Nonlinear Functional Analysis and Applications
Vol. 15, No. 3 (2010), pp. 435-458

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

OSCILLATION CRITERIA OF SECOND-ORDER NONLINEAR NEUTRAL FUNCTIONAL DYNAMIC EQUATIONS ON TIME SCALES

S. H. Saker

Department of Mathematics Skills, PY, King Saud University,
Riyadh 11451, Saudi Arabia,
Department of Mathematics, Faculty of Science,
Mansoura University, Mansoura, 35516, Egypt.
e-mail: `shsaker@mans.edu.eg`

Abstract. The purpose of this paper is to establish some new sufficient conditions for oscillation of the second-order neutral functional dynamic equation

$$\left[r(t)\left[y(t)+p(t)y(\tau(t))\right]^{\Delta}\right]^{\Delta}+q(t)f(y(\delta(t)))=0,$$

on a time scale T. The main investigation of the results depends on the generalized Riccati substitution and the analysis of the associated Riccati dynamic inequality. The results improve some oscillation results for neutral dynamic equations in the sense that our results do not require that $r^{\Delta}(t)\geq 0$ and $\int_{t_0}^{\infty}\delta(s)q(s)[1-p(\delta(s))]\Delta s=\infty$.

1. Introduction

In this paper, we are concerned with oscillation of the nonlinear neutral functional dynamic equation

$$\left(r(t)\left[y(t)+p(t)y(\tau(t))\right]^{\Delta}\right)^{\Delta}+q(t)f(y(\delta(t)))=0, \tag{1.1}$$

on a time scale $\mathbb{T}$. Throughout this paper, we will assume the following hypotheses:

---

[0]Received Jan. 4, 2009. Revised June 2, 2009.
[0]2000 Mathematics Subject Classification: 34K11, 39A10, 39A99.
[0]Keywords: Oscillation, neutral dynamic equations, time scales

(h_1) $r(t)$, $p(t)$ and $q(t)$ are real valued $rd-$continuous positive functions defined on $\mathbb{T}$, $\tau(t):\mathbb{T}\to\mathbb{T}$, $\delta(t):\mathbb{T}\to\mathbb{T}$, $\tau(t)\leq t$, for all $t\in\mathbb{T}$ and $\lim_{t\to\infty}\delta(t)=\lim_{t\to\infty}\tau(t)=\infty$;

(h_2) $\int_{t_0}^{\infty}\left(\frac{1}{r(t)}\right)\Delta t=\infty$, $0\leq p(t)<1$;

(h_3) $f:\mathbb{R}\to\mathbb{R}$ is continuous function such that $uf(u)>0$ for all $u\neq 0$ and $|f(u)|\geq K\,|u|$ for $K>0$.

Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup\mathbb{T}=\infty$, and define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$. Throughout this paper these assumptions will be supposed to hold. Let $\tau^*(t)=\min\{\tau(t),\delta(t)\}$ and let $T_0=\min\{\tau^*(t):t\geq 0\}$ and $\tau^*_{-1}(t)=\sup\{s\geq 0:\tau^*(s)\leq t\}$ for $t\geq T_0$. Clearly if $\tau^*(t)\leq t$, then $\tau^*_{-1}(t)\geq t$ for $t\geq T_0$, where $\tau^*_{-1}(t)$ is nondecreasing and coincides with the inverse of $\tau^*(t)$ when the latter exists. Throughout the paper, we will use the following notations:

$$x(t):=y(t)+p(t)y(\tau(t)),\ x^{[1]}:=r\left(x^{\Delta}\right),\ \text{and}\quad x^{[2]}:=\left(x^{[1]}\right)^{\Delta}. \tag{1.2}$$

By a solution of (1.1), we mean a nontrivial real-valued function $y(t)$ which has the properties $x(t)\in C^1_{rd}[\tau^*_{-1}(t_0),\infty)$, and $x^{[1]}\in C^1_{rd}[\tau^*_{-1}(t_0),\infty)$ where C_r is the space of $rd-$continuous functions. Our attention is restricted to those solutions of (1.1) which exist on some half line $[t_y,\infty)$ and satisfy $\sup\{|y(t)|:t>t_1\}>0$ for any $t_1\geq t_y$. A solution $y(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

The study of dynamic equations on time scales, which goes back to its founder Stefan Hilger [23], is an area of mathematics that has recently received a lot of attention. It has been created in order to unify the study of differential and difference equations. This way results not only related to the set of real numbers or set of integers but those pertaining to more general time scales are obtained.

The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus (see Kac and Cheung [28]), i.e, when $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{N}$ and $\mathbb{T}=q^{\mathbb{N}_0}=\{q^t:\ t\in\mathbb{N}_0\}$ where $q>1$. Dynamic equations on a time scale have an enormous potential for applications such as in population dynamics. There are applications of dynamic equations on time scales to *quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatorics.* A recent cover story article in New Scientist [42] discusses several possible applications. The book on the subject of time scale, i.e., measure chain, by Bohner and Peterson [10] summarizes and organizes much of time scale calculus. For completeness, we recall the following concepts related to the notion of time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The forward jump operator and the backward jump operator are defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$, where $\sup \emptyset = \inf \mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right–dense continuous (rd–continuous) provided g is continuous at right–dense points and at left–dense points in T, left hand limits exist and are finite. The set of all such $rd-$continuous functions is denoted by $C_{rd}(T)$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$.

One of the important techniques used in studying oscillations of dynamic equations on time scales is the averaging function method. On the other hand, the oscillatory properties can be described by the so called Reid Roundabout Theorem (cf.[10]). This theorem shows the connection among the concepts of disconjugacy, positive definiteness of the quadratic functional, and the solvability of the corresponding Riccati equation (or inequality) which in turn implies the existence of nonoscillatory solutions. The Reid Roundabout theorem provides two powerful tools for the investigation of oscillatory properties, namely the Riccati techniques (Riccati and generalized Riccati techniques) and the variational principle. The main investigation of the two techniques depends on the reduction of the equation to a Riccati equation (or inequality). For oscillation of second-order dynamic equations, we refer the reader to the papers [1, 2, 7, 8, 9, 12, 13, 15, 17, 18, 19, 20, 30, 31, 32, 34, 35, 38, 39] and the references cited therein. For oscillation of second-order neutral dynamic equations we refer the reader to the papers [3], [4], [5], [26], [27], [33], [36], [37], [41] and [43].

Agarwal et al.[3] considered the second-order nonlinear neutral delay dynamic equation

$$\left[r(t)([y(t)+p(t)y(t-\tau)]^{\Delta})^{\gamma}\right]^{\Delta} + f(t, y(t-\delta)) = 0, \tag{1.3}$$

on a time scale $\mathbb{T}$; here $\gamma > 0$ is a quotient of odd positive integers, τ and δ are positive constants such that the delay functions $\tau(t) := t - \tau < t$ and $\delta(t) := t - \delta < t$ satisfy $\tau(t) : \mathbb{T} \to \mathbb{T}$ and $\delta(t) : \mathbb{T} \to \mathbb{T}$ for all $t \in \mathbb{T}$, $p(t)$ and $r(t)$ are real valued $rd-$continuous positive functions defined on $\mathbb{T}$, and the following conditions are satisfied:

- (A_1) $\int_{t_0}^{\infty} (1/r(t))^{\frac{1}{\gamma}} \Delta t = \infty$, $0 \le p(t) < 1$,
- (A_2) $f(t, u) : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is continuous function such that $uf(t, u) > 0$ for all $u \ne 0$ and there exists a positive $rd-$continuous function $q(t)$ defined on $\mathbb{T}$ such that $|f(t, u)| \ge q(t) |u^{\gamma}|$.

In [3] the authors considered the case when $\gamma > 0$ is an odd positive integer and proved that the oscillation of (1.3) is equivalent to the oscillation of a first order delay dynamic inequality and established some sufficient conditions for oscillation. Also they considered the case when $\gamma \geq 1$ and established some sufficient conditions for oscillation by employing the Riccati technique. The results were applied only in discrete time scales, i.e., when the graininess function $\mu(t) \neq 0$.

Saker [33] considered (1.3) where $\gamma \geq 1$ is an odd positive integer, $(A_1)-(A_2)$ hold and established some new sufficient conditions for oscillation of (1.3) by employed the Riccati transformation technique. However the results established in [3, 33] can be applied only on the time scales $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{N}$, $\mathbb{T}=h\mathbb{N}$ and $\mathbb{T}=q^{\mathbb{N}}=\{t: t=q^k,\ k\in\mathbb{N},\ q>1\}$, and cannot be applied on the time scales $\mathbb{T}=\mathbb{N}^2=\{t^2: t\in\mathbb{N}\}$, $\mathbb{T}_2=\{\sqrt{n}: n\in\mathbb{N}_0\}$, $\mathbb{T}_3=\{\sqrt[3]{n}: n\in\mathbb{N}_0\}$, and $\mathbb{T}=\mathbb{T}_n=\{t_n: n\in\mathbb{N}_0\}$ where $\{t_n\}$ is the set of harmonic numbers. This follows from the fact that when $t\in\mathbb{T}$, the functions $t-\tau$ and $t-\delta$ may be not belong to the time scales $\mathbb{T}=\mathbb{N}^2$, $\mathbb{T}=\mathbb{T}_2$, $\mathbb{T}=\mathbb{T}_3$ and $\mathbb{T}=\mathbb{T}_n$.

