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ON A NONLINEAR MIXED VOLTERRA-FREDHOLM INTEGRODIFFERENTIAL EQUATION WITH NONLOCAL CONDITION IN BANACH SPACES

M. B. Dhakne[1] and H. L. Tidke[2]

[1]Department of Mathematics
Dr. Babasaheb Ambedkar Marathwada University,
Aurangabad-431 004, India
e-mail: mbdhakne@yahoo.com

[2]Department of Mathematics, North Maharashtra University,
Jalgaon-425 001, India
e-mail: tharibhau@gmail.com

Abstract. In the present paper we study the existence of solutions of nonlinear mixed Volterra-Fredholm integrodifferential equations in Banach spaces with nonlocal condition. The technique used in our analysis is based on an application of the topological transversality theorem known as Leray-Schauder alternative, rely on a priori bounds of solutions and the inequality established by Pachpatte.

1. Introduction

Let X be a Banach space with norm $\|\cdot\|$. Let $B = C([t_0, t_0 + \beta], X)$ be the Banach space of all continuous functions from $[t_0, t_0 + \beta]$ into X endowed with supremum norm

$$\|x\|_B = \sup\{\|x(t)\| : t \in [t_0, t_0 + \beta]\}.$$

Consider the nonlinear mixed Volterra-Fredholm integrodifferential equation of the type

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$$x'(t) + Ax(t) = f(t, x(t), \int_{t_0}^{t} a(t,s)k(s,x(s))ds, \int_{t_0}^{t_0+\beta} b(t,s)h(s,x(s))ds),$$
$$t \in [t_0, t_0 + \beta] \tag{1.1}$$
$$x(t_0) + g(t_1, t_2, \cdots, t_p, x(\cdot)) = x_0, \tag{1.2}$$

where $-A$ is an infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t),\ t \geq 0$ in X, $0 \leq t_0 < t_1 < t_2 < ... < t_p \leq t_0 + \beta,\ f : [t_0, t_0 + \beta] \times X \times X \times X \to X,\ \ k,\ h : [t_0, t_0 + \beta] \times X \to X$, $g(t_1, t_2, ..., t_p, .) : X \to X,\ a,\ b : [t_0, t_0 + \beta] \times [t_0, t_0 + \beta] \to R$ are functions and x_0 is a given element of X. The symbol $g(t_1, t_2, ..., t_p, x(\cdot))$ is used in the sense that in the place of '.' we can substitute only elements of the set $\{t_1, t_2, ..., t_p\}$. For example $g(t_1, t_2, ..., t_p, x(\cdot))$ can be defined by the formula

$$g(t_1, t_2, ..., t_p, x(\cdot)) = C_1 x(t_1) + C_2 x(t_2) + ... + C_p x(t_p), \tag{1.3}$$

where $C_i (i = 1, 2, ..., p)$ are given constants. In this case, (1.3) allows the measurements at $t = t_0, t_1, ..., t_p$, rather that just at $t = t_0$. So more information is available. These equations (1.1)–(1.2) can be applied in physics with better effect than equation (1.1) with classical initial condition.

The work in nonlocal initial value problem was first introduced by Byszewski. In [5], Byszewski using the method of semigroups and the Banach fixed point theorem, investigated the existence and uniqueness of mild, strong and classical solution of first order initial value problem

$$u'(t) + Au(t) = f(t, u(t)), \quad t \in [t_0, t_0 + a], \tag{1.4}$$
$$u(t_0) + g(t_1, t_2, ..., t_p, u(\cdot)) = u_0, \tag{1.5}$$

where $-A$ is the infinitesimal generator of C_0 semigroup of $T(t)$ on a Banach space X, $0 \leq t_0 < t_1 < ... < t_p \leq t_0 + a,\ a > 0,\ u_0 \in X$ and $f : [t_0, t_0 + a] \times X \to X,\ g(t_1, t_2, ..., t_p, .) : X \to X$ are given functions.

