Nonlinear Functional Analysis and Applications
Vol. 24, No. 4 (2019), pp. 827-836
ISSN: 1229-1595(print), 2466-0973(online)

http://nfaa.kyungnam.ac.kr/journal-nfaa

SOME NEW UNIQUENESS RESULTS FOR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

Khawlah H. Hussain[1], Ahmed A. Hamoud[2] and Nedal M. Mohammed[3]

[1]Department of Mechanical Technology, Basra Technical Institute
Southern Technical University, AL-Basrah, Iraq
e-mail: `khawlah.hussain@stu.edu.iq`

[2]Department of Mathematics
Dr. Babasaheb Ambedkar Marathwada University Aurangabad-431 004, India
e-mail: `drahmedselwi985@gmail.com`

[3]Department of Computer Science
Dr. Babasaheb Ambedkar Marathwada University, Aurangabad-431 004, India

Abstract. The main objective of this paper is to investigate and prove the existence and uniqueness theorems of the solution of nonlinear Volterra-Fredholm integro-differential equations of fractional order. The theorems of two types for the nonlinear fractional order Volterra-Fredholm integro-differential equations such as one- and two-dimensional are proved by applying the fixed point theorem of Banach space couple with contraction mapping principle, in which the sufficient conditions are presented in order to ensure the existence and uniqueness of a unique fixed point related to the Volterra-Fredholm integro-differential equation in operator form.

1. Introduction

The topic fractional calculus can be measured as an old as well as a new subject. Started from some speculations of Leibniz and Euler, followed by other

[0]Received April 22, 2019. Revised August 23, 2019.
[0]2010 Mathematics Subject Classification: 26A33, 47G20, 46E15, 34A12.
[0]Keywords: Caputo fractional derivative, integro-differential equation, Banach fixed point theorem, contraction mapping principle.
[0]Corresponding author: Ahmed A. Hamoud(`drahmedselwi985@gmail.com`).

important mathematicians like Laplace, Fourier, Abel, Liouville, Riemann and Holmgren [22]. In the fractional calculus the various integral inequalities plays an important role in the study of qualitative and quantitative properties of solution of differential and integral equations.

In recent years, many authors focus on the development of techniques for discussing the solutions of fractional differential and integro-differential equations. For instance, we can remember the following works:

Ibrahim and Momani [20] studied the existence and uniqueness of solutions of a class of fractional order differential equations, Karthikeyan and Trujillo [21] proved existence and uniqueness of solutions for fractional integro-differential equations with boundary value conditions, Bahuguna and Dabas [2] applied the method of lines to establish the existence and uniqueness of a strong solution for the partial integrodifferential equations, Ahmad and Sivasundaram [1] studied some existence and uniqueness results in a Banach space for the fractional integro-differential equation, Matar [24] deliberated the existence of solutions for nonlocal fractional semilinear integro-differential equations in Banach spaces via Banach fixed point theorem, Momani et al. [25] proved the Local and global uniqueness result by using Bihari's inequality for the fractional integro-differential equation, Wu and Liu [30] discussed the existence and uniqueness of solutions for fractional integro-differential equations,

Recently, in [3, 6, 7, 11, 12, 14, 15, 16, 17, 18, 19, 25, 31] the author's obtained the result on uniqueness of solutions for fractional integro-differential equations with nonlocal conditions using the fixed point theorem of Banach space couple with contraction mapping principle.

Motivated by above work, in this paper we discuss new uniqueness results for Caputo fractional Volterra-Fredholm integro-differential equation of the form:

$$ {}_0^cD_t^\alpha u(x,t) = g(x,t) + \int_a^x Z_1(y,t,u(y,t))dy + \int_a^b Z_2(y,t,u(y,t))dy, \tag{1.1} $$

$$ {}_0^cD_t^\alpha u(x,t) = g(x,t) + \int_a^x \int_0^t Z_1(y,t,u(y,t))dsdy + \int_a^b \int_0^t Z_2(y,t,u(y,t))dsdy, \tag{1.2} $$

with initial condition:

$$ u(x,0) = u_0(x) \tag{1.3} $$

where ${}_0^cD_t^\alpha$ is the Caputo's fractional derivative, $0 < \alpha \leq 1,\ x \in [a,b],\ t \in [0,T]$.

