4), (1.5) and (1.6).

For the proof of theorems, we need following lemmas. The first lemma is a special case of a result due to Aziz and Rather[3, 4].

**Lemma 1.1.** *If $P \in \mathcal{P}_n$ and $P(z)$ has its all zeros in $|z| \le 1$ and $Q(z) = z^n \overline{P(1/\overline{z})}$, then for $|z| = 1$,*

$$|Q'(z)| \le |P'(z)|.$$

**Lemma 1.2.** If all the zeros of $P \in \mathcal{P}_n$ lie in a circular region C and $w$ is any zero of $D_\alpha P(z)$, the polar derivative of $P(z)$, then at most one of the points $w$ and $\alpha$ may lie outside C.

The above Lemma is due to Laguerre (see [10]). Next lemma is due to Frappier, Rahman and Ruscheweyh [6].

**Lemma 1.3.** *If $P \in \mathcal{P}_n$ is a polynomial of degree $n \ge 1$, then for $R \ge 1$,*

$$\max_{|z|=R} |P(z)| \le R^n \max_{|z|=1} |P(z)| - (R^n - R^{n-2})|P(0)| \quad \text{if} \quad n > 1 \tag{1.7}$$

and

$$\max_{|z|=R} |P(z)| \le R \max_{|z|=1} |P(z)| - (R - 1)|P(0)| \quad \text{if} \quad n = 1. \tag{1.8}$$

From above lemma, we deduce:

**Lemma 1.4.** *If $P \in \mathcal{P}_n = a_n \prod_{j=1}^{n}(z - z_j)$ is a polynomial of degree $n \ge 2$ having no zeros in $|z| < 1$, then for every $\gamma \in \mathbb{C}$ with $|\gamma| \le 1$ and $R \ge 1$,*

$$\begin{aligned} \max_{|z|=R} |P(z)| \le \frac{R^n + 1}{2} \max_{|z|=1} |P(z)| - |\gamma| \frac{R^n - 1}{2} \min_{|z|=1} |P(z)| \\ - \left( \frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{n - 2} \right) |P'(0)| \qquad \text{if} \quad n > 2 \end{aligned} \tag{1.9}$$

and

$$\max_{|z|=R}|P(z)| \le \frac{R^2+1}{2}\max_{|z|=1}|P(z)| - |\gamma|\frac{R^2-1}{2}\min_{|z|=1}|P(z)| - \frac{(R-1)^2}{2}|P'(0)| \quad \text{if} \quad n=2. \tag{1.10}$$

*Proof.* By hypothesis, all the zeros of $P(z)$ lie in $|z| \ge 1$. Let $m = \min_{|z|=1}|P(z)|$. Then $m \le |P(z)|$ for $|z|=1$. Applying Rouche's theorem, it follows that the polynomial $G(z) = P(z) + \gamma m z^n$ has all its zeros in $|z| \ge 1$ for every $\gamma$ with $|\gamma| < 1$ (this is trivially true for $m=0$). Now for each $\theta$, $0 \le \theta < 2\pi$, we have

$$G(Re^{i\theta}) - G(e^{i\theta}) = \int_1^R e^{i\theta}G'(te^{i\theta})dt. \tag{1.11}$$

This gives with the help of (1.7) of Lemma 1.3 and inequality (1.1) for $n>2$,

$$\begin{aligned}\left|G(Re^{i\theta}) - G(e^{i\theta})\right| &\le \int_1^R |G'(te^{i\theta})|dt \\ &\le \frac{n}{2}\left(\int_1^R t^{n-1}dt\right)\max_{|z|=1}|G(z)| - \int_1^R \left(t^{n-1} - t^{n-3}\right)dt|G'(0)| \\ &= \frac{R^n-1}{2}\max_{|z|=1}|G(z)| - \left(\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\right)|P'(0)|,\end{aligned}$$

so that for $n>2$ and $0 \le \theta < 2\pi$, we have

$$\begin{aligned}\left|G(Re^{i\theta})\right| &\le \left|G(Re^{i\theta}) - G(e^{i\theta})\right| + \left|G(e^{i\theta})\right| \\ &= \frac{R^n+1}{2}\max_{|z|=1}|G(z)| - \left(\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\right)|P'(0)|.\end{aligned}$$