Şahiner [27] considered the general equation

$$\left[r(t)\left([y(t)+p(t)y(\tau(t))]^{\Delta}\right)^{\gamma}\right]^{\Delta}+f(t,y(\delta(t)))=0, \tag{1.4}$$

on a time scale $\mathbb{T}$ and followed the argument in [33] by reducing the oscillation of (1.4) to oscillation of a first order delay dynamic inequality and established some sufficient conditions for oscillation, when the following conditions are satisfied:

(B_1) δ, τ are positive $rd-$continuous functions, $\delta,\ \tau:\mathbb{T}\to\mathbb{T}$,

(B_2) $\int_{t_0}^{\infty}(1/r(t))^{\frac{1}{\gamma}}\Delta t=\infty$, $\gamma\geq 1$,and $0\leq p(t)<1$;

(B_3) $f(t,u):\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is a continuous function with $uf(t,u)>0$ for all $u\neq 0$ and there exists a positive $rd-$continuous function $q(t)$ defined on $\mathbb{T}$ such that $|f(t,u)|\geq q(t)\,|u^{\gamma}|$.

However one can easily see that the two examples that are given in [27] to illustrate the main results are valid only when $\mathbb{T}=\mathbb{R}$ and cannot be applied when $\mathbb{T}=\mathbb{N}$ since the delay functions that are considered in this paper are given by $t/2$, $\sqrt{t}$ and $t/64$ which are not in $C_{rd}(\mathbb{T},\ \mathbb{T})$ for a general time scale $\mathbb{T}$. Also the results cannot give a sharp sufficient condition for oscillation of (1.4) when $q(t)=\gamma/t^2$.

Wu et al. [43] considered also (1.4) on a time scale $\mathbb{T}$. They followed the argument in [33] by using the Riccati transformation technique and the Chain rule $(\omega\circ\nu)^{\Delta}(t)=(w^{\bar{\Delta}}\circ\nu)\nu^{\Delta}$, where $\bar{\Delta}$ is the delta derivative defined on $\tilde{\mathbb{T}}$ and $\nu(t)$ is strictly increasing, and established some sufficient conditions for oscillation of (1.4), when the following conditions are satisfied:

(C_1) $\delta:\mathbb{R}\to\mathbb{R}$ is continuous $\delta:\mathbb{T}\to\mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}}=\delta(\mathbb{T})\subset\mathbb{T}$ is a time scale;

(C_2) $(\delta \circ \sigma)(t) = (\sigma \circ \delta)(t)$;

(C_3) $\int_{t_0}^{\infty} (1/r(t))^{\frac{1}{\gamma}} \Delta t = \infty$, $\gamma \geq 1$, and $0 \leq p(t) < 1$;

(C_4) $f(t,u) : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is a continuous function with $uf(t,u) > 0$ for all $u \neq 0$ and there exists a positive $rd-$continuous function $q(t)$ defined on $\mathbb{T}$ such that $|f(t,u)| \geq q(t) |u^{\gamma}|$.

We note that the results in [43], which are based on the Chain rule, can only be applied if $\widetilde{\mathbb{T}}$ is a time scale and if $\tau(t) \leq t$ and $\delta(t) \geq \tau(\delta(t))$. The condition (C_2) also can be a restrictive condition, since on the time scale $\mathbb{T}{=}q^{\mathbb{N}}$ by choosing $\delta(t) = t - q^{n_0}$ one can easily see that $\delta(\sigma(t)) = \delta(qt) = qt - q^{n_0} \neq \sigma(\delta(t)) = q(t - q^{n_0}) = qt - q^{n_0+1}$, so the results in [43] cannot be applied on the time scale $\mathbb{T}{=}q^{\mathbb{N}}$ when $\delta(t) = t - q^{n_0}$. Also in the proof of the main results in [43, Lemma 2.5] the authors used the Chain rule $(f(g(t)))^{\Delta} = f^{\Delta}(g(t))g^{\Delta}(t)$ which is not true on general time scales. Of course trivially $(x \circ \tau)^{\Delta} = (x^{\Delta} \circ \tau)\tau^{\Delta}$ if δ is a constant with $\tau(t) = t - \delta \in \mathbb{T}$ for $t \in \mathbb{T}$.

Agarwal, O'Regan and Saker [4] considered the general nonlinear neutral delay dynamic equation (1.4) where $\gamma \geq 1$ is an odd positive integer,

(D_1) $\tau(t) : \mathbb{T} \to \mathbb{T}$, $\delta(t) : \mathbb{T} \to \mathbb{T}$, $\tau(t) \leq t$, $\delta(t) \leq t$ for all $t \in \mathbb{T}$ and $\lim_{t\to\infty} \delta(t) = \lim_{t\to\infty} \tau(t) = \infty$;

(D_2) $\int_{t_0}^{\infty} (1/r(t))^{\frac{1}{\gamma}} \Delta t = \infty$, $r^{\Delta}(t) \geq 0$, $0 \leq p(t) < 1$;

(D_3) $f(t,u) : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is continuous function such that $uf(t,u) > 0$ for all $u \neq 0$ and there exists a positive $rd-$continuous function $q(t)$ defined on $\mathbb{T}$ such that $|f(t,u)| \geq q(t) |u^{\gamma}|$,

and employed the Riccati technique and established some new oscillation criteria which can be applied on any time scale $\mathbb{T}$ and improved the results established in [3], [27], [33] and [43].

Agarwal, O'Regan and Saker [5] considered the nonlinear neutral delay dynamic equation

$$\left[r(t) \left[y(t) + p(t)y(\tau(t))\right]^{\Delta}\right]^{\Delta} + q(t)f(y(\delta(t))) = 0, \tag{1.5}$$

on a time scale $\mathbb{T}$ and assumed that $r(t)$, $p(t)$ and $q(t)$ are real valued $rd-$continuous positive functions defined on $\mathbb{T}$ and

(E_1) $\tau(t) : \mathbb{T} \to \mathbb{T}$, $\delta(t) : \mathbb{T} \to \mathbb{T}$, $\tau(t) \leq t$, $\delta(t) \leq t$ for all $t \in \mathbb{T}$ and $\lim_{t\to\infty} \delta(t) = \lim_{t\to\infty} \tau(t) = \infty$;

(E_2) $\int_{t_0}^{\infty} (1/r(t)) \Delta t = \infty$, $r^{\Delta}(t) \geq 0$, $0 \leq p(t) < 1$;

(E_3) $f(u) : \mathbb{R} \to \mathbb{R}$ is continuous function such that $uf(u) > 0$, and $f(u)/u \geq K > 0$ for all $u \neq 0$

They employed the generalized Riccati technique and established some sufficient conditions for oscillation and studied the asymptotic behavior of the nonoscillatory solutions. These results in the special case when $\mathbb{T} = \mathbb{R}$ extend and improve the results established by Li and Liu [24] for neutral delay differential equations. In the case when $\mathbb{T} = \mathbb{N}$ the results extended and improved the results established by Li and Yeh [25].

Remark 1.1. *We note that all the above results are given in the case when*

$$r^{\Delta}(t) \geq 0 \text{ and } \int_{t_0}^{\infty} \delta^{\gamma}(s)q(s)[1-p(\delta(s))]^{\gamma}\Delta s = \infty, \ \gamma \geq 1, \tag{1.6}$$

and $\delta(t) \leq t$ and nothing is known regarding the oscillation of neutral dynamic equations when (1.6) does not hold and and $\delta(t) > t$. So the natural question now is: If it is possible to find new oscillation criteria for (1.1) when (1.6) does not hold and $\delta(t) > t$? One of our aims in this paper is to give an affirmative answer to this question.

As a special case of (1.1), if $p(t) = 0$ and $\delta(t) = \sigma(t)$, then (1.1) becomes the second-order dynamic equation

$$(r(t)x^{\Delta}(t))^{\Delta} + q(t)(f \circ x^{\sigma}) = 0. \tag{1.7}$$

Erbe and Peterson [16] considered the equation (1.7), when $f(u) = u$ and supposed that there exists $t_0 \in \mathbb{T}$, such that $r(t)$ is bounded above on $[t_0, \infty)$, $h_0 = \inf\{\mu(t) : t \in [t_0, \infty)\} > 0$, and showed via Riccati techniques that

$$\int_{t_0}^{\infty} q(t)\Delta t = \infty.$$

Bohner and Saker [12], considered (1.7), when $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $xf(x) > 0$, $|f(x)| \geq K|x|$ for $x \neq 0$ and$K > 0$,

$$\int_{t_0}^{\infty} \frac{\Delta t}{r(t)} = \infty,$$

and employed the Riccati transformation and established some sufficient conditions for oscillation.

Erbe and Peterson and Saker [15] considered (1.7) and employed the generalized Riccati transformation techniques and generalized exponential functions, and established some oscillation criteria. The main results depend on an $rd-$continuous function $r(t)$ such that $(p \cdot r)$ is a differentiable function, and

$$(A) \quad \text{There exists } M > 0 \text{ such that } r(t)e_r(t, t_0)p(t) \leq M, \text{ for all large } t. \tag{1.8}$$

Also one can easily see that the results that has been established in these papers cannot be applied on the equation

$$\left(\frac{1}{t}x^{\Delta}(t)\right)^{\Delta} + \frac{1}{t^3}x^{\sigma}(t) = 0. \tag{1.9}$$

Our results in this paper as a special case can be applied on this type of equations when $p^{\Delta}(t) \leq 0$.