The main objective of this paper is to investigate and prove the uniqueness theorems of the solution of nonlinear partial Volterra-Fredholm integro-differential equations of fractional order. The theorems of two types for the nonlinear fractional order partial integro-differential equations such as one- and two-dimensional are proved by applying the fixed point theorem of Banach space couple with contraction mapping principle and some properties of fractional calculus.

The rest of the paper is organized as follows: In Section 2, some essential notations, definitions and Lemmas related to fractional calculus are recalled. In Section 3, the new existence and uniqueness results of the solution for Caputo fractional Volterra-Fredholm integro-differential equation have been proved. Finally, we will give a report on our paper and a brief conclusion is given in Section 4.

2. Preliminaries

The mathematical definitions of fractional derivative and fractional integration are the subject of several different approaches. The most frequently used definitions of the fractional calculus involves the Riemann-Liouville fractional derivative, Caputo derivative and we give some basic definitions and theorem which are used later on in this paper [4, 5, 8, 9, 10, 13, 23, 26, 28, 30].

Definition 2.1. ([22], **Riemann-Liouville fractional integral**) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function h is defined as

$$\begin{aligned} {}_aJ_t^{\alpha} h(x,t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(x,s)ds, \qquad \alpha \in \mathbb{R}^+, \\ {}_aJ_t^{0} h(x,t) &= h(x,t), \end{aligned}$$

where $\mathbb{R}^+$ is the set of positive real numbers.

Definition 2.2. ([22], **Caputo fractional derivative**) The fractional derivative of $h(x)$ in the Caputo sense is defined by

$$\begin{aligned} {}_aD_t^{\alpha} h(x,t) &= J^{m-\alpha} D^m h(x,t) \\ &= \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} \frac{\partial^m h(x,s)}{\partial s^m} ds, & m-1 < \alpha < m, \\ \frac{\partial^m h(x,t)}{\partial t^m}, & \alpha = m, \quad m \in \mathbb{N}, \end{cases} \end{aligned}$$

where the parameter α is the order of the derivative and is allowed to be real or even complex.

In this paper, only real and positive α will be considered. Hence, we have the following properties:

(1) ${}_aJ_x^{\alpha}\ {}_aJ_x^{v} h(x) = J^{\alpha+v} h(x), \quad \alpha, v > 0,$

(2) ${}_aJ_x^{\alpha} h^{\beta}(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} h^{\beta+\alpha}(x),$

(3) ${}_aD_x^\alpha h^\beta(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} h^{\beta-\alpha}(x), \quad \alpha > 0, \quad \beta > -1.$

(4) ${}_aJ_x^\alpha \ {}_aD_x^\alpha h(x) = h(x) - \sum_{k=0}^{m-1} h^{(k)}(0^+)\frac{(x-a)^k}{k!}, \quad x > 0.$

Definition 2.3. ([22], **Riemann-Liouville fractional derivative**) The Riemann Liouville fractional derivative of order $\alpha > 0$ is normally defined as

$$D^\alpha h(t,x) = D^m J^{m-\alpha} h(t,x), \qquad m-1 < \alpha \le m, \quad m \in \mathbb{N}. \tag{2.1}$$

Definition 2.4. ([27]) Let $T : X \longrightarrow X$ be a mapping on a normed space $(X, \|.\|)$. A point $x \in X$ for which $Tx = x$ is called a fixed point of T.

Definition 2.5. ([29]) The mapping T on a normed space $(X, \|.\|)$ is called contractive if there is a non-negative real number $c \in (0,1)$, such that

$$\|Tx - Ty\| \le c\|x - y\|$$

for all $x, y \in X$.

Lemma 2.6. ([32], **Banach fixed point theorem**) *Let $(X, \|.\|)$ be a complete normed space, and let $T : X \longrightarrow X$ be a contraction mapping. Then T has exactly one fixed point.*

3. Main results

In this section, we shall give an existence and uniqueness results of Eq.(1.1) and Eq.(1.2), with the initial condition (1.3) and prove it. Before starting and proving the main results, we introduce the following hypotheses:

(A1) There exist two constants $L_1, L_2 > 0$ such that, for any $u_1, u_2 \in C_t([a,b] \times [0,T])$

$$\|Z_1(y,s,u_1(y,s)) - Z_1(y,s,u_2(y,s))\| \le L_1 \|u_1 - u_2\|,$$
$$\|Z_2(y,s,u_1(y,s)) - Z_2(y,s,u_2(y,s))\| \le L_2 \|.u_1 - u_2\|.$$

(A2) The function $g : [a,b] \to [0,T]$ is continuous.