Replacing $G(z)$ by $P(z) + \gamma m z^n$, we get for $|z|=1$,

$$|P(Rz) + \gamma m R^n z^n| \le \frac{R^n+1}{2}\max_{|z|=1}|P(z) + \gamma m z^n| - \left(\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\right)|P'(0)|. \tag{1.12}$$

Choosing argument of $\gamma$ in the left hand side of (1.12) suitably, we obtain for $n>2$ and $|z|=1$,

$$|P(Rz)| + |\gamma| m R^n \le \frac{R^n+1}{2}\left\{\max_{|z|=1}|P(z)| + |\gamma| m\right\} - \left(\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\right)|P'(0)|,$$

equivalently for $n>2$, $|\gamma|<1$ and $|z|=1$, we have

$$|P(Rz)| \le \frac{R^n+1}{2}\max_{|z|=1}|P(z)| - |\gamma|\frac{R^n-1}{2}\min_{|z|=1}|P(z)| - \left(\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\right)|P'(0)|,$$

which proves inequality (1.9) for $n>2$ and $|\gamma|<1$. Similarly we can prove inequality (1.10) for $n=2$ by using (1.8) of Lemma 1.3 instead of (1.7). For $|\gamma|=1$, the result follows by continuity. This completes the proof of Lemma 1.4. □

Next we prove the following lemma:

**Lemma 1.5.** *If $P\in\mathcal{P}_n$ has all its zeros in $|z|\le k$ where $k\ge 1$, then for $0\le l<1$,*

$$\max_{|z|=k}|P(z)| \ge \frac{2k^n}{1+k^n}\max_{|z|=1}|P(z)| + l\left(\frac{k^n-1}{k^n+1}\right)\min_{|z|=k}|P(z)| + \frac{2k^{n-1}|a_{n-1}|}{k^n+1}\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right), \quad \text{if } n>2 \tag{1.13}$$

*and*

$$\max_{|z|=k}|P(z)| \ge \frac{2k^2}{1+k^2}\max_{|z|=1}|P(z)| + l\left(\frac{k^2-1}{k^2+1}\right)\min_{|z|=k}|P(z)| + \frac{k(k-1)^2|a_1|}{k^2+1}, \quad \text{if } n=2. \tag{1.14}$$

*Proof.* Since all the zeros of $P\in\mathcal{P}_n$ lie in $|z|\le k, k\ge 1$, therefore, all the zeros of $g(z)=P(kz)$ lie in $|z|\le 1$ and hence $g^*(z)=z^n\overline{g(1/\overline{z})}=z^n\overline{P(k/\overline{z})}$ does not vanish in $|z|<1$. Applying (1.9) of Lemma 1.4 to $g^*(z)$ with $R=k\le 1, |\gamma|<1$ and $m=\min_{|z|=k}|P(z)|=\min_{|z|=1}|g^*(z)|$, we obtain for $n>2$,

$$\max_{|z|=k}|g^*(z)| \le \frac{k^n+1}{2}\max_{|z|=1}|g^*(z)| - |\gamma|\left(\frac{k^n-1}{2}\right)m - \left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right)k^{n-1}|a_{n-1}|.$$

This implies for $n>2$,

$$k^n\max_{|z|=1}|P(z)| \le \frac{k^n+1}{2}\max_{|z|=k}|P(z)| - |\gamma|\left(\frac{k^n-1}{2}\right)m - \left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right)k^{n-1}|a_{n-1}|,$$

which on simplification yields inequality (1.13). In a similar manner we can prove inequality (1.14) by applying inequality (1.10) of lemma 1.4 instead of inequality (1.9) to the polynomial $g^*(z)$. This proves Lemma 1.5. □

Finally we need the following lemma due to Osserman [12], known as boundary Schwarz lemma.