We note that (1.1) in its general form involve some different types of differential and difference equations depending on the choice of the time scale $\mathbb{T}$. For example, when $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = t$, $\mu(t) = 0$, $f^{\Delta}(t) = f'(t)$ and (1.1) becomes the second-order neutral differential equation

$$\left(r(t)\left[y(t) + p(t)y(\tau(t))\right]'\right)' + q(t)f(y(\delta(t))) = 0. \tag{1.10}$$

Numerous oscillation criteria have been established for the second-order neutral delay differential equation (1.10) and some special cases of it, we refer the reader to the papers [22, 21] and the references cited therein.
Grammatikopoulos et al. [22] considered the second-order linear neutral delay differential equation

$$[y(t) + p(t)y(t-\tau)]'' + q(t)y(t-\delta) = 0, \qquad t \geq t_0, \tag{1.11}$$

and proved that: If $q(t) > 0$, $0 \leq p(t) < 1$ and

$$\int_{t_0}^{\infty} q(s)[1 - p(s-\delta)]ds = \infty, \tag{1.12}$$

then every solution of (1.11) oscillates.

Graef et al. [21] considered the second-order nonlinear delay neutral equation

$$[y(t) + p(t)y(t-\tau)]'' + q(t)f(y(t-\delta)) = 0, \qquad t \geq t_0, \tag{1.13}$$

and extended the condition (1.12) and proved that: If $q(t) > 0$, $0 \leq p(t) < 1$ and

$$\int_{t_0}^{\infty} q(s)f((1 - p(s-\delta)c)ds = \infty, \;\; c > 0, \tag{1.14}$$

then every solution of (1.13) oscillates. Note that the conditions (1.12) and (1.14) cannot be applied on the case when $q(t) = \beta/t^2$ where β is a positive constant.

When $\mathbb{T} = \mathbb{N}$, we have $\sigma(n) = n + 1$, $\mu(n) = 1$, $y^{\Delta}(n) = \Delta y(n) = y(n + 1) - y(n)$ and (1.1) becomes the second-order neutral difference equation

$$\Delta(r(n)(\Delta\left[y(n) + p(n)y(\tau(n))\right]\!)) + q(n)f(y(\delta(n))) = 0. \tag{1.15}$$

For oscillation of second-order neutral delay difference equations, as a special case of (1.15), Zhang and Cheng [44] considered the equation

$$\Delta[r(n)\Delta(y(n) + p(n)y(n-\tau))] + q(n)y(n-\delta) = 0, \qquad n \geq n_0, \tag{1.16}$$

and proved that: If $r(n) > 0$, $q(n) > 0$, $0 \leq p(n) < 1$,

$$\sum_{i=n_0}^{\infty} r^{-1}(i) = \infty, \text{ and } \sum_{i=n_0}^{\infty} q(i)[1 - p(i - \delta)] = \infty, \tag{1.17}$$

then every solution of (1.16) oscillates. Note that the condition (1.17) can not be applied to the second-order neutral delay difference equation (1.16) when $q(n) = \beta/n^2$ where β is a positive constant.

When $\mathbb{T} = h\mathbb{N}$, $h > 0$, we have $\sigma(t) = t + h$, $\mu(t) = h$, $y^{\Delta}(t) = \Delta_h y(t) = (y(t + h) - y(t))/h$, and (1.1) becomes the second-order neutral difference equation

$$\Delta_h(r(t)\Delta_h [y(t) + p(t)y(\tau(t))]) + q(t)f(y(\delta(t))) = 0. \tag{1.18}$$

When $\mathbb{T}=q^{\mathbb{N}} = \{t : t = q^n, \ n \in \mathbb{N}, \ q > 1\}$, we have $\sigma(t) = qt$, $\mu(t) = (q - 1)t$,

$$y^{\Delta}(t) = \Delta_q y(t) = (y(q\,t) - y(t))/((q - 1)t),$$

and (1.1) becomes the second-order $q-$neutral difference equation

$$\Delta_q(r(t)\Delta_q [y(t) + p(t)y(\tau(t))]) + q(t)f(y(\delta(t))) = 0. \tag{1.19}$$

When $\mathbb{T} = \mathbb{N}^2 = \{t^2 : t \in \mathbb{N}\}$, we have $\sigma(t) = (\sqrt{t} + 1)^2$ and $\mu(t) = 1 + 2\sqrt{t}$,

$$y^{\Delta}(t) = \Delta_{N^2} y(t) = (y((\sqrt{t} + 1)^2) - y(t))/(1 + 2\sqrt{t}),$$

and (1.1) becomes the second-order neutral difference equation

$$\Delta_{N^2}(r(t)\Delta_{N^2} [y(t) + p(t)y(\tau(t))]) + q(t)f(y(\delta(t))) = 0. \tag{1.20}$$

When $\mathbb{T} = \mathbb{T}_n = \{t_n : n \in \mathbb{N}\}$ where $\{t_n\}$ is the set of harmonic numbers defined by

$$t_0 = 0, \quad t_n = \sum_{k=1}^{n} \frac{1}{k}, \ n \in \mathbb{N}_0,$$

we have $\sigma(t_n) = t_{n+1}$, $\mu(t_n) = \frac{1}{n+1}$, $y^{\Delta}(t) = \Delta_{t_n} y(t_n) = (n + 1)y(t_n)$, and (1.1) becomes the second-order neutral difference equation

$$\Delta_{t_n}(r(t_n)\Delta_{t_n} [y(t_n) + p(t_n)y(\tau(t_n))]) + q(t_n)f(y(\delta(t_n))) = 0. \tag{1.21}$$

When $\mathbb{T} = \mathbb{T}_2 = \{\sqrt{n} : n \in \mathbb{N}_0\}$, we have $\sigma(t) = \sqrt{t^2 + 1}$ and $\mu(t) = \sqrt{t^2 + 1} - t$, $x^{\Delta}(t) = \Delta_2 x(t) = (x(\sqrt{t^2 + 1}) - x(t))/\sqrt{t^2 + 1} - t$, and (1.1) becomes the second-order neutral difference equation

$$\Delta_2(r(t)\Delta_2 [y(t) + p(t)y(\tau(t))]) + q(t)f(y(\delta(t))) = 0. \tag{1.22}$$

When $\mathbb{T} = \mathbb{T}_3 = \{\sqrt[3]{n} : n \in \mathbb{N}_0\}$, we have $\sigma(t) = \sqrt[3]{t^3 + 1}$ and $\mu(t) = \sqrt[3]{t^3 + 1} - t$, $x^{\Delta}(t) = \Delta_3 x(t) = (x(\sqrt[3]{t^3 + 1}) - x(t))/\sqrt[3]{t^3 + 1} - t$, and (1.1) becomes the second-order neutral difference equation

$$\Delta_3(r(t)(\Delta_3 [y(t) + p(t)y(\tau(t))]) + q(t)f(y(\delta(t))) = 0. \tag{1.23}$$

In this paper, we establish some new sufficient conditions for oscillation of (1.1). The main investigation of the main oscillation results depends on the generalized Riccati substitution and the analysis of the associated Riccati dynamic inequality. The results in the Subsection 2.1 cover the case when $\delta(t) > t$ and the results in the Subsection 2.2 cover the case when $\delta(t) \leq t$. The results in this paper are different from the results that has been established in the literature for second order neutral dynamic equations, in the sense that the results do not require the condition (1.6) when $\gamma = 1$ and can be applied on the case when $\delta(t) > t$.

2. Main Results

In this section, we state and prove the main oscillation results. We note that if $y(t)$ is a solution of (1.1) then $z(t) = -y(t)$ is also solution of (1.1), since $uf(u) > 0$ for $u \neq 0$. Thus, concerning nonoscillatory solutions of (1.1) we can restrict our attention to the positive ones. We start with the following Lemma which will play an important role in the proof of the main results.

Lemma 2.1. *Assume that* $(h_1) - (h_3)$ *hold and* $y(t)$ *is a positive solution of (1.1) on* $[t_0, \infty)_{\mathbb{T}}$. *Let* $x(t)$ *is defined as in (1.2). Then there exists* $T > t_0$ *such that* $x^{[1]}(t) > 0$, *for* $t \geq T$.

Proof. Since $y(t)$ is a positive solution of (1.1) on $[t_0, \infty)_{\mathbb{T}}$, we can pick $t_1 \in [t_0, \infty)_{\mathbb{T}}$ so that $t_1 > t_0$ and so that $y(t) > 0$, $y(\tau(t)) > 0$ and $y(\delta(t))$ on $[t_1, \infty)_{\mathbb{T}}$. (Note that in the case when $y(t)$ is negative the proof is similar, since the transformation $y(t) = -z(t)$ transforms (1.1) into the same form). Since $y(t)$ is a positive solution of (1.1), then from (1.2), since $q(t) > 0$, $x(t)$ is also positive and satisfies

$$(x^{[1]}(t))^{\Delta} \leq -q(t)f(y(\delta(t))) < 0, \quad \text{for} \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.1}$$

Then $x^{[1]}(t)$ is strictly decreasing on for $t \geq t_1$. We claim that $x^{[1]}(t) > 0$ for $t \geq t_1$. Assume not. Then there is a $t_2 > t_1$ such that $x^{[1]}(t_2) =: c < 0$. Then from (2.1), we have $x^{[1]}(t) \leq c$, for $t \geq t_2$, and therefore

$$x^{\Delta}(t) \leq \frac{c}{r(t)}, \quad \text{for} \quad t \geq t_2. \tag{2.2}$$

Integrating the last inequality form t_2 to t, we find by (h_2) that

$$x(t) = x(t_2) + \int_{t_2}^{t} x^{\Delta}(s)\Delta s \leq x(t_2) + c\int_{t_2}^{t} \frac{\Delta s}{r(s)} \to -\infty \text{ as } t \to \infty, \tag{2.3}$$

which implies that $x(t)$ is eventually negative. This contradiction completes the proof. □

Lemma 2.2. *Assume that* $(h_1)-(h_3)$ *hold and* $y(t)$ *is a positive solution of (1.1) on* $[t_0,\ \infty)_{\mathbb{T}}$. *Let* $x(t)$ *is defined as in (1.2). Then there exists* $T \geq t_0$ *such that*

$$(r(t)\left(x^{\Delta}(t)\right))^{\Delta}+Q(t)x(\delta(t))\leq 0,\ \textit{for}\ t\geq T, \tag{2.4}$$

where $Q(t)=Kq(t)(1-p(\delta(t)))^{\gamma}$.