Lemma 3.1. *Let u be defined on $C_t([a,b] \times [0,T])$ with continuous nth order partial derivatives with respect to t. Then $u(x,t)$ is a solution of the problem $(1.1) - (1.3)$ iff u satisfies*

$$\begin{aligned} u(x,t) &= u_0(x) + {}_0I_t^\alpha g(x,t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[\int_a^x Z_1(y,s,u(y,s))dy \\ &\quad + \int_a^b Z_2(y,s,u(y,s))dy \Big] ds \end{aligned}$$

for $x \in [a,b]$, $t \in [0,T]$.

Theorem 3.2. *If the hypotheses* $(A1)-(A2)$ *hold, and*

$$\frac{T^{\alpha}(b-a)(L_1+L_2)}{\Gamma(\alpha+1)}<1, \tag{3.1}$$

then the fractional integro-differential equation (1.1)-(1.3) *has a unique solution.*

Proof. By Lemma 3.1, we know that a function u is a solution to (1.1)-(1.3) iff u satisfies

$$\begin{aligned} u(x,t) &= u_0(x)+ {}_0I_t^{\alpha}g(x,t)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[\int_a^x Z_1(y,s,u(y,s))dy \\ &\quad +\int_a^b Z_2(y,s,u(y,s))dy\Big]ds. \end{aligned}$$

We transform the Cauchy problem (1.1)-(1.3) to be applicable to fixed point problem and define the operator $T: C_t([a,b]\times[0,T]) \longrightarrow C_t([a,b]\times[0,T])$ by

$$\begin{aligned} (Tu)(x,t) = u_0(x)+ {}_0I_t^{\alpha}g(x,t)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[\int_a^x Z_1(y,s,u(y,s))dy \\ +\int_a^b Z_2(y,s,u(y,s))dy\Big]ds, \end{aligned}$$

we can see that, if u is a fixed point of T, then u is a solution of (1.1)-(1.3).

Now we prove that T has a fixed point u in $C_t([a,b]\times[0,T])$. For that, let $u_1,u_2\in C_t([a,b]\times[0,T])$ and for any $x\in[a,b]$ such that

$$\begin{aligned} u_1(x,t) &= u_0(x)+ {}_0I_t^{\alpha}g(x,t)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[\int_a^x Z_1(y,s,u_1(y,s))dy \\ &\quad +\int_a^b Z_2(y,s,u_1(y,s))dy\Big]ds, \end{aligned}$$

and

$$\begin{aligned} u_2(x,t) &= u_0(x)+ {}_0I_t^{\alpha}g(x,t)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[\int_a^x Z_1(y,s,u_2(y,s))dy \\ &\quad +\int_a^b Z_2(y,s,u_2(y,s))dy\Big]ds. \end{aligned}$$

Consequently, we get

$$
\begin{aligned}
&\Big\| Tu_1(x,t) - Tu_2(x,t) \Big\| \\
&\le \Big\| u_0(x) + {}_0I_t^{\alpha} g(x,t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\
&\quad \times \Big[\int_a^x Z_1(y,s,u_1(y,s))dy + \int_a^b Z_2(y,s,u_1(y,s))dy \Big] ds \\
&\quad - u_0(x) - {}_0I_t^{\alpha} g(x,t) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\
&\quad \times \Big[\int_a^x Z_1(y,s,u_2(y,s))dy + \int_a^b Z_2(y,s,u_2(y,s))dy \Big] ds \Big\| \\
&\le \Big\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[\int_a^x \Big[Z_1(y,s,u_1(y,s)) - Z_1(y,s,u_2(y,s)) \Big] dy \\
&\quad + \int_a^b \Big[Z_2(y,s,u_1(y,s)) - Z_2(y,s,u_2(y,s)) \Big] dy \Big] ds \Big\| \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[\int_a^x L_1 \Big\| u_1(y,s) - u_2(y,s) \Big\| dy \\
&\quad + \int_a^b L_2 \Big\| u_1(y,s) - u_2(y,s) \Big\| dy \Big] ds \\
&\le \frac{(L_1+L_2)}{\Gamma(\alpha)} \frac{t^{\alpha}}{\alpha} (b-a) \Big\| u_1 - u_2 \Big\| \\
&\le \frac{(L_1+L_2)(b-a)T^{\alpha}}{\Gamma(\alpha+1)} \Big\| u_1 - u_2 \Big\|.
\end{aligned}
$$

