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FIXED POINT RESULTS WITH SIMULATION FUNCTIONS

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Abstract. In this paper, we prove some fixed point results through Ω-distance mappings in sense of Saadati et al. [21] by utilizing the concept of simulation functions in sense of Khojasteh et al. [17] as well as we support our result by introducing an example.

1. Introduction

It is known that the outstanding result in fixed point theory was the Banach contraction principle which introduced by Banach [10]. Then after many researchers study the fixed point theory in various directions, for instance we refer the reader to [3, 5, 13, 20, 25, 26, 27, 28, 29, 30, 31, 32, 33] and references therein.

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In 2006, Mustafa and Sims [19] introduced a new generalization for the notion of metric spaces namely generalized metric spaces or G-metric spaces as well as they prove some fixed point results. After that many authors proved several fixed point results in the setting on G-metric spaces. For more work on metric and G-metric spaces, we refer the reader to [4, 6, 7, 8, 9, 11, 12, 16, 34].

The definition of G-metric spaces is given as follows:

Definition 1.1. ([19]) Let X be a nonempty set and let $G : X \times X \times X \to [0, \infty)$ be a function satisfying:

(G1) $G(x, y, z) = 0$ if $x = y = z$,
(G2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$,
(G3) $G(x, y, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,
(G4) $G(x, y, z) = G(p\{x, y, z\})$, where $p\{x, y, z\}$ is the all possible permutations of (x, y, z) (symmetry),
(G5) $G(x, y, z) \leq G(x, a, a)+G(a, y, z), \forall\, x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric or more specifically a G-metric on X and the pair (X, G) is called a G-metric space.

Recently, Saadati et al. [21] introduced the concept of Ω-distance mapping related to a G-metric space and used it to prove some fixed point theorems. For more results on Ω-distance mappings we refer the reader to [1, 2, 14, 15, 22, 23, 24].

The definition of Ω-distance is given as follows:

Definition 1.2. ([21]) Let (X, G) be a G-metric space. Then a function $\Omega : X \times X \times X \to [0, \infty)$ is called an Ω-distance on X if the following conditions satisfied:

(a) $\Omega(x, y, z) \leq \Omega(x, a, a) + \Omega(a, y, z), \ \forall\, x, y, z, a \in X$,
(b) $for\, any\, x, y \in X, \Omega(x, y, .), \Omega(x, ., y) : X \to X$ are lower semicontinuous,
(c) for each $\epsilon > 0$, there exists a $\delta > 0$ such that $\Omega(x, a, a) \leq \delta$ and $\Omega(a, y, z) \leq \delta$ imply $G(x, y, z) \leq \epsilon$.

Definition 1.3. ([21]) Let (X, G) be a G-metric space and Ω be an Ω-distance on X. Then we say that X is Ω-bounded if there exists $\varrho \geq 0$ such that $\Omega(x, y, z) \leq \varrho$ for all $x, y, x \in X$.

The following lemma is an important tool in the development of our results.

Lemma 1.4. ([21]) *Let X be a metric space with metric G and Ω be an Ω-distance on X. Let $\{x_n\}$,$\{y_n\}$ be sequences in X, $\{\alpha_n\}$,$\{\beta_n\}$ be sequences in $[0, \infty)$ converging to zero and let $x, y, z, a \in X$. Then we have the followings:*

(1) *If* $\Omega(y, x_n, x_n) \leq \alpha_n$ *and* $\Omega(x_n, y, z) \leq \beta_n$ *for* $n \in \mathbb{N}$, *then* $G(y, y, z) < \epsilon$ *and hence* $y = z$;
(2) *If* $\Omega(y_n, x_n, x_n) \leq \alpha_n$ *and* $\Omega(x_n, y_m, z) \leq \beta_n$ *for any* $m > n \in \mathbb{N}$, *then* $G(y_n, y_m, z) \to 0$ *and hence* $y_n \to z$;
(3) *If* $\Omega(x_n, x_m, x_l) \leq \alpha_n$ *for any* $m, n, l \in \mathbb{N}$ *with* $n \leq m \leq l$, *then* $\{x_n\}$ *is a G-Cauchy sequence;*
(4) *If* $\Omega(x_n, a, a) \leq \alpha_n$ *for any* $n \in \mathbb{N}$, *then* $\{x_n\}$ *is a G-Cauchy sequence.*

In 2015, Khojasteh et al. [17] introduced the concept of simulation functions in which they used it to unify several fixed point results in the literature [18].