**Lemma 1.6.** *Let the following conditions satisfy:*

(a) $f(z)$ *is analytic for* $|z| < 1$,
(b) $|f(z)| < 1$ *for* $|z| < 1$,
(c) $f(0) = 0$,
(d) *for some* $b$ *with* $|b| = 1, f(z)$ *extends continuously to* $b$, $|f(b)| = 1$ *and* $f'(b)$ *exists.*

*Then we have*

$$|f'(b)| \geq \frac{2}{1 + |f'(0)|}.$$

## 2. Main results

**Theorem 2.1.** *If all the zeros of polynomial* $P \in \mathcal{P}_n$ *of degree* $n \geq 2$ *lie in* $|z| \leq k, k \geq 1$, *then for every* $\alpha \in \mathbb{C}$ *with* $|\alpha| \geq k$,

$$\max_{|z|=1} |D_\alpha P(z)| \geq \frac{|\alpha| - k}{1 + k^n}\left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)\left\{\max_{|z|=1} |P(z)| + \frac{|a_{n-1}|}{k}\phi(k)\right\} + |na_0 + \alpha a_1|\psi(k), \tag{2.1}$$

*where* $\phi(k) = \left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right)$ *or* $\frac{(k-1)^2}{2}$ *and* $\psi(k) = (1 - 1/k^2)$ or $(1 - 1/k)$ *according as* $n > 2$ *or* $n = 2$.

*Proof.* Let $f(z) = P(kz)$. Since $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k$ where $k \geq 1$, therefore, $f \in \mathcal{P}_n$ and $f(z)$ has all its zeros in $|z| \leq 1$. If $Q(z) = z^n\overline{f(1/\overline{z})}$, then it is easy to verify that

$$|Q'(z)| = |nf(z) - zf'(z)| \qquad \text{for} \quad |z| = 1.$$

By using Lemma 1.1, we get

$$|f'(z)| \geq |nf(z) - zf'(z)| \qquad \text{for} \quad |z| = 1. \tag{2.2}$$

Now for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$, we have for $|z| = 1$,

$$\begin{aligned} |D_{\alpha/k}f(z)| &= |nf(z) + (\alpha/k - z)f'(z)| \\ &\geq |\alpha/k||f'(z)| - |nf(z) - zf'(z)|, \end{aligned}$$

which gives with the help of (2.2) for $|z| = 1$ and $|\alpha| \geq k$,

$$|D_{\alpha/k}f(z)| \geq \left(\frac{|\alpha| - k}{k}\right)|f'(z)|, \tag{2.3}$$

consequently,

$$\max_{|z|=k}|D_\alpha P(z)| \geq (|\alpha| - k)\max_{|z|=k}|P'(z)|. \tag{2.4}$$

Again since all the zeros of $f(z) = P(kz)$ lie in $|z| \leq 1$ and hence all the zeros of polynomial $z^n\overline{f(1/\bar{z})}$ lie in $|z| \geq 1$. Therefore, the function

$$F(z) = \frac{f(z)}{z^{n-1}\overline{f(1/\bar{z})}} = z\frac{a_n}{\bar{a_n}}\prod_{j=1}^{n}\left(\frac{kz - z_j}{k - z\bar{z}_j}\right) \tag{2.5}$$

is analytic in $|z| < 1$ with $F(0) = 0$ and $|F(z)| = 1$ for $|z| = 1$. Further for $|z| = 1$, this gives

$$\frac{zF'(z)}{F(z)} = 1 - n + \frac{zf'(z)}{f(z)} + \overline{\left(\frac{zf'(z)}{f(z)}\right)}$$

so that

$$Re\left(\frac{zF'(z)}{F(z)}\right) = 1 - n + 2Re\left(\frac{zf'(z)}{f(z)}\right). \tag{2.6}$$

Also, we have from (2.5),

$$\frac{zF'(z)}{F(z)} = 1 + \sum_{j=1}^{n}\left(\frac{k^2 - |z_j|^2}{|kz - z_j|^2}\right) > 0 \quad \text{for} \quad |z| = 1,$$

as such,

$$\frac{zF'(z)}{F(z)} = \left|\frac{zF'(z)}{F(z)}\right| = |F'(z)| \qquad \text{for} \quad |z| = 1.$$