Proof. Since $y(t)$ is a positive solution of (1.1) on $[t_0,\infty)_{\mathbb{T}}$, we can pick $t_1\in[t_0,\infty)_{\mathbb{T}}$ so that $t_1>t_0$ and so that $y(t)>0$, $y(\tau(t))>0$, $y(\tau(\tau(t)))>0$ and $y(\delta(t))$ $t\geq t_1$. (Note that in the case when $y(t)$ is negative the proof is similar, since the transformation $y(t)=-z(t)$ transforms (1.1) into the same form). Since $y(t)$ is a positive solution of (1.1), then from Lemma 1, we see that $x(t)$ satisfies

$$x(t)>0,\ x^{[1]}(t)\geq 0,\ (x^{[1]}(t))^{\Delta}<0,\ \text{for}\ t\geq t_1. \tag{2.5}$$

This implies that $x^{\Delta}(t)>0$, and accordingly

$$\begin{aligned} y(t) &= x(t)-r(t)y(\tau(t))=x(t)-r(t)[x(\tau(t))-r(\tau(t))y(\tau(\tau(t)))] \\ &\geq x(t)-r(t)x(\tau(t))\geq(1-r(t))x(t). \end{aligned} \tag{2.6}$$

Then for $t\geq t_2$ where $t_2>t$ is chosen large enough, we have

$$y(\delta(t))\geq(1-r(\delta(t)))x(\delta(t)). \tag{2.7}$$

From (2.1) and the last inequality, we have the inequality (2.4) and this completes the proof. □

2.1. **The case when** $\delta(t)>\sigma(t)\geq t$**.** In this subsection, we establish new oscillation criteria for (1.1) when $\delta(t)>\sigma(t)\geq t$.

We define the function space $\Re$ as follows: $H\in\Re$ provided H is defined for $t_0\leq s\leq\sigma(t)$, $t,s\in[t_0,\infty)_{\mathbb{T}}$ $H(t,s)\geq 0$, $H(\sigma(t),t)=0$, $H^{\Delta_s}(t,s)\leq 0$ for $t\geq s\geq t_0$, and for each fixed t, $H^{\Delta_s}(t,s)$ is delta integrable with respect to s. Important examples of H when $\mathbb{T}=\mathbb{R}$ are $H(t,s)=(t-s)^m$ for $m\geq 1$. When $\mathbb{T}=\mathbb{Z}$, $H(t,s)=(t-s)^{\underline{k}}$, $k\in\mathbb{N}$, where $t^{\underline{k}}=t(t-1)...(t-k+1)$.

Suppose that there exist two positive functions $\phi(t)$ and $\varphi(t)$ such that

$$\begin{aligned} \phi^{\Delta}(t) &= -2\varphi(t)\eta(t)\phi(t),\ \text{where}\ \eta(t):=\frac{r(t)R(t,T)}{r(t)R(t,T)+\sigma(t)-t}>0, \\ R(t,T) &:= \int_T^t\frac{1}{r(s)}\Delta s>0. \end{aligned} \tag{2.8}$$

We define

$$\bar{H}(t,s):=H(\sigma(t),\sigma(s)),\ \ A(t):=\frac{r(t)\phi^2(t)}{\eta(t)\phi^{\sigma}},$$

and assume that

$$\Phi(t):=\phi^{\sigma}\left[Q(t)-(\varphi(t)r(t))^{\Delta}+\eta(t)r(t)\varphi^2(t)\right]>0.$$

Theorem 2.3. *Assume that* $(h_1)-(h_3)$ *hold and* $H \in \Re$, *such that for sufficiently large* $T \geq t_0$

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t),t_2)} \int_{t_2}^{t} \left[\bar{H}(t,s)\Phi(s) - \frac{1}{4}\frac{A(s)(H^{\Delta_s}(\sigma(t),s))^2}{\bar{H}(t,s)} \right] \Delta s = \infty. \quad (2.9)$$

Then every solution of (1.1) is oscillatory.

Proof. Without loss of generality, we may assume that $y(t)$ is an eventually positive solution of (1.1) with $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$ and $y(\delta(t)) > 0$ for all $t \geq t_1 > t_0$ sufficiently large. Let $x(t)$ be as defined by (1.2). Then from Lemma 2, we see that $x(t)$ is positive and there exists $t \geq T$ such that (2.5) holds for $t \geq T$. Define

$$w(t) := \phi(t)\left[\frac{r(t)x^{\Delta}(t)}{x(t)} + r(t)\varphi(t)\right], \quad \text{for } t \geq T. \quad (2.10)$$

By the quotient rule [10, Theorem 1.20], and the definition of $w(t)$, we have

$$\begin{aligned} w^{\Delta}(t) &= \phi^{\Delta}(t)\left[\frac{r(t)x^{\Delta}(t)}{x(t)} + r(t)\varphi(t)\right] + \phi^{\sigma}\left[\frac{r(t)x^{\Delta}(t)}{x(t)} + r(t)\varphi(t)\right]^{\Delta} \\ &= -2\varphi(t)\eta(t)w(t) + \phi^{\sigma}\left(r(t)\varphi(t)\right)^{\Delta} + \phi^{\sigma}\left[\frac{r(t)x^{\Delta}(t)}{x(t)}\right]^{\Delta} \\ &= -2\varphi(t)\eta(t)w(t) + \phi^{\sigma}\left(r(t)\varphi(t)\right)^{\Delta} \\ &\quad +\phi^{\sigma}\left[\frac{x(t)(r(t)x^{\Delta}(t))^{\Delta} - r(t)\left(x^{\Delta}(t)\right)^2}{x(t)x^{\sigma}}\right] \\ &= -2\varphi(t)\eta(t)w(t) + \phi^{\sigma}\left(r(t)\varphi(t)\right)^{\Delta} \\ &\quad +\phi^{\sigma}\frac{(r(t)x^{\Delta}(t))^{\Delta}}{x^{\sigma}} - \phi^{\sigma}\frac{r(t)\left(x^{\Delta}(t)\right)^2}{x(t)x^{\sigma}}. \end{aligned} \quad (2.11)$$

Then from (2.4), and (2.11), we have

$$\begin{aligned} w^{\Delta} &\leq -\phi^{\sigma}Q(t)\frac{x^{\delta}}{x^{\sigma}} - 2\varphi(t)\eta(t)w(t) + \phi^{\sigma}\left(r(t)\varphi(t)\right)^{\Delta} - \phi^{\sigma}r(t)\frac{(x^{\Delta}(t))^2}{x(t)x^{\sigma}} \\ &\leq -\phi^{\sigma}Q(t)\frac{x^{\delta}}{x^{\sigma}} - 2\varphi(t)\eta(t)w(t) + \phi^{\sigma}\left(r(t)\varphi(t)\right)^{\Delta} \\ &\quad -\phi^{\sigma}r(t)\frac{x(t)}{x^{\sigma}}\left(\frac{x^{\Delta}(t)}{x(t)}\right)^2. \end{aligned} \quad (2.12)$$

From the definition of $w(t)$, we have

$$\left(\frac{x^{\Delta}(t)}{x(t)}\right)^2 = \left[\frac{w(t)}{r(t)\phi(t)} - \varphi(t)\right]^2 = \left[\frac{w(t)}{r(t)\phi(t)}\right]^2 + \varphi^2(t) - 2\frac{w(t)\varphi(t)}{r(t)\phi(t)}. \quad (2.13)$$

Substituting (2.13) into (2.12), and using the fact that $\phi^\sigma \le \phi$, we obtain

$$\begin{aligned} w^\Delta(t) &\le -\phi^\sigma Q(t)\frac{x^\delta}{x^\sigma} - 2\varphi(t)\eta(t)w(t) + \phi^\sigma (r\varphi)^\Delta(t) \\ &\quad -\phi^\sigma r(t)\frac{x(t)}{x^\sigma}\left[\left[\frac{w(t)}{r(t)\phi(t)}\right]^2 + \varphi^2(t) - 2\frac{\varphi(t)w(t)}{r(t)\phi(t)}\right] \\ &= -\phi^\sigma Q(t)\frac{x^\delta}{x^\sigma} - 2\varphi(t)\eta(t)w(t) + \phi^\sigma (r(t)\varphi(t))^\Delta - \frac{x(t)}{x^\sigma}\frac{\phi^\sigma w^2(t)}{r(t)\phi^2(t)} \\ &\quad -\phi^\sigma r(t)\varphi^2(t)\frac{x(t)}{x^\sigma} + 2\frac{\phi^\sigma\varphi(t)}{\phi(t)}\frac{x(t)}{x^\sigma}w(t) \\ &\le -\phi^\sigma Q(t)\frac{x^\delta}{x^\sigma} - 2\varphi(t)\eta(t)w(t) + \phi^\sigma (r(t)\varphi(t))^\Delta - \frac{x(t)}{x^\sigma}\frac{\phi^\sigma w^2(t)}{r(t)\phi^2(t)} \\ &\quad -\phi^\sigma r(t)\varphi^2(t)\frac{x(t)}{x^\sigma} + 2\varphi(t)\frac{x(t)}{x^\sigma}w(t). \end{aligned}$$