Several authors have studied the problems such as existence, uniqueness, boundedness and other properties of solutions of these equations (1.1)–(1.2) or their special forms by using different techniques, see [1], [2], [8], [9], [11], [12], [14], [15]. The equations (1.1)–(1.2) or their special forms serve as mathematical models for various partial differential equations or partial integrodifferential equations arising in heat flow in material with memory and viscoelasticity problems, see [3], [4], [6], [7]. Recently, in an interesting paper [9], Dhakne M. B. and Kendre S. D. have studied the global existence of solutions of (1.1) with initial condition when $a(t, s)k(s, x(s)) = k_1(t, s, x(s))$ and $b(t, s)h(s, x(s)) = h_1(t, s, x(s))$. We are motivated by the work of Dhakne and Kendre S. D. [9] and influenced by the work of Byszewski [5].

The objective of the present paper is to study the global existence of solutions of the equations (1.1)–(1.2). The main tool used in our analysis is based on an application of the topological transversality theorem known as Leray-Schauder alternative, rely on a priori bounds of solutions and the inequality established by B. G. Pachpatte. The interesting and useful aspect of the method employed here is that it yields simultaneously the global existence of solutions and the maximal interval of existence.

The paper is organized as follows. In section 2, we present the preliminaries and the statement of our main result. Section 3 deals with proof of the theorem. Finally in section 4 we give an example to illustrate the application of our result.

2. Preliminaries and Main Result

Before proceeding to the statement of our main result, we give the following preliminaries and hypotheses used in our subsequent discussion.

Definition 2.1. Let $-A$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t),\ t \geq 0$ in X. Let $f \in L^1(t_0, t_0 + \beta; X)$. The function $x \in B$ given by

$$x(t) = T(t-t_0)(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))) + \int_{t_0}^{t} T(t-s) f(s, x(s), \int_{t_0}^{s} a(s,\tau)k(\tau, x(\tau))d\tau, \int_{t_0}^{t_0+\beta} b(s,\tau)h(\tau, x(\tau))d\tau)ds, \quad t \in [t_0, t_0+\beta] \tag{2.1}$$

is called the mild solution of the initial value problem (1.1)–(1.2).

Theorem 2.2. *([13]) Let $z(t),\ u(t),\ v(t),\ w(t) \in C([\alpha, \beta], R_+)$ and $k \geq 0$ be a real constant and*

$$z(t) \leq k + \int_{\alpha}^{t} u(s)[z(s) + \int_{\alpha}^{s} v(\sigma)z(\sigma)d\sigma + \int_{\alpha}^{\beta} w(\sigma)z(\sigma)d\sigma]ds, \ \ t \in [\alpha, \beta].$$

If

$$r = \int_{\alpha}^{\beta} w(\sigma)\exp(\int_{\alpha}^{\sigma}[u(\tau) + v(\tau)]d\tau)d\sigma < 1,$$

then

$$z(t) \leq \frac{k}{1-r}\exp(\int_{\alpha}^{t}[u(s) + v(s)]ds), \quad \text{for} \quad t \in [\alpha, \beta].$$

Theorem 2.3. *([10]) Let S be a convex subset of a normed linear space E and assume $0 \in S$. Let $F : S \to S$ be a completely continuous operator, and* *let* $\varepsilon(F) = \{x \in S : x = \lambda Fx \quad \text{for some} \quad 0 < \lambda < 1\}$. *Then either* $\varepsilon(F)$ *is unbounded or* F *has a fixed point.*

We list the following hypotheses for our convenience.

(H_1) $-A$ is the infinitesimal generator of a semigroup of continuous bounded linear operators $T(t)$ in X. Then there exists a constant $K \geq 1$ such that

$$\|T(t)\| \leq K.$$

(H_2) There exists a constant G such that

$$G = \max_{x \in B} \|g(t_1, t_2, ..., t_p, x(\cdot))\|.$$

(H_3) There exists a continuous function $p : [t_0, t_0 + \beta] \to R_+$ such that

$$\|k(t, , x(t))\| \leq p(t)\|x(t)\|,$$

for every $t \in [t_0, t_0 + \beta]$ and $x \in X$.

(H_4) There exists a continuous function $q : [t_0, t_0 + \beta] \to R_+$ such that

$$\|h(t, x(t))\| \leq q(t)\|x(t)\|,$$

for every $t \in [t_0, t_0 + \beta]$ and $x \in X$

(H_5) There exists a continuous function $l : [t_0, t_0 + \beta] \to R_+$ such that

$$\|f(t, x, y, z)\| \leq l(t)(\|x\| + \|y\| + \|z\|),$$

for every $t \in [t_0, t_0 + \beta]$ and $x, y, z \in X$.