Using this fact in (2.6), we get for points $z$ on $|z| = 1$ with $f(z) \neq 0$,

$$1 - n + 2Re\left(\frac{zf'(z)}{f(z)}\right) = |F'(z)|. \tag{2.7}$$

Applying Lemma 1.6 to $F(z)$, we obtain for all points $z$ on $|z| = 1$ with $f(z) \neq 0$,

$$1 - n + 2Re\left(\frac{zf'(z)}{f(z)}\right) \geq \frac{2}{1 + |F'(0)|},$$

that is, for $|z| = 1$ with $f(z) \neq 0$,

$$Re\left(\frac{zf'(z)}{f(z)}\right) \geq \frac{1}{2}\left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right).$$

This implies

$$\left|\frac{zf'(z)}{f(z)}\right| \geq \frac{1}{2}\left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right) \qquad \text{for} \quad |z| = 1,\ f(z) \neq 0,$$

and hence,

$$|f'(z)| \geq \frac{1}{2}\left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)|f(z)|, \qquad \text{for} \quad |z| = 1. \tag{2.8}$$

Replacing $f(z)$ by $P(kz)$, we obtain

$$k|P'(kz)| \geq \frac{1}{2}\left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)|P(kz)|, \qquad \text{for} \quad |z| = 1,$$

which implies,

$$\max_{|z|=k}|P'(z)| \geq \frac{1}{2k}\left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)\max_{|z|=k}|P(z)|.$$

Combining this with inequality (2.4), we get

$$\max_{|z|=k}|D_\alpha P(z)| \geq \frac{(|\alpha| - k)}{2k}\left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)\max_{|z|=k}|P(z)|. \tag{2.9}$$

Further since $D_\alpha P(z)$ is a polynomial of degree at most $n-1$, by inequality (1.7) of Lemma 1.3, we have for $n > 2$

$$\max_{|z|=R}|D_\alpha P(z)| \leq R^{n-1}\max_{|z|=1}|D_\alpha P(z)| - (R^{n-1} - R^{n-3})|na_0 + \alpha a_1|. \tag{2.10}$$

Using inequality (2.10) with $R = k \geq 1$ and (1.13) of Lemma 1.5 with $l = 0$ in (2.9), we obtain for $n > 2$,

$$\begin{aligned}
&k^{n-1}\max_{|z|=1}|D_\alpha P(z)| - (k^{n-1} - k^{n-3})|na_0 + \alpha a_1| \\
&\geq \frac{(|\alpha| - k)}{2k}\left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right) \\
&\quad \times \left\{\frac{2k^n}{1 + k^n}\max_{|z|=1}|P(z)| + \frac{2k^{n-1}|a_{n-1}|}{k^n + 1}\left(\frac{k^n - 1}{n} - \frac{k^{n-2} - 1}{n-2}\right)\right\},
\end{aligned}$$

which on simplification gives for $n > 2$,

$$\begin{aligned}
\max_{|z|=1}|D_\alpha P(z)| \geq{}& \frac{|\alpha| - k}{1 + k^n}\left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right) \\
&\times \left\{\max_{|z|=1}|P(z)| + \frac{|a_{n-1}|}{k}\left(\frac{k^n - 1}{n} - \frac{k^{n-2} - 1}{n-2}\right)\right\} \\
&+ \left(1 - 1/k^2\right)|na_0 + \alpha a_1|.
\end{aligned}$$

This proves Theorem 2.1 for $n > 2$. Similarly we can prove Theorem 2.1 for the case $n = 2$ by using (1.8) of Lemma 1.3 instead of (1.7) and (1.14) of Lemma 1.5 instead of (1.13). This proves Theorem 2.1 completely. □

**Remark 2.2.** Since all the zeros of $P(z)$ lie in $|z| \le k$, $k \ge 1$, it follows that $|a_0| \le k^n|a_n|$. In view of this inequality (2.1) refines inequality (1.6).

If we divide the two sides of (2.1) by $|\alpha|$ and let $|\alpha| \to \infty$, we get The following result.