This implies that

$$\begin{aligned} w^\Delta(t) &\le -\phi^\sigma Q(t)\frac{x^\delta}{x^\sigma} - 2\varphi(t)\eta(t)w(t) + \phi^\sigma (r(t)\varphi(t))^\Delta \\ &\quad -\frac{x(t)}{x^\sigma}\frac{\phi^\sigma w^2(t)}{r(t)\phi^2(t)} - \phi^\sigma r(t)\varphi^2(t)\frac{x(t)}{x^\sigma} + 2\varphi(t)\frac{x(t)}{x^\sigma}w(t). \end{aligned} \tag{2.14}$$

Since $x^\sigma = x(t) + \mu(t)x^\Delta$, we have

$$\frac{x^\sigma}{x(t)} = 1 + \mu(t)\frac{x^\Delta}{x(t)} = 1 + \frac{\mu(t)}{r(t)}\frac{x^{[1]}(t)}{x(t)}.$$

Also since $x^{[1]}(t)$ is decreasing, we get

$$\begin{aligned} x(t) &= x(T) + \int_T^t \frac{x^{[1]}(u)}{r(u)}\Delta u \ge x(T) + x^{[1]}(t)\int_T^t \frac{1}{r(u)}\Delta u \\ &> x^{[1]}(t)\int_T^t \left(\frac{1}{r(u)}\right)\Delta u. \end{aligned}$$

It follows that

$$\frac{x(t)}{x^{[1]}(t)} \ge \int_T^t \left(\frac{1}{r(u)}\right)\Delta u = R(t,T). \tag{2.15}$$

Hence

$$\frac{x^\sigma}{x(t)} = 1 + \mu(t)\frac{x^\Delta}{x(t)} = 1 + \frac{\mu(t)}{r(t)}\frac{x^{[1]}(t)}{x(t)} \le \frac{r(t)R(t,T) + \mu(t)}{r(t)R(t,T)}.$$

Hence, we have

$$\frac{x(t)}{x^\sigma} \ge \frac{r(t)R(t,T)}{r(t)R(t,T) + \mu(t)} = \frac{r(t)R(t,T)}{r(t)R(t,T) + \sigma(t) - t} = \eta(t). \tag{2.16}$$

Now, since $\delta(t) > \sigma(t) \geq t$ and $x(t)$ is increasing, we have

$$x^{\delta}(t) > x^{\sigma}(t). \tag{2.17}$$

Substituting from (2.16) and (2.17) into (**??**), we have

$$\begin{aligned} w^{\Delta}(t) \quad &\leq \quad -\phi^{\sigma}Q(t) - 2\varphi(t)\eta(t)w(t) + \phi^{\sigma}(r(t)\varphi(t))^{\Delta} - \frac{\eta(t)\phi^{\sigma}w^{2}(t)}{r(t)\phi^{2}(t)} \\ &\quad -\phi^{\sigma}r(t)\varphi^{2}(t)\eta(t) + 2\varphi(t)\eta(t)w(t). \end{aligned}$$

This and (2.8) imply after simplification that

$$w^{\Delta}(t) \leq -\phi^{\sigma}\left[Q(t) + (r(t)\varphi(t))^{\Delta} - \eta(t)r(t)\varphi^{2}(t)\right] - \frac{1}{A(t)}w^{2}(t).$$

Using the definition of $\Phi(t)$ we obtain

$$w^{\Delta}(t) \leq -\Phi(t) - \frac{1}{A(t)}w^{2}(t). \tag{2.18}$$

Evaluating both sides of (2.18) at s, multiplying by $H(\sigma(t), \sigma(s))$ and integrating we get

$$\begin{aligned} &\int_{T}^{t} H(\sigma(t), \sigma(s))\Phi(s)\Delta s \\ &\leq \; -\int_{T}^{t} H(\sigma(t), \sigma(s))w^{\Delta}(s)\Delta s - \int_{T}^{t} \frac{H(\sigma(t), \sigma(s))}{A(s)}w^{2}(s)\Delta s. \end{aligned} \tag{2.19}$$

Integrating by parts and using the fact that $H(\sigma(t), t) = 0$, we get

$$\int_{T}^{t} H(\sigma(t), \sigma(s))w^{\Delta}(s)\Delta s = -H(\sigma(t), T)w(T) - \int_{T}^{t} H^{\Delta_s}(\sigma(t), s)w(s)\Delta s.$$

Substituting this into (2.19), we have

$$\begin{aligned} &\int_{T}^{t} H(\sigma(t), \sigma(s))\Phi(s)\Delta s \\ &\leq \; H(\sigma(t), T)w(T) + \int_{T}^{t} H^{\Delta_s}(\sigma(t), s)w(s)\Delta s \\ &\quad - \int_{T}^{t} \frac{H(\sigma(t), \sigma(s))}{A(s)}w^{2}(s)\Delta s. \end{aligned} \tag{2.20}$$

This implies, after using the inequality $bu - au^2 \leq \frac{1}{4}\frac{b^2}{a}$ with $b = H^{\Delta_s}(\sigma(t), s)$ and $a = \frac{H(\sigma(t),\sigma(s))}{A(s)}$, that

$$\int_{T}^{t} \bar{H}(t, s)\Phi(s)\Delta s \leq H(\sigma(t), T)w(T) + \int_{T}^{t} \frac{A(s)(H^{\Delta_s}(\sigma(t), s))^2}{4\bar{H}(t, s)}\Delta s.$$

Thus

$$\frac{1}{H(\sigma(t),T)}\int_T^t\left[\bar{H}(t,s)\Phi(s)-\frac{1}{4}\frac{A(s)(H^{\Delta_s}(\sigma(t),s))^2}{\bar{H}(t,s)}\right]\Delta s\le w(T),$$

which contradicts (2.9). The proof is complete. □

From Theorem 3 by choosing the function $H(t,s)$, appropriately, we can obtain different sufficient conditions for oscillation of (1.1). For instance, if we define a function $h(t,s)$ by

$$H^{\Delta_s}(\sigma(t),s)=-h(t,s)\sqrt{H(\sigma(t),\sigma(s))}, \tag{2.21}$$

we have the following oscillation result. Note that when $\mathbb{T}=\mathbb{R}$, we have $H(t,\sigma(s))=H(t,s)$ and when $\mathbb{T}=\mathbb{N}$, we have $H(t,\sigma(s))=H(t,s+1)$.

Corollary 2.4. *Assume that $(h_1)-(h_3)$ hold and $H\in\Re$, such that for sufficiently large $T\ge t_0$*

$$\limsup_{t\to\infty}\frac{1}{H(\sigma(t),T)}\int_T^t\left[\bar{H}(t,s)\Phi(s)-\frac{h^2(t,s)}{4}\right]\Delta s=\infty, \tag{2.22}$$

where $\Phi(t)$ is defined as in Theorem 3, then every solution of (1.1) is oscillatory.

If we define $H(t,s)$ for $t_0\le s\le\sigma(t)$ by $H(\sigma(t),t)=0$ and $H(t,s)=1$ otherwise, $\phi(t)$ and $\varphi(t)$ are defined as in (2.8), we have $h(t,s)=0$ and from Corollary 4 we have the following oscillation result for *(1.1)*.

Corollary 2.5. *Assume that $(h_1)-(h_3)$ hold . Furthermore assume that for sufficiently large t_1*

$$\lim_{t\to\infty}\sup\int_{t_1}^t\phi^\sigma(s)\Phi(s)\Delta s=\infty.$$

Then every solution of (1.1) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.

In the following, we assume that

$$\int_{t_0}^\infty\Phi(s)\Delta s<\infty,$$

and establish new oscillation criteria for (1.1). We introduce the following notations:

$$q_*:=\liminf_{t\to\infty}\frac{1}{t}\int_T^t\frac{s^2}{A(s)}\Phi(s)\Delta s,\quad p_*:=\liminf_{t\to\infty}\frac{t}{A(t)}\int_{\sigma(t)}^\infty\Phi(s)\Delta s,$$

and assume that $l:=\liminf_{t\to\infty}\frac{t}{\sigma(t)}$. From the definition of $\sigma(t)$ it is clear that $0\le l\le 1$.