(H_6) There exists a constant M such that

$$|a(t, s)| \leq M, \quad \text{for} \quad t \geq s \geq t_0.$$

(H_7) There exists a constant N such that

$$|b(t, s)| \leq N, \quad \text{for} \quad t, s \in [t_0, t_0 + \beta].$$

(H_8) For each $t \in [t_0, t_0 + \beta]$ the function $f(t, ., ., .) : [t_0, t_0 + \beta] \times X \times X \times X \to X$ is continuous and for each $x,\ y,\ z \in X$ the function $f(., x, y, z) : [t_0, t_0 + \beta] \times X \times X \times X \to X$ is strongly measurable.

(H_9) For each $t \in [t_0, t_0 + \beta]$ the functions $k(t, .),\ h(t, .) : [t_0, t_0 + \beta] \times X \to X$ are continuous and for each $x \in X$ the functions $k(., x),\ h(., x) : [t_0, t_0 + \beta] \times X \to X$ are strongly measurable.

(H_{10}) For every positive integer m there exists $\alpha_m \in L^1(t_0, t_0 + \beta)$ such that

$$\sup_{\|x\| \leq m, \|y\| \leq m, \|z\| \leq m} \|f(t, x, y, z)\| \leq \alpha_m(t), \quad \text{for} \quad t \in [t_0, t_0 + \beta] \quad \text{a. e.}$$

Our main result is established in the following theorem.

Theorem 2.4. *Suppose that the hypotheses $(H_1)-(H_{10})$ hold. If*

$$r^* = \int_{t_0}^{t_0+\beta} Nq(\sigma)\exp(\int_{t_0}^{\sigma}[Kl(\tau)+Mp(\tau)]d\tau)d\sigma < 1, \tag{2.2}$$

then the initial value problem (1.1)–(1.2) has a mild solution on $[t_0, t_0+\beta]$.

Remark 2.5. We remark that K. Balachandran in [2] has studied the existence and uniqueness of mild and strong solutions of special forms of the equations (1.1)–(1.2) by using Banach fixed point theorem. Here our approach to the problem and conditions on functions are different from those in [2].

3. Proof of Theorem

First we establish the priori bounds for the initial value problem

$$\begin{aligned} x'(t) + Ax(t) = \lambda f(t, x(t), &\int_{t_0}^{t} a(t,s)k(s,x(s))ds, \\ &\int_{t_0}^{t_0+\beta} b(t,s)h(s,x(s))ds), \quad [t_0, t_0+\beta] \end{aligned} \tag{3.1}$$

under the initial conditions (1.2) for $\lambda \in (0,1)$. Let $x(t)$ be a solution of the problem (3.1)–(1.2), then it satisfies the equivalent integral equation

$$\begin{aligned} x(t) &= T(t-t_0)(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))) \\ &\quad + \lambda \int_{t_0}^{t} T(t-s) f(s, x(s), \int_{t_0}^{s} a(s,\tau)k(\tau, x(\tau))d\tau, \\ &\qquad \int_{t_0}^{t_0+\beta} b(s,\tau)h(\tau, x(\tau))d\tau)ds, \quad t \in [t_0, t_0+\beta]. \end{aligned} \tag{3.2}$$

Using (3.2), hypotheses $(H_1)-(H_7)$ and the fact that $\lambda \in (0,1)$, we have

$$\begin{aligned} \|x(t)\| \le\ & K\|x_0 - g(t_1, t_2, ..., t_p, x(\cdot))\| + \int_{t_0}^{t} \|T(t-s) f(s, x(s), \\ & \int_{t_0}^{s} a(s,\tau)k(\tau,x(\tau))d\tau, \int_{t_0}^{t_0+\beta} b(s,\tau)h(\tau,x(\tau))d\tau)\|ds \\ \le\ & K\|x_0 - g(t_1, t_2, ..., t_p, x(\cdot))\| + \int_{t_0}^{t} \|T(t-s)\| l(s)[\|x(s)\| \\ & + \int_{t_0}^{s} \|a(s,\tau)k(\tau,x(\tau))\|d\tau + \int_{t_0}^{t_0+\beta} \|b(s,\tau)h(\tau,x(\tau))\|d\tau]ds \end{aligned}$$

$$\leq K(\|x_0\| + G) + \int_{t_0}^{t} Kl(s)[\|x(s)\| + \int_{t_0}^{s} Mp(\tau)\|x(\tau)\|d\tau$$
$$+ \int_{t_0}^{t_0+\beta} Nq(\tau)\|x(\tau)\|d\tau]ds. \tag{3.3}$$