**Corollary 2.3.** *If all the zeros of polynomial $P \in \mathcal{P}_n$ of degree $n \ge 2$ lie in $|z| \le k$, $k \ge 1$, then*

$$\begin{aligned}\max_{|z|=1}|P'(z)| \ge \frac{1}{1+k^n}\left(n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\max_{|z|=1}|P(z)| \\ + \frac{|a_{n-1}|}{k(1+k^n)}\left(n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\phi(k)+|a_1|\psi(k),\end{aligned} \tag{2.11}$$

*where $\phi(k)$ and $\psi(k)$ are same as defined in Theorem* 2.1.

The result is best possible and equality in (2.11) holds for $P(z) = z^n + k^n$.

**Remark 2.4.** As in Remark 2.2, it can be easily seen that inequality 2.11 constitutes a refinement of inequality (1.3). Further, inequality (2.11) reduces to inequality (1.5) for $k = 1$.

**Theorem 2.5.** *If all the zeros of polynomial $P \in \mathcal{P}_n$ of degree $n \ge 2$ lie in $|z| \le k$ where $k \ge 1$ and $m = \min_{|z|=k}|P(z)|$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \ge k$ and $0 \le l < 1$,*

$$\begin{aligned}\max_{|z|=1}|D_\alpha P(z)| \ge \frac{n}{1+k^n}\left\{(|\alpha|-k)\max_{|z|=1}|P(z)|+(|\alpha|+1/k^{n-1})lm\right\} \\ + \frac{(|\alpha|-k)}{k^n(k^n+1)}\left(\frac{k^n|a_n|-lm-|a_0|}{k^n|a_n|-lm+|a_0|}\right)\left(k^n\max_{|z|=1}|P(z)|-lm\right) \\ + \frac{(|\alpha|-k)|a_{n-1}|}{k(1+k^n)}\left(n+\frac{k^n|a_n|-lm-|a_0|}{k^n|a_n|-lm+|a_0|}\right)\phi(k) \\ + |na_0+\alpha a_1|\psi(k).\end{aligned} \tag{2.12}$$

*where $\phi(k)$ and $\psi(k)$ are same as defined in Theorem* 2.1.

*Proof.* By hypothesis $P \in \mathcal{P}_n$ has all zeros in $|z| \le k, k \ge 1$. If $P(z)$ has a zero on $|z| = k$, then $m = \min_{|z|=k}|P(z)| = 0$ and result follows from Theorem 2.1. Henceforth, we suppose that $P(z)$ has all its zeros in $|z| < k, k \ge 1$, so that

$m > 0$. Now if $f(z) = P(kz)$, then $f \in \mathcal{P}_n$ and $f(z)$ has all zeros in $|z| < 1$ and $m = \min_{|z|=k} |P(z)| = \min_{|z|=1} |f(z)|$. This implies,

$$m \le |f(z)| \qquad \text{for} \quad |z| = 1.$$

By the Rouche's Theorem, we conclude that for every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$, the polynomial $g(z) = f(z) - \lambda m z^n$ has all zeros in $|z| < 1$. Applying inequality (2.3) to the polynomial $g(z)$, it follows for $|z| = 1$ and $|\alpha| \ge k$,

$$|D_{\alpha/k} g(z)| \ge \left(\frac{|\alpha| - k}{k}\right) |g'(z)|. \tag{2.13}$$

Since all the zeros of $g(z)$ lie in $|z| < 1$, therefore in view of inequality (2.8), we have

$$|g'(z)| \ge \frac{1}{2}\left(n + \frac{|k^n a_n - \lambda m| - |a_0|}{|k^n a_n - \lambda m| + |a_0|}\right)|g(z)|, \qquad \text{for} \quad |z| = 1. \tag{2.14}$$

Combining (2.13) and (2.14), we obtain for $|z| = 1$ and $|\alpha| \ge k$,

$$|D_{\alpha/k} g(z)| \ge \frac{1}{2}\left(\frac{|\alpha| - k}{k}\right)\left(n + \frac{|k^n a_n - \lambda m| - |a_0|}{|k^n a_n - \lambda m| + |a_0|}\right)|g(z)|.$$