Since $\frac{(L1+L_2)(b-a)T^{\alpha}}{\Gamma(\alpha+1)} < 1$, which implies T is a contractive mapping and therefore T has a unique fixed point, which means that equation (1.1) has a unique solution. □

Lemma 3.3. *Let u be defined on $C_t([a,b]\times[0,T])$ with continuous nth order partial derivatives with respect to t. Then $u(x,t)$ is a solution of the problem $(1.2)-(1.3)$ iff u satisfies*

$$
\begin{aligned}
u(x,t) = u_0(x) &+ {}_0I_t^{\alpha} g(x,t) \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x \int_0^{\xi} (t-\xi)^{\alpha-1} Z_1(y,s,u(y,s)) dy ds d\xi \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^b \int_0^{\xi} (t-\xi)^{\alpha-1} Z_2(y,s,u(y,s)) dy ds d\xi
\end{aligned}
$$

for $x \in [a, b]$.

Theorem 3.4. *If the hypotheses (A1)-(A2) hold, and*

$$\frac{T^{\alpha+1}(b-a)(L_1+L_2)}{\Gamma(\alpha+2)} < 1, \tag{3.2}$$

then the fractional integro-differential equation (1.2)-(1.3) *has a unique solution.*

Proof. By Lemma 3.3. we know that a function u is a solution to (1.2)-(1.3) iff u satisfies

$$\begin{aligned} u(x,t) = u_0(x) + {}_0I_t^\alpha g(x,t) \\ + \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x \int_0^\xi (t-\xi)^{\alpha-1} Z_1(y,s,u(y,s)) dy ds d\xi \\ + \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^b \int_0^\xi (t-\xi)^{\alpha-1} Z_2(y,s,u(y,s)) dy ds d\xi. \end{aligned}$$

We transform the Cauchy problem (1.2)-(1.3) to be applicable to fixed point problem and define the operator $T : C_t([a,b] \times [0,T]) \longrightarrow C_t([a,b] \times [0,T])$ by

$$\begin{aligned} (Tu)(x,t) = u_0(x) + {}_0I_t^\alpha g(x,t) \\ + \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x \int_0^\xi (t-\xi)^{\alpha-1} Z_1(y,s,u(y,s)) dy ds d\xi \\ + \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^b \int_0^\xi (t-\xi)^{\alpha-1} Z_2(y,s,u(y,s)) dy ds d\xi, \end{aligned}$$

we can see that, if u is a fixed point of T , then u is a solution of (1.2)-(1.3). Now we prove T has a fixed point u in $C_t([a,b] \times [0,T])$. For that, let $u_1, u_2 \in C_t([a,b] \times [0,T])$ and for any $x \in [a,b]$ such that

$$\begin{aligned} u_1(x,t) = u_0(x) + {}_0I_t^\alpha g(x,t) \\ + \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x \int_0^\xi (t-\xi)^{\alpha-1} Z_1(y,s,u_1(y,s)) dy ds d\xi \\ + \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^b \int_0^\xi (t-\xi)^{\alpha-1} Z_2(y,s,u_1(y,s)) dy ds d\xi \end{aligned}$$

and

$$\begin{aligned} u_2(x,t) = u_0(x) + {}_0I_t^\alpha g(x,t) \\ + \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x \int_0^\xi (t-\xi)^{\alpha-1} Z_1(y,s,u_2(y,s)) dy ds d\xi \\ + \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^b \int_0^\xi (t-\xi)^{\alpha-1} Z_2(y,s,u_2(y,s)) dy ds d\xi. \end{aligned}$$