Using the hypothesis (2.2) and Pachpatte's inequality given in Theorem 2.2 with $z(t) = \|x(t)\|$ in (3.3), we have

$$\begin{aligned}\|x(t)\| &\leq \frac{K(\|x_0\| + G)}{1 - r^*} \exp(\int_{t_0}^{t} [Kl(s) + Mp(s)]ds) \\ &\leq \frac{k^*}{1 - r^*} \exp(\beta[KL + MP]) = \gamma,\end{aligned} \tag{3.4}$$

where

$$k^* = K(\|x_0\| + G), \qquad L = \sup_{t \in [t_0, t_0+\beta]} \{l(t)\}, \qquad P = \sup_{t \in [t_0, t_0+\beta]} \{p(t)\}.$$

Thus, there exists a constant γ independent of $\lambda \in (0, 1)$ such that $\|x(t)\| \leq \gamma$ and consequently

$$\|x\|_B = \sup\{\|x(t)\| : t \in [t_0, t_0 + \beta]\} \leq \gamma.$$

Now, we rewrite the problem (1.1)–(1.2) as follows: If $y \in B$ and $x(t) = T(t - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))) + y(t), \quad t \in [t_0, t_0 + \beta]$, where $y(t)$ satisfies

$$\begin{aligned} y(t) = \int_{t_0}^{t} & T(t - s)f(s, y(s) + T(s - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))), \\ & \int_{t_0}^{s} a(s, \tau)k(\tau, y(\tau) + T(\tau - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))))d\tau, \\ & \int_{t_0}^{t_0+\beta} b(s, \tau)h(\tau, y(\tau) + T(\tau - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))))d\tau)ds, \end{aligned}$$
$$t \in [t_0, t_0 + \beta]$$

if and only if $x(t)$ satisfies

$$\begin{aligned} x(t) = T(t - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))) + \int_{t_0}^{t} T(t - s)f(s, x(s), \\ \int_{t_0}^{s} a(s, \tau)k(\tau, x(\tau))d\tau, \int_{t_0}^{t_0+\beta} b(s, \tau)h(\tau, x(\tau))d\tau)ds, \ t \in [t_0, t_0 + \beta]. \end{aligned}$$

Define $F : B_0 \to B_0,\ B_0 = \{y \in B : y(t_0) = 0\}$ by

$$\begin{aligned}
&(Fy)(t)\\
&= \int_{t_0}^{t} T(t-s) f(s, y(s) + T(s-t_0)(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))),\\
&\quad \int_{t_0}^{s} a(s,\tau) k(\tau, y(\tau) + T(\tau - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))))d\tau,\\
&\quad \int_{t_0}^{t_0+\beta} b(s,\tau) h(\tau, y(\tau) + T(\tau - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))))d\tau)ds,\\
&t \in [t_0, t_0 + \beta].
\end{aligned} \tag{3.5}$$

Now, we prove that $F : B_0 \to B_0$ is continuous. Let $\{u_n\}$ be a sequence of elements of B_0 converging to u in B_0. Then

$$\begin{aligned}
&(Fu_n)(t)\\
&= \int_{t_0}^{t} T(t-s) f(s, u_n(s) + T(s-t_0)(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))),\\
&\quad \int_{t_0}^{s} a(s,\tau) k(\tau, u_n(\tau) + T(\tau - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))))d\tau,\\
&\quad \int_{t_0}^{t_0+\beta} b(s,\tau) h(\tau, u_n(\tau) + T(\tau - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))))d\tau)ds,\\
&t \in [t_0, t_0 + \beta].
\end{aligned} \tag{3.6}$$