Using the fact that the function $S(x) = \frac{x - |a_0|}{x + |a_0|}$, $x > 0$ is non-decreasing function of $x$ and $|k^n a_n - \lambda m| \ge k^n |a_n| - |\lambda| m > 0$, we get for every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ and $|z| = 1$,

$$|D_{\alpha/k} g(z)| \ge \frac{1}{2}\left(\frac{|\alpha| - k}{k}\right)\left(n + \frac{k^n |a_n| - |\lambda| m - |a_0|}{k^n |a_n| - |\lambda| m + |a_0|}\right)|g(z)|. \tag{2.15}$$

Replacing $g(z)$ by $f(z) - \lambda m z^n$ in (2.15), we get for $|z| = 1$ and $|\alpha| \ge k$,

$$\begin{aligned} &\left| D_{\alpha/k} f(z) - \frac{nm\alpha\lambda}{k} z^{n-1} \right| \\ &\ge \frac{1}{2}\left(\frac{|\alpha| - k}{k}\right)\left(n + \frac{k^n |a_n| - |\lambda| m - |a_0|}{k^n |a_n| - |\lambda| m + |a_0|}\right)\big(|f(z) - \lambda m z^n|\big). \end{aligned} \tag{2.16}$$

Since all the zeros of $f(z) - \lambda m z^n = g(z)$ lie in $|z| < 1$ and $|\alpha/k| \ge 1$, it follows by Lemma 1.5 that all the zeros of

$$D_{\alpha/k}(f(z) - m\lambda z^n) = D_{\alpha/k} f(z) - \frac{nm\alpha\lambda}{k} z^{n-1}$$

lie in $|z| < 1$. This implies that

$$\big|D_{\alpha/k} f(z)\big| \ge \frac{nm|\alpha|}{k} |z|^{n-1} \qquad \text{for} \quad |z| \ge 1. \tag{2.17}$$

In view of this inequality, choosing argument of $\lambda$ in the left hand side of inequality (2.16) such that

$$\begin{aligned}&\left|D_{\alpha/k}f(z)-\frac{nm\alpha\lambda}{k}z^{n-1}\right|\\&=\left|D_{\alpha/k}f(z)\right|-\frac{nm|\alpha||\lambda|}{k}\qquad\text{for}\quad |z|=1.\end{aligned}$$

Hence we get for $|z|=1$ and $|\alpha|\geq k$,

$$\begin{aligned}&\left|D_{\alpha/k}f(z)\right|-\frac{nm|\alpha||\lambda|}{k}\\&\geq\frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(n+\frac{k^n|a_n|-|\lambda|m-|a_0|}{k^n|a_n|-|\lambda|m+|a_0|}\right)\big(|f(z)|-|\lambda|m\big),\end{aligned}$$

which on simplification yields,

$$\begin{aligned}\left|D_{\alpha/k}f(z)\right|\geq&\frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(n+\frac{k^n|a_n|-|\lambda|m-|a_0|}{k^n|a_n|-|\lambda|m+|a_0|}\right)|f(z)|\\&-\frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(\frac{k^n|a_n|-|\lambda|m-|a_0|}{k^n|a_n|-|\lambda|m+|a_0|}\right)|\lambda|m\\&+\frac{n}{2}\left(\frac{|\alpha|+k}{k}\right)|\lambda|m.\end{aligned}$$

This implies for $|z|=1$ and $|\alpha|\geq k$,

$$\begin{aligned}\max_{|z|=k}|D_\alpha P(z)|\geq&\frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(n+\frac{k^n|a_n|-|\lambda|m-|a_0|}{k^n|a_n|-|\lambda|m+|a_0|}\right)\max_{|z|=k}|P(z)|\\&-\frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(\frac{k^n|a_n|-|\lambda|m-|a_0|}{k^n|a_n|-|\lambda|m+|a_0|}\right)|\lambda|m\\&+\frac{n}{2}\left(\frac{|\alpha|+k}{k}\right)|\lambda|m.\end{aligned}$$

Moreover since $D_\alpha P(z)$ is a polynomial of degree at most $n-1$, applying (1.7) of Lemma 1.3 with $R=k\geq 1$ and (1.13) of Lemma 1.5 with $|\alpha|\geq k, 0\leq$