Theorem 2.6. *Assume that (h_1)–(h_3) hold, $A^{\Delta}(s) \geq 0$ and $\int_{t_0}^{\infty}(1/A(s))\Delta s = \infty$. Let $y(t)$ be a positive solution of (1.1), and $x(t)$ is defined as in (1.2). Let $w(t)$ is defined as in (2.10), and define*

$$r_* := \liminf_{t\to\infty} \frac{tw^{\sigma}(t)}{A(t)}, \quad R := \limsup_{t\to\infty} \frac{tw^{\sigma}(t)}{A(t)}.$$

Then

$$p_* \leq r_* - r_*^2 l. \tag{2.23}$$

and

$$p_* + q_* \leq 1/l^2. \tag{2.24}$$

Proof. Assume that $y(t)$ is a positive solution of (1.1) on $[t_0, \infty)_{\mathbb{T}}$. Pick $t_1 \in [t_0, \infty)_{\mathbb{T}}$ so that $t_1 > t_0$ and so that $y(t) > 0$, $y(\tau(t)) > 0$ $y(\tau(\tau(t))) > 0$ and $y(\delta(t))$ on $[t_1, \infty)_{\mathbb{T}}$. (Note that in the case when $y(t)$ is negative the proof is similar, since the transformation $y(t) = -z(t)$ transforms (1.1) into the same form). Since $y(t)$ is a positive solution of (1.1), then from Lemma 2, we see that $x(t) > 0$ and satisfies (2.5). Define the function $w(t)$ by the Riccati substitution as in Theorem 3. Then, we get from (2.18) that

$$-w^{\Delta}(t) > \Phi(t) + \frac{1}{A(t)}w(t)w^{\sigma}(t), \quad \text{for} \quad t \geq T. \tag{2.25}$$

Since $\Phi(t) > 0$, we have

$$\frac{w^{\Delta}(t)}{w(t)w^{\sigma}} < -1/A(t), \quad \text{for} \quad t \geq T,$$

which implies that $(-1/w(t))^{\Delta} \leq -1/A(t)$. Integrating the last inequality from t_2 to t, we have

$$-\frac{1}{w(t)} < -\frac{1}{w(t)} + \frac{1}{w(T)} < -\int_T^t \frac{1}{A(s)}\Delta s, \quad \text{for } t \geq T,$$

which implies, using $\int_{t_0}^{\infty}(1/A(s))\Delta s = \infty$, that $\lim_{t\to\infty} w(t) = 0$. First, we prove that (2.23) holds. Integrating (2.25) from $\sigma(t)$ to ∞ and using $\lim_{t\to\infty} w(t) = 0$, we have

$$w^{\sigma}(t) \geq \int_{\sigma(t)}^{\infty} \Phi(s)\Delta s + \int_{\sigma(t)}^{\infty} \frac{1}{A(s)}(w^{\sigma}(s))^2\Delta s. \tag{2.26}$$

It follows from (2.26) that

$$\frac{tw^{\sigma}(t)}{A(t)} \geq \frac{t}{A(t)}\int_{\sigma(t)}^{\infty} \Phi(s)\Delta s + \frac{t}{A(t)}\int_{\sigma(t)}^{\infty} \frac{1}{A(s)}(w^{\sigma}(s))^2\Delta s. \tag{2.27}$$

Let $\epsilon > 0$, then by the definition of p_* and r_* we can pick $t_1 \in [T,\infty)_{\mathbb{T}}$, sufficiently large, so that

$$\frac{t}{A(t)}\int_{\sigma(t)}^{\infty}\Phi(s)\Delta s \geq p_* - \epsilon, \quad \text{and} \quad \frac{tw^{\sigma}(t)}{A(t)} \geq r_* - \epsilon, \text{ for } t \in [t_1,\infty)_{\mathbb{T}}. \tag{2.28}$$

From (2.27) and (2.28) and using the fact $A^{\Delta}(t) \geq 0$, we get that

$$\begin{aligned}
\frac{tw^{\sigma}(t)}{A(t)} &\geq (p_* - \epsilon) + \frac{t}{A(t)}\int_{\sigma(t)}^{\infty}\frac{1}{A(s)}\frac{sw^{\sigma}(s)sw^{\sigma}(s)}{s^2}\Delta s \\
&\geq (p_* - \epsilon) + (r_* - \epsilon)^2\frac{t}{A(t)}\int_{\sigma(t)}^{\infty}\frac{A(s)}{s^2}\Delta s \\
&\geq (p_* - \epsilon) + (r_* - \epsilon)^2\, t\int_{\sigma(t)}^{\infty}\frac{1}{s^2}\Delta s \\
&\geq (p_* - \epsilon) + (r_* - \epsilon)^2\, t\int_{\sigma(t)}^{\infty}\frac{1}{s\sigma(s)}\Delta s \\
&= (p_* - \epsilon) + (r_* - \epsilon)^2\, t\int_{\sigma(t)}^{\infty}\left(\frac{-1}{s}\right)^{\Delta}\Delta s.
\end{aligned} \tag{2.29}$$

Then, we have

$$\frac{tw^{\sigma}(t)}{A(t)} \geq (p_* - \epsilon) + (r_* - \epsilon)^2\left(\frac{t}{\sigma(t)}\right).$$

Taking the lim inf of both sides as $t \to \infty$ we get that $r_* \geq p_* - \epsilon + (r_* - \epsilon)^2\, l$. Since $\epsilon > 0$ is arbitrary, we get the desired inequality (2.23). Next, we prove that (2.24) holds. Multiplying both sides (2.25) by $\frac{t}{A(t)}$, and integrating from T to t $(t \geq T)$, we get

$$\int_T^t \frac{s^2}{A(s)}w^{\Delta}(s)\Delta s \leq -\int_T^t \frac{s^2}{A(s)}\Phi(s)\Delta s - \int_T^t\left(\frac{sw^{\sigma}(s)}{A(s)}\right)^2\Delta s.$$

Using integration by parts, we obtain

$$\begin{aligned}
\frac{t^2w(t)}{A(t)} \leq\; & \frac{T^2w(T)}{A(T)} + \int_T^t\left(\frac{s^2}{A(s)}\right)^{\Delta} w^{\sigma}(s)\Delta s \\
& - \int_T^t \frac{s^2}{A(s)}\Phi(s)\Delta s - \int_T^t\left(\frac{sw^{\sigma}(s)}{A(s)}\right)^2\Delta s.
\end{aligned}$$

By the quotient rule and applying the Pötzsche chain rule,

$$\left(\frac{s^2}{A(s)}\right)^{\Delta} = \frac{(s^2)^{\Delta}}{A^{\sigma}} - \frac{s^2A^{\Delta}(s)}{A(s)A^{\sigma}} \leq \frac{2\sigma(s)}{A^{\sigma}(s)} \leq \frac{2\sigma(s)}{A(s)}. \tag{2.30}$$

Hence

$$\begin{aligned} \frac{t^2 w(t)}{A(t)} &\leq \frac{T^2 w(T)}{A(T)} - \int_T^t \frac{s^2}{A(s)}\Phi(s)\Delta s + \int_T^t 2\left(\frac{\sigma(s)w^\sigma(s)}{A(s)}\right)\Delta s \\ &- \gamma \int_T^t \left(\frac{s w^\sigma(s)}{A(s)}\right)^2 \Delta s. \end{aligned}$$

Let $\epsilon > 0$ be given, then using the definition of l, we can assume, without loss of generality, that T is sufficiently large so that $\frac{s}{\sigma(s)} > l - \epsilon$, $s \geq T$. It follows that

$$\sigma(s) \leq Ls, \quad s \geq T \quad \text{where} \quad L := \frac{1}{l-\epsilon}.$$

We then get that

$$\frac{t^2 w(t)}{A(t)} \leq \frac{T^2 w(T)}{A(T)} - \int_T^t \frac{s^2}{A(s)}\Phi(s)\Delta s + \int_T^t \left[2L\frac{s w^\sigma(s)}{A(s)} - \left(\frac{s w^\sigma(s)}{A(s)}\right)^2\right]\Delta s.$$

Let $u(s) := \frac{s w^\sigma(s)}{A(s)}$, then $u^2(s) = \left(\frac{s w^\sigma(s)}{A(s)}\right)^2$. It follows that

$$\frac{t^2 w(t)}{A(t)} \leq \frac{T^2 w(T)}{A(T)} - \int_T^t \frac{s^2}{A(s)}\Phi(s)\Delta s + \int_T^t \{2Lu(s) - u^2(s)\}\Delta s.$$

Using the inequality $Bu - Au^2 \leq \frac{1}{4}\frac{B^2}{A}$, where A, B are constants, we get

$$\begin{aligned} \frac{t^2 w(t)}{A(t)} &\leq \frac{T^2 w(T)}{A(T)} - \int_T^t \frac{s^2}{A(s)}\Phi(s)\Delta s + \int_T^t \frac{1}{4}[2L]^2\Delta s \\ &\leq \frac{T^2 w(T)}{A(T)} - \int_T^t \frac{s^2}{A(s)}\Phi(s)\Delta s + L^2(t-T). \end{aligned}$$

It follows from this that

$$\frac{t w(t)}{A(t)} \leq \frac{T^2 w(T)}{tA(T)} - \frac{1}{t}\int_T^t \frac{s^2}{A(s)}\Phi(s)\Delta s + L^2(1 - \frac{T}{t}).$$

Since $w^\sigma(t) \leq w(t)$, we get

$$\frac{t w^\sigma(t)}{A(t)} \leq \frac{T^2 w(T)}{tA(T)} - \frac{1}{t}\int_T^t \frac{s^2}{A(s)}\Phi(s)\Delta s + L^2(1 - \frac{T}{t}).$$

Taking the lim sup of both sides as $t \to \infty$ we obtain $R \leq -q_* + L^2 = -q_* + \frac{1}{(l-\epsilon)^2}$. Since $\epsilon > 0$ is arbitrary, we get that $R \leq -q_* + \frac{1}{l^2}$. Using this and the inequality (2.23), we get

$$p_* \leq r_* - l^\gamma r_*^2 \leq r_* \leq R \leq -q_* + \frac{1}{l^2}.$$

Therefore

$$p_* + q_* \leq \frac{1}{l^2},$$

which is the desired inequality (2.24). The proof is complete. □

Theorem 2.7. *Assume that* $(h_1)-(h_3)$ *hold,* $A^{\Delta}(s) \geq 0$ *and* $\int_{t_0}^{\infty}(1/A(s))\Delta s = \infty$. *Furthermore, assume that*

$$p_* > \frac{1}{4l},$$

Then every solution of (1.1) is oscillatory.