Definition 1.5. ([17]) Let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a mapping. Then ζ is called a simulation function if it satisfies the following conditions:
($\zeta 1$) $\zeta(0, 0) = 0$,
($\zeta 2$) $\zeta(t, s) < s - t$ for all $s, t > 0$,
($\zeta 3$) If $\{t_n\}$ and $\{s_n\}$ are sequences in $[0, \infty)$ with $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.
Henceforth, we denote by $\mathcal{Z}$ the set of all simulation functions.

Next, we list some examples of simulation functions wherein ζ is defined from $[0, \infty) \times [0, \infty)$ to $\mathbb{R}$.

Example 1.6. ([17]) Let $h_1, h_2 : [0, \infty) \to [0, \infty)$ be two continuous functions such that $h_1(t) = h_2(t) = 0$ if and only if $t = 0$ and $h_2(t) < t \leq h_1(t)$ for all $t \in [0, \infty)$ and define $\zeta(t, s) = h_2(s) - h_1(t)$ for all $t, s \in [0, \infty)$. Then ζ is a simulation function.

Example 1.7. ([17]) Let $g : [0, \infty) \to [0, \infty)$ be a continuous function such that $g(t) = 0$ if and only if t=0 and define $\zeta(t, s) = s - g(s) - t$ for all $t, s \in [0, \infty)$. Then ζ is a simulation function.

Definition 1.8. ([35]) Let Θ denotes the set of all functions $\theta : (0, \infty) \to (1, \infty)$ that satisfying the following conditions:
(Θ_1) θ is nondecreasing,
(Θ_2) For each sequence $\{t_n\}$ in $(0, \infty)$, $\lim\limits_{n\to\infty} \theta(t_n) = 1$ if and only if
$$\lim_{n\to\infty} t_n = 0,$$
(Θ_3) θ is continuous on $(0, \infty)$.

Definition 1.9. ([35]) Let Φ denotes the set of all functions $\phi : [1, \infty) \to [1, \infty)$ that satisfying the following conditions:
(Φ_1) ϕ is nondecreasing,

(Φ_2) For each $t > 1$, $\lim_{n\to\infty} \phi^n(t) = 1$,

(Φ_3) θ is continuous on $[1, \infty)$.

Remark 1.10. ([35]) If $\phi \in \Phi$, then $\phi(1) = 1$ and $\phi(t) < t$ for each $t > 1$.

In fact, Zheng et al. [35] used the above classes of functions to generalize some previous fixed point theorems.

Now, we introduce the definition of symmetric Ω-distance mappings at 0.

Definition 1.11. Let (X, G) be a G-metric space and Ω be an Ω-distance on X. We say that Ω is symmetric at 0 if $\Omega(a, b, c) = 0$ implies that $\Omega(p\{a, b, c\}) = 0$ for any permutation p of (a, b, c).

Next, we provide some examples of symmetric Ω-distance mappings at 0.

Example 1.12. Let (X, d) be a metric space and let $G : X \times X \times X \to [0, \infty)$ be defined by

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}$$

for all $x, y, z \in X$. Then G is symmetric at 0.

Example 1.13. Let $X = [0, 1]$. Define $G, \Omega : X \times X \times X \to [0, \infty)$ by $G(x, y, z) = |x - y| + |y - z| + |x - z|$ and $\Omega(x, y, z) = |x - y| + |x - z|$. Then Ω is symmetric at 0.

2. Main Results

To facilitate our work, we introduce the following definition:

Definition 2.1. Let (X, G) be a G-metric space, $\zeta \in \mathcal{Z}$ and Ω be an Ω-distance on X such that Ω is symmetric at 0. A self mapping $f : X \to X$ is said to be (Ω, θ, ϕ)-contraction with respect to ζ if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that $\Omega(fx, fy, fz) \neq 0$, then

$$\zeta(\theta\Omega(fx, fy, fz), \phi\theta\Omega(x, y, z)) \geq 0 \quad for\ all \quad x, y, z \in X. \tag{2.1}$$

Lemma 2.2. *Let (X, G) be a G-metric space and Ω be an Ω-distance on X. Let $f : X \to X$ be an (Ω, θ, ϕ)-contraction with respect to $\zeta \in \mathcal{Z}$. If f has a fixed point (say) $u \in X$, then it is unique.*