Consequently, we get

$$\begin{aligned}
&\Big\| Tu_1(x,t) - Tu_2(x,t) \Big\| \\
&\leq \Big\| u_0(x) + {}_0I_t^\alpha g(x,t) + \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x \int_0^\xi (t-\xi)^{\alpha-1} Z_1(y,s,u_1(y,s)) dyds d\xi \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^b \int_0^\xi (t-\xi)^{\alpha-1} Z_2(y,s,u_1(y,s)) dyds d\xi \\
&\quad - u_0(x) - {}_0I_t^\alpha g(x,t) - \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x \int_0^\xi (t-\xi)^{\alpha-1} Z_1(y,s,u_2(y,s)) dyds d\xi \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^b \int_0^\xi (t-\xi)^{\alpha-1} Z_2(y,s,u_2(y,s)) dyds d\xi \Big\| \\
&\leq \Big\| \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x \int_0^\xi (t-\xi)^{\alpha-1} [Z_1(y,s,u_1(y,s)) - Z_1(y,s,u_2(y,s))] dyds d\xi \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^b \int_0^\xi (t-\xi)^{\alpha-1} [Z_2(y,s,u_1(y,s)) - Z_2(y,s,u_2(y,s))] dyds d\xi \Big\| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^x \int_0^\xi (t-\xi)^{\alpha-1} L_1 \Big\| u_1(y,s) - u_2(y,s) \Big\| dyds d\xi \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \int_a^b \int_0^\xi (t-\xi)^{\alpha-1} L_2 \Big\| u_1(y,s) - u_2(y,s) \Big\| dyds d\xi \\
&\leq \frac{(L_1+L_2)}{\Gamma(\alpha)} \frac{t^{\alpha+1}}{\alpha(\alpha+1)} (b-a) \Big\| u_1 - u_2 \Big\| \\
&\leq \frac{(L_1+L_2)(b-a)T^{\alpha+1}}{\Gamma(\alpha+2)} \Big\| u_1 - u_2 \Big\|.
\end{aligned}$$

Since $\frac{(L_1+L_2)(b-a)T^{\alpha+1}}{\Gamma(\alpha+2)} < 1$, T is a contractive mapping and therefore T has a unique fixed point, which means that equation (1.2) has a unique solution. □

4. Conclusion

The main purpose of this paper was to present new existence and uniqueness results of the solution for Caputo fractional Volterra-Fredholm integro-differential. The techniques used to prove our results are a variety of tools such as the fixed point theorem of a Banach space couple with contraction mapping principle and some properties of fractional calculus. Moreover, the results of references [21, 25] appear as a special case of our results.

References

[1] B. Ahmad and S. Sivasundaram, *Some existence results for fractional integro-differential equations with nonlinear conditions*, Commun. Appl. Anal., **12** (2008), 107–112.

[2] D. Bahuguna and J. Dabas, *Existence and uniqueness of a solution to a partial integro-differential equation by the method of Lines*, Elect. J. of Qualitative Theory of Diffe. Equa., **4** (2008), 1-12.

[3] K. Balachandran and J. Trujillo, *The nonlocal Cauchy problem for nonlinear fractional integro-differential equations in Banach spaces,* Nonlinear Anal. TMA., **72** (2010), 4587-4593.

[4] A. Hamoud and K. Ghadle, *The approximate solutions of fractional Volterra-Fredholm integro-differential equations by using analytical techniques*, Probl. Anal. Issues Anal., **(25)**(1) (2018), 41-58.

[5] A. Hamoud and K. Ghadle, *Modified Laplace decomposition method for fractional Volterra-Fredholm integro-differential equations*, J. Math. Model., **6**(1) (2018), 91-104.

[6] A. Hamoud and K. Ghadle, *Usage of the homotopy analysis method for solving fractional Volterra-Fredholm integro-differential equation of the second kind,* Tamkang J. of Math., **49**(4) (2018), 301-315.

[7] A. Hamoud and K. Ghadle, *Homotopy analysis method for the first order fuzzy Volterra-Fredholm integro-differential equations*, Indonesian J. Elec. Eng. & Comp. Sci., **11**(3) (2018), 857-867.

[8] A. Hamoud and K. Ghadle, *Some new existence, uniqueness and convergence results for fractional Volterra-Fredholm integro-differential equations*, J. Appl. Comput. Mech., **5**(1) (2019), 58-69.

[9] A. Hamoud and K. Ghadle, *Existence and uniqueness of solutions for fractional mixed Volterra-Fredholm integro-differential equations*, Indian J. Math., **60**(3) (2018), 375-395.

[10] A. Hamoud K. Ghadle and S. Atshan, *The approximate solutions of fractional integro-differential equations by using modified Adomian decomposition method*, Khayyam J. Math., **5**(1) (2019), 21-39.