Now, $\|Fu_n - Fu\|_B = \sup_{t\in[t_0,t_0+\beta]} \|(Fu_n)(t) - (Fu)(t)\|$. Since $\{u_n\}$ is the sequence of elements of B_0 converging to u in B_0 and by hypotheses (H_8) – (H_{10}), we have

$$\begin{aligned}
&f(t, u_n(t) + T(t-t_0)(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))),\\
&\qquad \int_{t_0}^{t} a(t,s) k(s, u_n(s) + T(s-t_0)(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))))ds,\\
&\qquad \int_{t_0}^{t_0+\beta} b(t,s) h(s, u_n(s) + T(s-t_0)(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))))ds)\\
&\to f(t, u(t) + T(t-t_0)(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))),\\
&\qquad \int_{t_0}^{t} a(t,s) k(s, u(s) + T(s-t_0)(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))))ds,\\
&\qquad \int_{t_0}^{t_0+\beta} b(t,s) h(s, u(s) + T(s-t_0)(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))))ds)
\end{aligned}$$

for each $t \in [t_0, t_0 + \beta]$. Then by dominated convergence theorem, we get

$$\begin{aligned}
&\|(Fu_n)(t) - (Fu)(t)\| \\
&\leq \int_{t_0}^{t} \|T(t-s)\| \|f(s, u_n(s) + T(s-t_0)(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))), \\
&\qquad \int_{t_0}^{s} a(s,\tau)k(\tau, u_n(\tau) + T(\tau - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))))d\tau, \\
&\qquad \int_{t_0}^{t_0+\beta} b(s,\tau)h(\tau, u_n(\tau) + T(\tau - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))))d\tau) \\
&\quad - f(s, u(s) + T(s-t_0)(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))), \\
&\qquad \int_{t_0}^{s} a(s,\tau)k(\tau, u(\tau) + T(\tau - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))))d\tau, \\
&\qquad \int_{t_0}^{t_0+\beta} b(s,\tau)h(\tau, u(\tau) + T(\tau - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))))d\tau)\| ds \\
&\to 0
\end{aligned}$$

and consequently $\|Fu_n - Fu\|_B \to 0$ as $n \to \infty$ i.e., $Fu_n \to Fu$ in B_0 as $u_n \to u \in B_0$. Therefore, F is continuous.

Now, we prove that F maps a bounded set of B_0 into a precompact set of B_0. Let $B_m = \{y \in B_0 : \|y\|_B \leq m\}$ for $m \geq 1$. We first show that F maps B_m into an equicontinuous family of functions with values in X. Let $y \in B_m$ and $t_0 \leq s < t < t_0 + \beta$. Then by hypotheses (H_1)-(H_7), we have

$$\begin{aligned}
&\|(Fy)(s) - (Fy)(t)\| \\
&\leq \int_{t_0}^{s} \|T(s-\tau) - T(t-\tau)\| l(\tau)[\|y(\tau) + T(\tau - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(.)))\| \\
&\qquad + \| \int_{t_0}^{\tau} a(\tau,\sigma)k(\sigma, y(\sigma) + T(\sigma - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(.))))d\sigma\| \\
&\qquad + \| \int_{t_0}^{t_0+\beta} b(\tau,\sigma)h(\sigma, y(\sigma) + T(\sigma - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(.))))d\sigma\|]d\tau \\
&\qquad + \int_{s}^{t} Kl(\tau)[\|y(\tau) + T(\tau - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(.)))\| \\
&\qquad + \| \int_{t_0}^{\tau} a(\tau,\sigma)k(\sigma, y(\sigma) + T(\sigma - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(.))))d\sigma\| \\
&\qquad + \| \int_{t_0}^{t_0+\beta} b(\tau,\sigma)h(\sigma, y(\sigma) + T(\sigma - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(.))))d\sigma\|]d\tau \\
&\leq \int_{t_0}^{s} \|T(s-\tau) - T(t-\tau)\| l(\tau)[m + K(\|x_0\| + G)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_{t_0}^{\tau} Mp(\sigma)(\|y(\sigma) + T(\sigma - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(.)))\|)d\sigma \\
&\quad + \int_{t_0}^{t_0+\beta} Nq(\sigma)(\|y(\sigma) + T(\sigma - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(.)))\|)d\sigma]d\tau \\
&\quad + \int_{s}^{t} Kl(\tau)[m + K(\|x_0\| + G) \\
&\quad + \int_{t_0}^{\tau} Mp(\sigma)(\|y(\sigma) + T(\sigma - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(.)))\|)d\sigma \\
&\quad + \int_{t_0}^{t_0+\beta} Nq(\sigma)(\|y(\sigma) + T(\sigma - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(.)))\|)d\sigma]d\tau \\
&\le \int_{t_0}^{s} \|T(s-\tau) - T(t-\tau)\| l(\tau)[m + k^* + \int_{t_0}^{\tau} MP(m + K(\|x_0\| + G))d\sigma \\
&\quad + \int_{t_0}^{t_0+\beta} NQ(m + K(\|x_0\| + G))d\sigma]d\tau \\
&\quad + \int_{s}^{t} Kl(\tau)[m + k^* + \int_{t_0}^{\tau} MP(m + K(\|x_0\| + G))d\sigma \\
&\quad + \int_{t_0}^{t_0+\beta} NQ(m + K(\|x_0\| + G))d\sigma]d\tau \\
&\le (m + k^*) \int_{t_0}^{s} \|T(s-\tau) - T(t-\tau)\| L[1 + \beta MP + \beta NQ]d\tau \\
&\quad + (m + k^*) \int_{s}^{t} KL[1 + \beta MP + \beta NQ]d\tau, \qquad (3.7)
\end{aligned}
$$