Proof. Assume that there is $v \in X$ such that $fv = v$. We show that $\Omega(u, u, v) = 0$. If $\Omega(u, u, v) \neq 0$, by substituting $x = y = u$ and $z = v$ in (2.1) and taking into account ($\zeta 2$), we have

$$\begin{aligned}0 &\leq \zeta(\theta\Omega(fu, fu, fv), \phi\theta\Omega(u, u, v)) \\ &= \zeta(\theta\Omega(u, u, v), \phi\theta\Omega(u, u, v)) \\ &< \phi\theta\Omega(u, u, v) - \theta\Omega(u, u, v) \\ &< \theta\Omega(u, u, v) - \theta\Omega(u, u, v) = 0,\end{aligned}$$

a contradiction and so $\Omega(u, u, v) = 0$. By the same argument we can show that $\Omega(u, u, u) = 0$. Thus, $G(u, u, v) = 0$ which implies that $u = v$. □

Let (X, G) be a G-metric space, $x_0 \in X$ and $f : X \to X$ be a self mapping. Then the sequence $\{x_n\}$, where $x_n = fx_{n-1}$, $n \in \mathbb{N}$ is called the Picard sequence generated by f with initial point x_0.

Lemma 2.3. *Let (X, G) be a G-metric space, $\zeta \in \mathcal{Z}$ and Ω be an Ω-distance on X. If $f : X \to X$ is an (Ω, θ, ϕ)-contraction with respect to ζ, then*

$$\begin{aligned}&\Omega(x_n, x_{n+1}, x_{n+1}) > 0 \quad \textit{for each } n \in \mathbb{N} \quad \textit{implies} \lim_{n\to\infty} \Omega(x_n, x_{n+1}, x_{n+1}) = 0, \\ &\Omega(x_{n+1}, x_n, x_n) > 0 \quad \textit{for each } n \in \mathbb{N} \quad \textit{implies} \lim_{n\to\infty} \Omega(x_{n+1}, x_n, x_n) = 0\end{aligned} \tag{2.2}$$

for any initial point $x_0 \in X$, where $\{x_n\}$ is the Picard sequence generated by f at x_0.

Proof. Let $x_0 \in X$ be any point and $\{x_n\}$ be the Picard sequence generated by f at x_0. From (2.1) and ($\zeta 2$), we have

$$\begin{aligned}0 &\leq \zeta(\theta\Omega(fx_{n-1}, fx_n, fx_n), \phi\theta\Omega(x_{n-1}, x_n, x_n)) \\ &= \zeta(\theta\Omega(x_n, x_{n+1}, x_{n+1}), \phi\theta\Omega(x_{n-1}, x_n, x_n)) \\ &< \phi\theta\Omega(x_{n-1}, x_n, x_n) - \theta\Omega(x_n, x_{n+1}, x_{n+1}) \\ &< \theta\Omega(x_{n-1}, x_n, x_n) - \theta\Omega(x_n, x_{n+1}, x_{n+1}).\end{aligned}$$

Thus, $\{\Omega(x_n, x_{n+1}, x_{n+1})\}$ is a nonincreasing sequence in $[0, \infty)$ and so there is $\gamma \geq 0$ such that $\lim\limits_{n\to\infty} \Omega(x_n, x_{n+1}, x_{n+1}) = \gamma$. Suppose to the contrary that is, $\gamma > 0$. Then by (2.1) and ($\zeta 3$), we have

$$0 \leq \limsup_{n\to\infty} \zeta(\theta\Omega(x_n, x_{n+1}, x_{n+1}), \phi\theta\Omega(x_{n-1}, x_n, x_n)) < 0,$$

which is a contradiction and so $\lim\limits_{n\to\infty} \Omega(x_n, x_{n+1}, x_{n+1}) = 0$. By the same way we can show that $\lim_{n\to\infty} \Omega(x_{n+1}, x_n, x_n) = 0$. □

Lemma 2.4. *Let (X, G) be a G-metric space, $\zeta \in \mathcal{Z}$ and Ω be an Ω-distance on X such that Ω is symmetric at 0. Let $f : X \to X$ be an (Ω, θ, ϕ)-contraction* *with respect to* ζ*. If* $\Omega(x_{n_0}, x_{n_0+1}, x_{n_0+1}) = 0$ *or* $\Omega(x_{n_0+1}, x_{n_0}, x_{n_0}) = 0$ *for some* $n_0 \in \mathbb{N}$*, then* x_{n_0} *is a fixed point for* f*.*