Proof. Suppose to the contrary and assume that $y(t)$ is a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(t)) > 0$ and $y(\delta(t)) > 0$ for $t \geq T$ where T is chosen large enough. We consider only this case, because the proof when $y(t) < 0$ is similar. Let w and r_* be as defined in Theorem 6. Then from Theorem 6, we see that r_* satisfies the inequality

$$p_* \leq r_* - l^{\gamma} r_*^2.$$

Using Using the inequality $Bu - Au^2 \leq \frac{1}{4}\frac{B^2}{A}$ with $B = 1$ and $A = l$, we get that

$$p_* \leq \frac{1}{4l},$$

which contradicts (2.23). The proof is complete. □

Theorem 2.8. *Assume that* $(h_1)-(h_3)$ *hold,* $A^{\Delta}(s) \geq 0$ *and* $\int_{t_0}^{\infty}(1/A(s))\Delta s = \infty$. *Furthermore, assume that*

$$p_* + q_* > \frac{1}{l^2}.$$

Then every solution of (1.1) is oscillatory.

Proof. Suppose to the contrary and assume that y is a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(t)) > 0$ and $y(\delta(t)) > 0$ for $t \geq T$ where T is chosen large enough. We consider only this case, because the proof when $y(t) < 0$ is similar. Let w and r_* be as defined in Theorem 6. Then from Theorem 6, we see that r_* satisfies the inequality

$$p_* + q_* < \frac{1}{l^2},$$

which contradicts (2.23). The proof is complete. □

From Theorem 7, we have the following results immediately.

Corollary 2.9. *Assume that* $(h_1)-(h_3)$ *hold,* $A^{\Delta}(s) \geq 0$ *and* $\int_{t_0}^{\infty}(1/A(s))\Delta s = \infty$. *Furthermore, assume that*

$$\liminf_{t\to\infty} \frac{1}{t}\int_T^t \frac{s^2}{A(s)}\Phi(s)\Delta s > \frac{1}{l^2}. \tag{2.31}$$

Then every solution of (1.1) is oscillatory.

Corollary 2.10. *Assume that* (h_1)–(h_3) *hold,* $A^{\Delta}(s) \geq 0$ *and* $\int_{t_0}^{\infty}(1/A(s))\Delta s = \infty$. *Furthermore, assume that*

$$\liminf_{t\to\infty} \frac{t}{A(t)} \int_{\sigma(t)}^{\infty} \Phi(s)\Delta s > \frac{1}{l^2}. \tag{2.32}$$

Then every solution of (1.1) is oscillatory.

2.2. **The case when** $\delta(t) \leq t$. In this subsection, we establish some sufficient conditions for oscillation of (1.1) when $\delta(t) \leq t$. Suppose that there exist two positive functions $\psi(t)$ and $\pi(t)$ such that

$$\psi^{\Delta}(t) := -2\pi(t)\alpha(t)\psi(t), \tag{2.33}$$

and define

$$\Psi(t) := \psi^{\sigma}\left[Q(t)\theta(t) - (\pi(t)r(t))^{\Delta} + \alpha(t)r(t)\pi^2(t)\right] > 0,\; B(t) := \frac{r(t)\psi^2(t)}{\alpha(t)\psi^{\sigma}},$$

where

$$\theta(t) := \begin{cases} \int_T^{\sigma(t)} \frac{1}{r(s)}\Delta s \left(\int_T^{\delta(t)} \frac{1}{r(s)}\Delta s\right)^{-1}, & \delta(t) < \sigma(t), \\ 1, & \delta(t) = \sigma(t). \end{cases}$$

and

$$\alpha(t) := \begin{cases} \int_T^{t} \frac{1}{r(s)}\Delta s \left(\int_T^{\sigma(t)} \frac{1}{r(s)}\Delta s\right)^{-1}, & \sigma(t) > t \\ 1, & \sigma(t) = t. \end{cases}$$

Theorem 2.11. *Assume that* $(h_1) - (h_3)$ *hold and* $B^{\Delta}(t) \geq 0$. *Let* $y(t)$ *be a solution of (1.1),* $x(t)$ *is defined as in (1.2) and make the generalized Riccati substitution*

$$u(t) := \psi(t)\left[\frac{r(t)x^{\Delta}(t)}{x(t)} + r(t)\pi(t)\right], \qquad \text{for } t \geq T. \tag{2.34}$$

Then $u(t) > 0$ *for* $t \geq T$ *and satisfies*

$$u^{\Delta}(t) + \Psi(t) + \frac{1}{B(t)}u^2(t) \leq 0, \quad \text{for } \quad t \geq T. \tag{2.35}$$

Proof. Assume that $y(t)$ is a positive solution of (1.1) on $[t_0, \infty)_{\mathbb{T}}$. Pick $t_1 \in [t_0, \infty)_{\mathbb{T}}$ so that $t_1 > t_0$ and so that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$ and $y(\delta(t))$ on $[t_1, \infty)_{\mathbb{T}}$. (Note that in the case when $y(t)$ is negative the proof is similar, since the transformation $y(t) = -z(t)$ transforms (1.1) into the same form). Since $y(t)$ is a positive solution of (1.1), then from Lemma 2, we see that $x(t)$ satisfies (2.5). Hence $x^{[1]}(t)$ is decreasing for $t \geq T$. Then for $t \geq T$, we have

$$x^{\sigma}(t) - x(\delta(t)) = \int_{\delta(t)}^{\sigma(t)} \frac{(rx^{\Delta})(s)}{r(s)}\Delta s \leq (rx^{\Delta})(\delta(t)) \int_{\delta(t)}^{\sigma(t)} \frac{1}{r(s)}\Delta s,$$

and this implies that

$$\frac{x^{\sigma}(t)}{x(\delta(t))} \leq 1 + \frac{(rx^{\Delta})(\delta(t))}{x(\delta(t))} \int_{\delta(t)}^{\sigma(t)} \frac{1}{r(s)} \Delta s. \tag{2.36}$$

On the other hand for $t \geq T$, we have that

$$x(\delta(t)) > x(\delta(t)) - x(T) = \int_{T}^{\delta(t)} \frac{(rx^{\Delta})(s)}{r(s)} \Delta s \geq (rx^{\Delta})(\delta(t)) \int_{T}^{\delta(t)} \frac{1}{r(s)} \Delta s,$$

which leads to

$$\frac{(rx^{\Delta})(\delta(t))}{x(\delta(t))} < \left(\int_{T}^{\delta(t)} \frac{1}{r(s)} \Delta s \right)^{-1}, \text{ for } t \geq T.$$

Using this last inequality and (2.36), we get that

$$\frac{x^{\sigma}(t)}{x(\delta(t))} < 1 + \frac{\int_{\delta(t)}^{\sigma(t)} \frac{1}{r(s)} \Delta s}{\int_{T}^{\delta(t)} \frac{1}{r(s)} \Delta s} = \frac{\int_{T}^{\sigma(t)} \frac{1}{r(s)} \Delta s}{\int_{T}^{\delta(t)} \frac{1}{r(s)} \Delta s} = \frac{1}{\theta(t)}, \qquad \text{for } t \geq T.$$

Hence, we get the desired inequality

$$x(\delta(t)) \geq \theta(t) x^{\sigma}(t), \text{ for } t \geq T. \tag{2.37}$$

Also, we can prove that

$$x(t) \geq \alpha(t) x^{\sigma}(t) \tag{2.38}$$

From the definition of $u(t)$, by quotient rule [10, Theorem 1.20] and continue as in the proof of Theorem 2.1, we get

$$\begin{aligned} u^{\Delta}(t) \quad \leq \quad & -\psi^{\sigma} Q(t) \left(\frac{x^{\delta}}{x^{\sigma}} \right) - 2\pi(t)\xi(t)u(t) + \psi^{\sigma}(r(t)\pi(t))^{\Delta} - \frac{x(t)}{x^{\sigma}} \frac{\psi^{\sigma} u^2(t)}{r(t)\psi^2(t)} \\ & -\psi^{\sigma} r(t)\pi^2(t) \frac{x(t)}{x^{\sigma}} + 2\pi(t) \frac{x(t)}{x^{\sigma}} u(t). \end{aligned} \tag{2.39}$$

From (2.37), (2.38) and (2.39), we have

$$\begin{aligned} u^{\Delta}(t) \quad \leq \quad & -\psi^{\sigma} Q(t)\theta(t) - 2\pi(t)\alpha(t)u(t) + \psi^{\sigma}(r(t)\pi(t))^{\Delta} - \alpha(t) \frac{\psi^{\sigma} u^2(t)}{r(t)\psi^2(t)} \\ & -\psi^{\sigma} r(t)\pi^2(t)\alpha(t) + 2\pi(t)\alpha(t)u(t). \end{aligned}$$

Simplifying this inequality, we have the inequality (2.35) and this completes the proof. □

The proofs of the following theorems are similar to the proofs of the above theorems by using the inequality (2.35) and due the limited space the details are omitted.

Theorem 2.12. *Assume that* $(h_1)-(h_3)$ *hold,* $H \in \Re$ *such that for sufficiently large* $T \geq t_0$

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t),T)} \int_T^t \left[\bar{H}(t,s)\Psi(s) - \frac{1}{4} \frac{A(s)(H^{\Delta_s}(\sigma(t),s))^2}{\bar{H}(t,s)} \right] \Delta s = \infty. \tag{2.40}$$

Then every solution of (1.1) is oscillatory.