where $Q = \sup_{t\in[t_0,t_0+\beta]}\{q(t)\}$. The compactness of $T(t)$, for $t > 0$ implies the continuity in uniform operator topology. Hence the right hand side of (3.7) tends to zero as $s - t \to 0$. Thus F_{B_m} is an equicontinuous family of functions with values in X.

We next show that F_{B_m} is uniformly bounded. From the definition of F in (3.5) and using hypotheses $(H_1) - (H_7)$ and the fact that $\|y\|_B \le m$, we obtain

$$
\begin{aligned}
&\|(Fy)(t)\| \\
&\le \int_{t_0}^{t} \|T(t-s)\| l(s)[\|y(s) + T(s - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(.)))\| \\
&\quad + \| \int_{t_0}^{s} a(s,\tau)k(\tau, y(\tau) + T(\tau - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(.))))d\tau\|
\end{aligned}
$$

$$
\begin{aligned}
&+ \| \int_{t_0}^{t_0+\beta} b(s,\tau)h(\tau, y(\tau) + T(\tau - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(.))))d\tau\|]ds \\
&\leq \int_{t_0}^{t} KL[m + k^* + \int_{t_0}^{s} MP(m + K(\|x_0\| + G))d\tau \\
&+ \int_{t_0}^{t_0+\beta} NQ(m + \|x_0\| + G))d\tau]ds \\
&\leq \beta(m + k^*)KL[1 + \beta MP + \beta NQ].
\end{aligned}
$$

This implies that the set $\{(Fy)(t) : \|y\|_B \leq m,\ t_0 \leq t \leq t_0 + \beta\}$ is uniformly bounded in X and hence $\{F_{B_m}\}$ is uniformly bounded.

We have already shown that F_{B_m} is an equicontinuous and uniformly bounded collection. To prove the set F_{B_m} is precompact in B, it is sufficient, by Arzela-Ascoli's argument, to show that the set $\{(Fy)(t) : y \in B_m\}$ is precompact in X for each $t \in [t_0, t_0 + \beta]$. Since $(Fy)(t_0) = 0$ for $y \in B_m$, it suffices to show this for $t_0 < t \leq t_0 + \beta$. Let $t_0 < t \leq t_0 + \beta$ be fixed and ϵ a real number satisfying $t_0 < \epsilon < t$. For $y \in B_m$, we define

$$
\begin{aligned}
(F_\epsilon y)(t) = \int_{t_0}^{t-\epsilon} & T(t-s)f(s, y(s) + T(s - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(.))), \\
& \int_{t_0}^{s} a(s,\tau)k(\tau, y(\tau) + T(\tau - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(.))))d\tau, \\
& \int_{t_0}^{t_0+\beta} b(s,\tau)h(\tau, y(\tau) + T(\tau - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(.))))d\tau)ds.
\end{aligned}
$$