Proof. The proof follows from part (c) of the definition of Ω and the assumption that Ω is symmetric at 0. □

Theorem 2.5. *Let* (X, G) *be a complete* G*-metric space,* $\zeta \in \mathcal{Z}$ *and* Ω *be an* Ω*-distance on* X *such that* Ω *is symmetric at 0. Suppose that* $f : X \to X$ *is* (Ω, θ, ϕ)*-contraction with respect to* ζ *that satisfies the following condition: for all* $u \in X$ *if* $fu \neq u$*, then*

$$\inf\{\Omega(x, fx, u) : x \in X\} > 0. \tag{2.3}$$

Then f *has a unique fixed point* $x \in X$*.*

Proof. Let $x_0 \in X$ and consider the Picard sequence $\{x_n\}$ in X generated by f at x_0. According to Lemma 2.4, if there exists $n_0 \in \mathbb{N}$ such that $\Omega(x_{n_0}, x_{n_0+1}, x_{n_0+1}) = 0$ or $\Omega(x_{n_0+1}, x_{n_0}, x_{n_0}) = 0$, then x_{n_0} is a fixed point for f. So, we may assume that for each $n \in \mathbb{N}$, $\Omega(x_n, x_{n+1}, x_{n+1}) \neq 0$ and $\Omega(x_{n+1}, x_n, x_n) \neq 0$. Thus, by Lemma 2.2 we have $\lim\limits_{n\to\infty} \Omega(x_n, x_{n+1}, x_{n+1}) = 0$ and $\lim\limits_{n\to\infty} \Omega(x_{n+1}, x_n, x_n) = 0$.

Now, we claim that $\lim\limits_{n,m\to\infty} \Omega(x_n, x_m, x_m) = 0$ for $m, n \in \mathbb{N}$ with $m > n$. Assume to the contrary that is, $\lim\limits_{n,m\to\infty} \Omega(x_n, x_m, x_m) \neq 0$. Thus, there is $\epsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that $\{x_{m_k}\}$ is chosen as the smallest index for which

$$\Omega(x_{n_k}, x_{m_k}, x_{m_k}) \geq \epsilon, \;\; m_k > n_k > k. \tag{2.4}$$

This implies that

$$\Omega(x_{n_k}, x_{m_k-1}, x_{m_k-1}) < \epsilon. \tag{2.5}$$

By using (2.4), (2.5) and part (a) of the definition of Ω, we get

$$\begin{aligned}
\epsilon &\leq \Omega(x_{n_k}, x_{m_k}, x_{m_k}) \\
&\leq \Omega(x_{n_k}, x_{m_k-1}, x_{m_k-1}) + \Omega(x_{m_k-1}, x_{m_k}, x_{m_k}) \\
&< \epsilon + \Omega(x_{m_k-1}, x_{m_k}, x_{m_k}).
\end{aligned}$$

By taking the limit as $k \to \infty$ and taking into account (2.2), we get

$$\lim_{k\to\infty} \Omega(x_{n_k}, x_{m_k}, x_{m_k}) = \epsilon.$$

Also,

$$\begin{aligned}
\epsilon &\leq \Omega(x_{n_k}, x_{m_k}, x_{m_k}) \\
&\leq \Omega(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + \Omega(x_{n_k+1}, x_{m_k+1}, x_{m_k+1}) + \Omega(x_{m_k+1}, x_{m_k}, x_{m_k})
\end{aligned}$$

and

$$\begin{aligned}\Omega(x_{n_k+1}, x_{m_k+1}, x_{m_k+1}) &\leq \Omega(x_{n_k+1}, x_{n_k}, x_{n_k}) + \Omega(x_n, x_{m_k}, x_{m_k}) \\ &+\Omega(x_{m_k}, x_{m_k+1}, x_{m_k+1}).\end{aligned}$$