In the following, we assume that

$$\int_{t_0}^{\infty} \Psi(s)\Delta s < \infty,$$

and establish new oscillation criteria for (1.1). We introduce the following notations:

$$A_* := \liminf_{t\to\infty} \frac{t}{B(t)} \int_{\sigma(t)}^{\infty} \Psi(s)\Delta s, \text{ and } B_* := \liminf_{t\to\infty} \frac{1}{t} \int_T^t \frac{s^2}{B(s)} \Psi(s)\Delta s,$$

Theorem 2.13. *Assume that* $(h_1)-(h_3)$ *hold,* $B^{\Delta}(t) \geq 0$ *and* $\int_{t_0}^{\infty}(1/B(s))\Delta s = \infty$*. Furthermore, assume that*

$$A_* > \frac{1}{4l}, \tag{2.41}$$

or

$$A_* + B_* > \frac{1}{l^2}. \tag{2.42}$$

Then every solution of (1.1) is oscillatory.

Corollary 2.14. *Assume that* $(h_1)-(h_3)$ *hold and* $B^{\Delta}(t) \geq 0$ *and* $\int_{t_0}^{\infty}(1/B(s))\Delta s = \infty$*. Furthermore, assume that*

$$\liminf_{t\to\infty} \frac{t}{B(t)} \int_{\sigma(t)}^{\infty} A(s)\Delta s > \frac{1}{l^2}. \tag{2.43}$$

Then every solution of (1.1) is oscillatory.

Corollary 2.15. *Assume that* $(h_1)-(h_3)$*hold and* $B^{\Delta}(t) \geq 0$ *and* $\int_{t_0}^{\infty}(1/B(s))\Delta s = \infty$*. Furthermore, assume that*

$$\liminf_{t\to\infty} \frac{1}{t} \int_T^t \frac{s^2}{B(s)} \Psi(s)\Delta s > \frac{1}{l^2}. \tag{2.44}$$

Then every solution of (1.1) is oscillatory.

References

[1] R. Agarwal, M. Bohner and S. H. Saker, *Oscillation criteria for second order delay dynamic equation*, Canad. Appl. Math. Quart., 13 (2005), 1-19.

[2] R. P. Agarwal, D. O'Regan and S. H. Saker, *Oscillation criteria for second-order nonlinear neutral delay dynamic equations*, J. Math. Anal. Appl., 300 (2004), 203-217.

[3] R. P. Agarwal, D. O'Regan and S. H. Saker, *Oscillation criteria for nonlinear perturbed dynamic equations of second-order on time scales*, J. Appl. Math. and Comput., 20 (2006), 133-147.

[4] R. P. Agarwal, D. O'Regan and S. H. Saker, *Oscillation results for second-order nonlinear neutral delay dynamic equations on time scales*, Appl. Anal., 86 (2007), 1-17.

[5] R. P. Agarwal, D. O'Regan and S. H. Saker, *Oscillation theorems for second-order nonlinear neutral delay dynamic equations on time scales*, Acta Math. Sinica, 24 (2008), 1409-1432.

[6] R. P. Agarwal, D. O'Regan and S. H. Saker, *Properties of bounded solutions of nonlinear dynamic equations on time scales*, Canadian Appl. Math., 14 (2006), 1-10.

[7] R. P. Agarwal, D. O'Regan and S. H. Saker, *Oscillation of second-order damped dynamic equations on time scales*, J. Math. Anal. Appl., 330 (2007), 1317-1337.

[8] E. Akin-Bohner, M. Bohner, and S. H. Saker, *Oscillation for a certain of class of second order Emden-Fowler dynamic equations*, Elect. Trans. Numer. Anal., 27 (2007), 1-12.

[9] E. Akın-Bohner and J. Hoffacker, *Oscillation properties of an Emden–Fowler type equation on discrete time scales*, J. Difference Eqns. Appl., 9 (2003), 603–612.

[10] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.

[11] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[12] M. Bohner and S. H. Saker, *Oscillation of second order nonlinear dynamic equations on time scales*, Rocky Mountain J. Math., 34(4) (2004), 1239-1254.

[13] M. Bohner and S. H. Saker, *Oscillation criteria for perturbed nonlinear dynamic equations,* Mathl. Comp. Modeling, 40 (2004), 249–260.

[14] L. Erbe, *Oscillation criteria for second order linear equations on a time scale*, Canad. Appl. Math. Quart., 9 (2001), 1–31.

[15] L. Erbe, A. Peterson and S. H. Saker, *Oscillation criteria for second-order nonlinear dynamic equations on time scales.* J. London Math. Soc., 76 (2003), 701–714.

[16] L. Erbe and A. Peterson, *Riccati equations on a measure chain*, In G. S. Ladde, N. G. Medhin, and M. Sambandham, editors, Proceedings of Dynamic Systems and Applications, volume 3, pages 193–199, Atlanta, 2001. Dynamic publishers.

[17] L. Erbe, A. Peterson and S. H. Saker, *Kamenev-type oscillation criteria for second-order linear delay dynamic equations*, Dynamic Syst. & Appl., 15 (2006), 65-78.

[18] L. Erbe, A. Peterson and S. H. Saker, *Oscillation criteria for a forced second order nonlinear dynamic equation*, J. Diff. Eqns. Appl., 14 (2008), 997-1009.

[19] L. Erbe, A. Peterson and S. H. Saker, *Hille-Kneser type criteria for second-order dynamic equations on time scales*, Advances in Difference Eqns., 2006 (2006), 1-18.

[20] L. Erbe, A. Peterson and S. H. Saker, *Oscillation criteria for second-order nonlinear delay dynamic equations*, J. Math. Anal. Appl., 333 (2007), 505-522.
[21] J. R. Graef, M. K. Grammatikopoulos and P. W. Spikes, *Asymptotic properties of solutions of nonlinear neutral delay differential equations of the second order*, Radovi Mat., 4 (1988), 133-149.
[22] M. K. Grammatikopoulos, G. Ladas and A. Meimaridou, *Oscillation of second order neutral delay differential equations*, Radovi Mat., 1 (1985), 267-274.
[23] S. Hilger, *Analysis on measure chains–a unified approach to continuous and discrete calculus*, Results Math., 18 (1990), 18–56.
[24] H. J. Li and W. L. Liu, *Oscillation criteria for second order neutral differential equations*, Canad. J. Math., 48 (1996), 871-886.
[25] H. J. Li and C. C. Yeh, *Oscillation criteria for second-order neutral delay difference equations*, Comp. Math. Appl., 36 (1998), 123-132.
[26] R. M. Mathsen, QI-RU Wang and Hong-Wu Wu, *Oscillation for neutral dynamic functional equations on time scales*, J. Diff. Eqns. Appl., 10 (2004), 651-659.
[27] Y. Şahiner, *Oscillation of second-order neutral delay and mixed-type dynamic equations on time scales*, Adv. Difference Eqns., 2006 (2006), 1-9.
[28] V. Kac and P. Cheung, *Quantum Calculus*, Springer, New York, 2001.
[29] W. Kelley and A. Peterson, *Difference Equations: An Introduction With Applications*, second edition, Harcourt/Academic Press, San Diego, 2001.
[30] S. H. Saker, *Oscillation of nonlinear dynamic equations on time scales*, Appl. Math. Comp., 148 (2004), 81-91.
[31] S. H. Saker, *Oscillation criteria of second-order half-linear dynamic equations on time scales*, J. Comp. Appl. Math., 177 (2005), 375-387.
[32] S. H. Saker, *Oscillation of second-order forced nonlinear dynamic equations on time scales*, E. J. of Qual. Theory Diff. Equs., 23 (2005), 1-17.
[33] S. H. Saker, *Oscillation of second-order nonlinear neutral delay dynamic equations on time scales*, J. Comp. Appl. Math., 177 (2005), 375–387.
[34] S. H. Saker, *Boundedness of solutions of second-order forced nonlinear dynamic equations*, Rocky Mountain J. Math., 36 (6) (2006), 2027-2039.
[35] S. H. Saker, *New oscillation criteria for second-order nonlinear dynamic equations on time scales*, Nonlinear Fun. Anal. Appl. 11 (3) (2006), 351-370.
[36] S. H. Saker, *Oscillation of second-order neutral delay dynamic equations of Emden-Fowler type*, Dyn. Sys. Appl., 15 (2006), 629-644.
[37] S. H. Saker, *Oscillation of second-order delay and neutral delay dynamic equations on time scales*, Dyn. Sys. Appl., 16 (2007), 345-360.
[38] S. H. Saker, *Oscillation of second-order neutral delay dynamic equations of Emden-Fowler type*, Dyn. Sys. Appl. 15 (2006), 629-644.
[39] S. H. Saker, *On oscillation of second-order delay dynamic equations on time scales*, Australian J. Mathl. Anal. & Appl., 5 (2008), 1-14.
[40] S. H. Saker, *Oscillation of second-order delay perturbed dynamic equations on time scales*, Comm. Appl. Anal., 12 (2008), 209-220.
[41] S. H. Saker, *Oscillation criteria for a certain class of second-order neutral delay dynamic equations*, Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications & Algorithms (to appear).
[42] V. Spedding, *Taming Nature's Numbers*, New Scientist, July 19, 2003, 28–31.
[43] H. -Wu Wu, R. K. Zhuang and R. M. Mathsen, *Oscillation criteria for second-order nonlinear neutral variable delay dynamic equations*, Appl. Math. Comp., 178 (2006), 321-331.

[44] B. G. Zhang and S. S. Cheng, *Oscillation criteria and comparison theorems for delay difference equations*, Fasciculi Mathematici 25 (1995), 13-32.