Since $T(t)$ is compact operator and the set FB_m is bounded in B, the set $Y_\epsilon(t) = \{(F_\epsilon y)(t) : y \in B_m\}$ is precompact in X for every ϵ, $t_0 < \epsilon < t$. Moreover for every $y \in B_m$, we have

$$
\begin{aligned}
&(Fy)(t) - (F_\epsilon y)(t) \\
&= \int_{t-\epsilon}^{t} T(t-s)f(s, y(s) + T(s - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(.))), \\
&\int_{t_0}^{s} a(s,\tau)k(\tau, y(\tau) + T(\tau - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(.))))d\tau, \\
&\int_{t_0}^{t_0+\beta} b(s,\tau)h(\tau, y(\tau) + T(\tau - t_0)(x_0 - g(t_1, t_2, ..., t_p, x(.))))d\tau)ds.
\end{aligned} \tag{3.8}
$$

By making use of hypotheses $(H_1) - (H_7)$ and the fact that $\|y(s)\| \leq m$, we have

$$
\|(Fy)(t) - (F_\epsilon y)(t)\|
$$

$$\leq \int_{t-\epsilon}^{t} Kl(s)[\|y(s)\| + \|T(s-t_0)\|(\|x_0 + \|g(t_1,t_2,...,t_p,x(.))\|)$$
$$+ \int_{t_0}^{s} |a(s,\tau)|\|k(\tau, y(\tau) + T(\tau - t_0)(x_0 - g(t_1,t_2,...,t_p,x(.))))\|d\tau$$
$$+ \int_{t_0}^{t_0+\beta} |b(s,\tau)|\|h(\tau, y(\tau) + T(\tau - t_0)(x_0 - g(t_1,t_2,...,t_p,x(.))))\|d\tau]ds$$
$$\leq \int_{t-\epsilon}^{t} Kl(s)[m + K(\|x_0 + G)$$
$$+ \int_{t_0}^{s} Mp(\tau)(\|y(\tau) + T(\tau - t_0)(x_0 - g(t_1,t_2,...,t_p,x(.)))\|)d\tau$$
$$+ \int_{t_0}^{t_0+\beta} Nq(\tau)(\|y(\tau) + T(\tau - t_0)(x_0 - g(t_1,t_2,...,t_p,x(.)))\|)d\tau]ds$$
$$\leq KL(m + k^*)[1 + \beta MP + \beta NQ]\epsilon.$$

This shows that there exists precompact sets arbitrarily close to the set $\{(Fy)(t) : y \in B_m\}$. Hence the set $\{(Fy)(t) : y \in B_m\}$ is precompact in X. Thus we have shown that F is completely continuous operator.

Moreover, the set

$$\varepsilon(F) = \{y \in B_0 : y = \lambda F y \quad \text{for some} \quad 0 < \lambda < 1\},$$

is bounded in B, since for every y in $\varepsilon(F)$, the function $x(t) = y(t) + T(t - t_0)(x_0 - g(t_1,t_2,...,t_p,x(.)))$ is a solution of (3.1)–(1.2) for which we have proved $\|x\|_B \leq \gamma$ and hence $\|y\|_B \leq \gamma + k^*$. Now, by virtue of Theorem 2.3, the operator F has a fixed point in B_0. Therefore, the initial value problem (1.1)–(1.2) has a solution on $[t_0, t_0 + \beta]$. This completes the proof of the Theorem 2.4.

4. Application

To illustrate the application of our main result, consider the following nonlinear mixed Volterra- Fredholm partial integrodifferential equation of the form

$$\frac{\partial}{\partial t} w(u,t) - \frac{\partial^2}{\partial^2 t} w(u,t) = H(t, w(u,t), \int_0^t a_1(t,s)k_1(s, w(u,s))ds, \int_0^\beta b_1(t,s)h_1(s, w(u,s))ds), \tag{4.1}$$

$$w(0,t) = w(\pi,t) = 0, \; 0 \leq t \leq \beta, \tag{4.2}$$

$$w(u,0) + \sum_{i=1}^{p} w(u,t_i) = w_0(u),$$

$$0 < t_1 < t_2 < ... < t_p \leq \beta, \quad 0 \leq u \leq \pi. \tag{4.3}$$

where $H : [0,\beta]\times R\times R\times R \to R$, $k_1,\ h_1 : [0,\beta]\times R \to R$, $a_1,\ b_1 : [0,\beta]\times[0,\beta] \to R$ are continuous functions. We assume that the functions $H,\ a_1, b_1,\ k_1$ and h_1 in (4.1)–(4.3) satisfy the following conditions:

(1) There exists a constant G_1 such that

$$G_1 = \max_{w\in R} \|\sum_{i=1}^{p} w(u,t_i)\|.$$

(2) There exists a nonnegative function p_1 defined on $[0,\beta]$ such that

$$|k_1(t,x)| \leq p_1(t)|x|$$

for $t \in [0,\beta]$ and $x \in R$.