If we pass the limit as $k \to \infty$ in the above two inequalities and taking into account (2.2), we get

$$\lim_{n\to\infty} \Omega(x_{n_k+1}, x_{m_k+1}, x_{m_k+1}) = \epsilon.$$

Now, by letting $s_{n_k} = \Omega(x_{n_k}, x_{m_k}, x_{m_k})$ and $t_{n_k} = \Omega(x_{n_k+1}, x_{m_k+1}, x_{m_k+1})$ then ($\zeta3$) and (2.1) yield that

$$0 \leq \limsup_{k\to\infty} \zeta(\Omega(x_{n_k+1}, x_{m_k+1}, x_{m_k+1}), \Omega(x_{n_k}, x_{m_k}, x_{m_k})) < 0$$

which is a contradiction. Therefore,

$$\lim_{n,m\to\infty} \Omega(x_n, x_m, x_m) = 0, \;\; m > n.$$

By the same argument we can show that

$$\lim_{n,m\to\infty} \Omega(x_n, x_n, x_m) = 0, \;\; m > n.$$

For $l > m > n$, we have

$$\Omega(x_n, x_m, x_l) \leq \Omega(x_n, x_m, x_m) + \Omega(x_m, x_m, x_l).$$

By taking the limit as $n, m, l \to \infty$, we get

$$\lim_{n,m,l\to\infty} \Omega(x_n, x_m, x_l) = 0.$$

Thus by Lemma 1.4, $\{x_n\}$ is a G-Cauchy sequence. So there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Since $\lim_{n,m,l\to\infty} \Omega(x_n, x_m, x_l) = 0$, for any $\epsilon > 0$ there is $k_0 \in \mathbb{N}$ such that

$$\Omega(x_n, x_m, x_l) \leq \epsilon, \quad \forall\, l > m > n \geq k_0.$$

The lower semi-continuity of Ω implies that

$$\Omega(x_n, x_m, u) \leq \liminf_{p\to\infty} \Omega(x_n, x_m, x_p) \leq \epsilon, \;\; \forall\, m > n \geq k_0.$$

Suppose that $fu \neq u$. Then we have

$$\begin{aligned} 0 &< \inf\{\Omega(x, fx, u) : x \in X\} \\ &\leq \inf\{\Omega(x_n, x_{n+1}, u) : n \in \mathbb{N}\} \\ &\leq \epsilon \end{aligned}$$

for every $\epsilon > 0$ which is a contradiction. Therefore $fu = u$. The uniqueness of u follows from Lemma 2.2. This completes the proof. □

We introduce the following example to support our main result.

Example 2.6. Let $X = \{0,1\} \cup [4,\infty)$ and let $G : X \times X \times X \to [0,\infty)$, $\Omega : X \times X \times X \to [0,\infty)$, $f : X \to X$, $\phi : [1,\infty) \to [1,\infty)$, $\theta : (0,\infty) \to (1,\infty)$ and $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined as follow:

$$\Omega(x,y,z) = G(x,y,z) = \begin{cases} 0 & , x = y = z, \\ \max\{x,y,z\}, & otherwise, \end{cases} \qquad fx = \begin{cases} 0, & x = 0,1, \\ 1, & x \in [4,\infty), \end{cases}$$

$\phi(t) = t^{\frac{1}{2}}$, $\theta(t) = e^t$ and $\zeta(t,s) = ks - t$, where $e^{-1} \leq k < 1$. Then,

(1) (X,G) is a complete G-metric space and Ω is an Ω-distance on X and symmetric at 0,
(2) $\zeta \in \mathcal{Z}$, $\phi \in \Phi$ and $\theta \in \Theta$,
(3) f is a (Ω,θ,ϕ)-contraction with respect to ζ,
(4) for every $u \in X$ if $fu \neq u$, then $\inf\{\Omega(x,fx,u) : x \in X\} > 0$.

We show (3) and (4). In order to see (3), that is, f is (Ω,θ,ϕ)-contraction with respect to ζ, let $x,y,z \in X$ be such that $\Omega(fx,fy,fz) \neq 0$. Then, $fx \neq fy$ or $fx \neq fz$ or $fy \neq fz$. We just discuss the case that $fx \neq fy$ and the other are the same.