[11] A. Hamoud, K. Hussain and K. Ghadle, *The reliable modified Laplace Adomian decomposition method to solve fractional Volterra-Fredholm integro differential equations*, Dynam. of Continu. Disc. and Impul. Sys. Series B: Appl. & Algo., **26** (2019), 171-184.

[12] A. Hamoud and K. Ghadle, *Modified Adomian decomposition method for solving fuzzy Volterra-Fredholm integral equations*, J. Indian Math. Soc., **85**(1-2) (2018), 52-69.

[13] A. Hamoud, K. Ghadle, M. Bani Issa and Giniswamy, *Existence and uniqueness theorems for fractional Volterra-Fredholm integro-differential equations*, Int. J. Appl. Math., **31**(3) (2018), 333-348.

[14] A. Hamoud and K. Ghadle, *Existence and uniqueness of the solution for Volterra-Fredholm integro-differential equations*, J. of Siberian Federal University. Math. & Phys., **11**(6) (2018), 692-701.

[15] A. Hamoud, A. Azeez and K. Ghadle, *A study of some iterative methods for solving fuzzy Volterra-Fredholm integral equations*, Indonesian J. Elec. Eng. & Comput. Sci., **11**(3) (2018), 1228-1235.

[16] A. Hamoud, L. Dawood, K. Ghadle and S. Atshan, *Usage of the modified variational iteration technique for solving Fredholm integro-differential equations*, Inter. J. of Mech. Produ. Eng. Res. Develo., **9**(2) (2019), 895-902.

[17] A. Hamoud, M. Bani Issa and K. Ghadle, *Existence and uniqueness results for nonlinear Volterra-Fredholm integro-differential equations*, Nonlinear Funct. Anal. Appl., **23**(4) (2018), 797-805.

[18] A. Hamoud, N. Mohammed, K. Ghadle and S. Dhondge, *Solving integro-differential equations by using numerical techniques,* Inter. J. Appl. Eng. Resea., **14**(14) (2019), 3219-3225.

[19] A. Hamoud and K. Ghadle, *Some new uniqueness results of solutions for fractional Volterra-Fredholm integro-differential equations,* Iranian J. Math. Sci. Inform., (To Appears).

[20] R. Ibrahim and S. Momani, *On the existence and uniqueness of solutions of a class of fractional differential equations,* J. Math. Anal. Appl., **334**(1) (2007), 1-10.

[21] K. Karthikeyan and J. Trujillo, *Existence and uniqueness results for fractional integro-differential equations with boundary value conditions*, Commun. Nonlinear Sci. Numer. Simu., **17** (2012), 4037-4043.

[22] A. Kilbas, H. Srivastava and J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Math. Stud. Elsevier, Amsterdam, 2006.

[23] V. Lakshmikantham and M. Rao, *Theory of integro-differential equations*, Gordon & Breach, London, 1995.

[24] M. Matar, *Controllability of fractional semilinear mixed Volterra-Fredholm integrodifferential equations with nonlocal conditions, Inter. J. Math. Anal.*, **4**(23) (2010), 1105-1116.

[25] S. Momani, A. Jameel and S. Al-Azawi, *Local and global uniqueness theorems on fractional integro-differential equations via Bihari's and Gronwall's inequalities*, Soochow J. Math., **33**(4) (2007), 619-627.

[26] K. Miller and B. Ross, *An introduction to the fractional calculus and differential equations*, John Wiley, New York, 1993.

[27] M. Reed and B. Simon, *Functional Analysis*, Academic Press, Inc., New York, 1980.

[28] S. Samko, A. Kilbas and O. Marichev, *Fractional integrals and derivatives, theory and applications*, Gordon and Breach, Yverdon, 1993.

[29] X. Su and L. Liu, *Existence of solution for boundary value problem of nonlinear fractional differential equations,* Appl. Math. J. Chinese Univ. Ser. B, **22**(3) (2007), 291-298.

[30] J. Wu and Y. Liu, *Existence and uniqueness of solutions for the fractional integro-differential equations in Banach spaces*, Electronic J. Diff. Equat., **2009** (2009), 1-8.

[31] J. Wu and Y. Liu, *Existence and uniqueness results for fractional integro-differential equations with nonlocal conditions,* 2nd IEEE International Conference on Information and Financial Engineering, (2010), 91-94.

[32] Y. Zhou, *Basic Theory of Fractional Differential Equations*, *Singapore: World Scientific*, 2014.