$l < 1$, we obtain for $n > 2$ and $|z| = 1$

$$\begin{aligned}
&k^{n-1}\max_{|z|=1}|D_\alpha P(z)| - (k^{n-1}-k^{n-3})|na_0+\alpha a_1| \\
&\geq \frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(n + \frac{k^n|a_n| - lm - |a_0|}{k^n|a_n| - lm + |a_0|}\right) \\
&\quad \times\left\{\frac{2k^n}{1+k^n}\max_{|z|=1}|P(z)| + \frac{k^n-1}{k^n+1}lm + \frac{2k^{n-1}|a_{n-1}|}{k^n+1}\left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right)\right\} \\
&\quad - \frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(\frac{k^n|a_n| - lm - |a_0|}{k^n|a_n| - lm + |a_0|}\right)lm \\
&\quad + \frac{n}{2}\left(\frac{|\alpha|+k}{k}\right)lm.
\end{aligned}$$

Equivalently, we have for $n > 2,\ |\alpha| \geq k,\ 0 \leq l < 1$ and $|z| = 1$,

$$\begin{aligned}
\max_{|z|=1}|D_\alpha P(z)| \geq \frac{n}{1+k^n}&\left\{(|\alpha|-k)\max_{|z|=1}|P(z)| + (|\alpha| + 1/k^{n-1})lm\right\} \\
&+ \frac{(|\alpha|-k)}{k^n(k^n+1)}\left(\frac{k^n|a_n| - lm - |a_0|}{k^n|a_n| - lm + |a_0|}\right)\left(k^n\max_{|z|=1}|P(z)| - lm\right) \\
&+ \frac{(|\alpha|-k)|a_{n-1}|}{k(1+k^n)}\left(n + \frac{k^n|a_n| - lm - |a_0|}{k^n|a_n| - lm + |a_0|}\right) \\
&\times\left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right) \\
&+ (1 - 1/k^2)|na_0 + \alpha a_1|.
\end{aligned}$$

This proves Theorem 2.5 for the case $n > 2$.

Similarly we can prove Theorem 2.5 for the case $n = 2$ by applying inequality (1.8) of Lemma 1.3 instead of inequality (1.7) and inequality (1.14) of Lemma 1.5 instead of inequality (1.13). This completes the proof of Theorem 2.5. □

**Remark 2.6.** As before, it is easy to see that Theorem 2.5 is refinement of Theorem 2.1. Also for $l = 0$, Theorem 2.5 reduces to Theorem 2.1. Further for $k = 1$, inequality (2.12) gives a refinement of inequality (1.6).

If we divide both sides of inequality (2.12) by $|\alpha|$ and let $|\alpha| \to \infty$, we get the following result:

**Corollary 2.7.** *If all the zeros of polynomial $P \in \mathcal{P}_n$ of degree $n \geq 2$ lie in $|z| \leq k$ where $k \geq 1$ and $m = \min_{|z|=k} |P(z)|$, then for $0 \leq l < 1$,*

$$\begin{aligned} \max_{|z|=1} |P'(z)| \geq & \frac{n}{1+k^n}\big(\max_{|z|=1}|P(z)| + lm\big) + \psi(k)|a_1| \\ & + \frac{1}{k^n(1+k^n)}\left\{\left(\frac{k^n|a_n| - lm - |a_0|}{k^n|a_n| - lm + |a_0|}\right)\big(k^n \max_{|z|=1}|P(z)| - lm\big) \right. \\ & \left. + k^{n-1}|a_{n-1}|\phi(k)\left(n + \frac{k^n|a_n| - lm - |a_0|}{k^n|a_n| - lm + |a_0|}\right)\right\}, \end{aligned} \tag{2.18}$$

*where $\phi(k)$ and $\psi(k)$ are same as defined in Theorem* 2.1*.*

The result is sharp and equality in (2.18) holds for $P(z) = z^n + k^n$.

**Remark 2.8.** Clearly Corollary 2.7 refines Corollary 2.3.

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