We consider the following cases:
Case(1): If $x = 0,1$, $y \geq 4$ and $z \in X$, then

$$\begin{aligned} \zeta(\theta\Omega(fx,fy,fz),\phi\theta\Omega(x,y,z)) &= k\ \phi\theta\Omega(x,y,z) - \theta\Omega(fx,fy,fz) \\ &= k\ e^{\frac{1}{2}\max\{x,y,z\}} - e^{\max\{0,1,fz\}} \\ &\geq k\ e^2 - e^1 \\ &\geq 0. \end{aligned}$$

Case(2): If $x \geq 4$, $y = 0,1$ and $z \in X$, then

$$\begin{aligned} \zeta(\theta\Omega(fx,fy,fz),\phi\theta\Omega(x,y,z)) &= k\ \phi\theta\Omega(x,y,z) - \theta\Omega(fx,fy,fz) \\ &= k\ e^{\frac{1}{2}\max\{x,y,z\}} - e^{\max\{1,0,fz\}} \\ &\geq k\ e^2 - e^1 \\ &\geq 0. \end{aligned}$$

Therefore, for all $x,y,z \in X$, we have

$$\zeta(\theta\Omega(fx,fy,fz),\phi\theta\Omega(x,y,z)) \geq 0.$$

This means that f is (Ω,θ,ϕ)-contraction with respect to ζ.

Next, to see (4), if $fu \neq u$, then $u \neq 0$. To find $\inf\{\Omega(x,fx,u) : x \in X\}$, we have two cases:
Case(1): If $x = 0,1$, then

$$\begin{aligned}\inf\{\Omega(x,fx,u):x\in\mathcal{U}\} &= \inf\{\Omega(x,0,u):x=0,1\}\\ &= \inf\{\max\{x,0,u\}:x=0,1\}\\ &\geq 1.\end{aligned}$$

Case(2): If $x \geq 4$, then

$$\begin{aligned}\inf\{\Omega(x,fx,u):x\in\mathcal{U}\} &= \inf\{\Omega(x,1,u):x\geq 4\}\\ &= \inf\{\max\{x,0,u\}:x\geq 4\}\\ &\geq 4.\end{aligned}$$

Therefore by Theorem 2.5 f has a unique fixed point in X.

Corollary 2.7. *Let (X,G) be a complete G-metric space, $\theta \in \Theta$ and Ω be an Ω-distance on X such that Ω is symmetric at 0. Suppose that $f : X \to X$ is a self-mapping and there are $k, \lambda \in [0,1)$ such that for all $x, y, z \in X$ if $\Omega(fx, fy, fz) \neq 0$, then*

$$\theta\Omega(fx,fy,fz) \leq k\ [\theta\Omega(x,y,z))]^{\lambda}. \tag{2.6}$$

Also, for all $u \in X$ if $fu \neq u$, then

$$\inf\{\Omega(x,fx,u):x\in X\} > 0. \tag{2.7}$$

Then f has a unique fixed point $x \in X$.

Proof. Define $\zeta : [0,\infty)\times[0,\infty) \to \mathbb{R}$ and $\phi : [1,\infty) \to [1,\infty)$ by $\zeta(t,s) = ks-t$ and $\phi(t) = t^{\lambda}$. Then $\zeta \in \mathcal{Z}$ and $\phi \in \Phi$. Clearly f is (Ω,θ,ϕ)-contraction and so the result follows from Theorem 2.5. □

Corollary 2.8. *Let (X,G) be a complete G-metric space, $\theta \in \Theta$ and Ω be an Ω-distance on X such that Ω is symmetric at 0. Suppose that $f : X \to X$ is a self mapping and there are $\lambda \in [0,1)$ and $\tau > 0$ such that for all $x, y, z \in X$ if $\Omega(fx, fy, fz) \neq 0$, then*

$$\Omega(fx,fy,fz) \leq \lambda\Omega(x,y,z) - \tau. \tag{2.8}$$

Also, for all $u \in X$ if $fu \neq u$, then

$$\inf\{\Omega(x,fx,u):x\in X\} > 0. \tag{2.9}$$

Then f has a unique fixed point $x \in X$.

Proof. Since the function $\delta t = e^t$ is strictly increasing on the set of real numbers, we have

$$\Omega(fx,fy,fz) \leq \lambda\Omega(x,y,z) - \tau \Leftrightarrow e^{\Omega(fx,fy,fz)} \leq e^{-\tau}e^{\lambda\Omega(x,y,z)}.$$

Now, if $k = e^{-\tau}$ and $\theta : (0,\infty) \to (1,\infty)$ is defined by $\theta(t) = e^t$, then $k < 1$ and $\theta \in \Theta$. Therefore,

$$\Omega(fx,fy,fz) \leq \lambda\Omega(x,y,z) - \tau \Leftrightarrow \theta\Omega(fx,fy,fz) \leq k[\theta\Omega(x,y,z)]^{\lambda}.$$

Thus the result follows from Corollary 2.7. □

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