(3) There exists a nonnegative function q_1 defined on $[0,\beta]$ such that

$$|h_1(t,x)| \leq q_1(t)|x|$$

for $t, s \in [0,\beta]$ and $x \in R$.

(4) There exists a constant M_1 such that

$$|a(t,s)| \leq M_1, \quad \text{for} \quad 0 \leq s \leq t \leq \beta.$$

(5) There exists a constant N_1 such that

$$|a(t,s)| \leq N_1, \quad \text{for} \quad 0 \leq s \leq t \leq \beta.$$

(6) There exists nonnegative real valued continuous function l_1 defined on $[0,\beta]$ such that

$$|H(t,x,y,z)| \leq l_1(t)(|x| + |y| + |z|)$$

for $t \in [0,\beta]$ and $x, y, z \in R$.

(7) For every positive integer m_1 there exists $\alpha_{m_1} \in L^1[0,\beta]$ such that

$$\sup_{|x|\leq m_1, |y|\leq m_1, |z|\leq m_1} |H(t,x,y,z)| \leq \alpha_{m_1}(t),$$

for $0 \leq t \leq \beta,\ a.\ e.$

Let us take $X = L^2[0,\pi]$. Define the operator $A : X \to X$ by $Az = -z''$ with domain $D(A) = \{z \in X : z,\ z'$ are absolutely continuous, $z'' \in X$ and $z(0) = z(\pi) = 0\}$. Then the operator A can be written as

$$Az = \sum_{n=1}^{\infty} n^2(z, z_n)z_n,\ z \in D(A)$$

where $z_n(u) = (\sqrt{2/\pi})\sin nu,\ n = 1, 2, ...$ is the orthogonal set of eigenvectors of A and A is the infinitesimal generator of an analytic semigroup $T(t),\ t \geq 0$

and is given by

$$T(t)z = \sum_{n=1}^{\infty} e^{-n^2 t}(z, z_n)z_n, \; z \in X.$$

Now, the analytic semigroup $T(t)$ being compact, there exists constant K_0 such that

$$|T(t)| \leq K_0, \quad \text{for each} \quad t \in [0, \beta].$$

Suppose that the condition

$$r_1^* = \int_0^\beta N_1 q_1(\sigma) \exp\Big(\int_0^\sigma [K_0 l_1(\tau) + M_1 p(\tau)]d\tau\Big)d\sigma < 1$$

is satisfied and $k_1^* = K_0(|w_0| + G_1)$. Define the functions $f : [0, \beta] \times X \times X \times X \to X$, $k : [0, \beta] \times X \to X$, $h : [0, \beta] \times X \to X$ and $a,\ b : [0, \beta] \times [0, \beta] \to R$ as follows

$$\begin{aligned}
&f(t, x, y, z)(u) = H(t, x(u), y(u), z(u)),\\
&k(t, x)(u) = k_1(t, x(u)),\\
&h(t, x)(u) = h_1(t, x(u)), \quad \text{and}\\
&a(t, s) = a_1(t, s),\; b(t, s) = b_1(t, s)
\end{aligned}$$

for $t \in [0, \beta],\ x, y, z \in X$ and $0 \leq u \leq \pi$. Then the above problem (4.1)–(4.3) can be formulated abstractly as nonlinear mixed Volterra-Fredholm integrodifferential equation in Banach space X:

$$x'(t) + Ax(t) = f\Big(t, x(t), \int_{t_0}^{t} a(t, s)k(s, x(s))ds, \int_{t_0}^{t_0+\beta} b(t, s)h(s, x(s))ds\Big), \quad t \in [t_0, t_0 + \beta] \tag{4.4}$$

$$x(t_0) + g(t_1, t_2, ..., t_p, x(\cdot)) = x_0. \tag{4.5}$$

Since all the hypotheses of the Theorem 2.4 are satisfied, the Theorem 2.4 can be applied to guarantee the mild solution of the nonlinear mixed Volterra-Fredholm partial integrodifferential equations (4.1)–(4.3).

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