EXISTENCE THEOREMS FOR THE GENERALIZED
RELAXED PSEUDOMONOTONE VARIATIONAL
INEQUALITIES

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Abstract. This work aims to suggest a generalized relaxed $\gamma$-pseudomonotone variational inequalities in Hilbert spaces and show that the iterative sequence defined by an algorithm weakly converges to a solution.

1. Introduction

The variational inequality theory plays a very important role in many areas, such as optimal control, mechanics, economics, transportation equilibrium and engineering sciences etc. An important part of the research focuses on the existence of solutions to variational inequality. The most basic methods for the solution of variational inequalities are projection method, extra-gradient method, Tikhonov regularization method and proximal point method; see,
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[9, 13, 16, 21]. The well-known gradient projection method can be successfully applied for solving strongly monotone variational inequalities and inverse strongly monotone variational inequalities [4, 7, 9, 12, 20]. Korpelevich introduced the extra-gradient method [18], and this method was applied for solving monotone variational inequalities in infinite-dimensional spaces. It is a known fact [9] that the extra-gradient method can be successfully applied for solving monotone variational inequalities in infinite-dimensional Hilbert spaces [3, 5, 6, 13, 15, 22, 23, 24]. Providing that the variational inequality has solutions and the assigned mapping is monotone and Lipschitz continuous, it is proved that the iterative sequence defined by the extra-gradient method weakly converges to a solution.

The goal of this work is to define the generalized relaxed \( \gamma \)-pseudomonotone variational inequalities in Hilbert spaces and prove that the iterative sequence suggested by the algorithm for solving generalized relaxed \( \gamma \)-pseudomonotone variational inequalities weakly converges to a solution.

2. Preliminaries

As a matter of convenience, we introduce the notation first. Let \( H \) be a real Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \), and \( \Omega \) be a nonempty, closed and convex subset of \( H \). Let \( x_n \to x \) and \( x_n \rightharpoonup x \) represent sequence \( \{x_n\} \) converging strongly and weakly to \( x \), respectively, and \( \mathfrak{W}_\omega(x_n) \) denote the set of weak cluster points of sequence \( \{x_n\} \).

Let \( Q : H \to H \) be a mapping. The variational inequality \( \text{VI}(\Omega, Q) \) defined by \( \Omega \) and \( Q \) consists in finding a point \( x^* \in \Omega \) such that

\[
\langle Q(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \Omega.
\]  

(2.1)

The solution set of (2.1) is denoted by \( \text{Sol}(\Omega, Q) \).

Let \( P_\Omega \) be a mapping from \( H \) onto a nonempty, closed and convex subset \( \Omega \) of \( H \). Then \( P_\Omega \) is called the orthogonal projection from \( H \) onto \( \Omega \), if

\[
P_\Omega(x) = \arg \min_{y \in \Omega} \| x - y \|, \quad \forall x \in H,
\]

Lemma 2.1. ([10]) Let \( \Omega \) be a nonempty, closed and convex subset of a real Hilbert space \( H \). For any \( x, y \in H \) and \( z \in \Omega \), it satisfies:

(i) \( \| x - P_\Omega(x) \| \leq \| x - y \| \);
(ii) \( \langle x - P_\Omega(x), z - P_\Omega(x) \rangle \leq 0 \);
(iii) \( \| P_\Omega(x) - P_\Omega(y) \|^2 \leq \| x - y \| \);
(iv) \( \| P_\Omega(x) - z \|^2 \leq \| x - z \|^2 - \| P_\Omega(x) - x \|^2 \).

Definition 2.2. ([2]) A mapping \( P_\Omega : H \to \Omega \) goes by the name of
(a) non-expansive if
\[ \| P_\Omega(x) - P_\Omega(y) \| \leq \| x - y \|, \forall x, y \in \mathcal{H}; \]
(b) firmly nonexpansive if
\[ \| P_\Omega(x) - P_\Omega(y) \|^2 \leq \langle x - y, P_\Omega(x) - P_\Omega(y) \rangle, \forall x, y \in \mathcal{H}. \]

Lemma 2.3. ([10, 16]) Let \( x \in \mathcal{H} \) and \( z \in \Omega \). Then \( z = P_\Omega(x) \) if and only if
\[ P_\Omega(x) \in \Omega \]
and
\[ \langle x - P_\Omega(x), y - P_\Omega(x) \rangle \leq 0, \forall y \in \Omega. \] (2.2)

Remark 2.4. \( x^* \in \mathcal{H} \) is a solution of (2.1) if and only if
\[ x^* = P_\Omega(x^* - \lambda Q(x^*)), \lambda > 0. \]

Definition 2.5. Let \( \Omega \) be a convex set in \( \mathbb{R}^n \) and \( Q : \Omega \to \Omega \) be a mapping. Then, \( Q \) is said to be:
(a) strongly monotone on \( \Omega \) with constant \( \gamma > 0 \), if
\[ \langle Q(x) - Q(y), x - y \rangle \geq \gamma \| x - y \|^2, \forall x, y \in \Omega; \]
(b) strictly monotone on \( \Omega \), if
\[ \langle Q(x) - Q(y), x - y \rangle > 0, \text{ distinct } x, y \in \Omega; \]
(c) monotone on \( \Omega \), if
\[ \langle Q(x) - Q(y), x - y \rangle \geq 0, \forall x, y \in \Omega; \]
(d) relaxed monotone on \( \Omega \), if
\[ \langle Q(x) - Q(y), x - y \rangle \geq -\gamma \| x - y \|^2, \forall x, y \in \Omega; \]
(e) relaxed \( \gamma \)-pseudomonotone on \( \Omega \), if \( \langle Q(y), x - y \rangle \geq 0 \) then
\[ \langle Q(x), x - y \rangle \geq -\gamma \| x - y \|^2, \forall x, y \in \Omega \text{ and } \gamma > 0; \]
(f) pseudo monotone on \( \Omega \), if \( \langle Q(y), x - y \rangle \geq 0 \) then
\[ \langle Q(x), x - y \rangle \geq 0, \forall x, y \in \Omega. \]

Remark 2.6. (i) We observe that \( Q \) is relaxed \( \gamma \)-pseudomonotone on \( \Omega \), if \( Q \) is relaxed \( \gamma \)-pseudomonotone for every \( x \in \Omega \). Therefore the relaxed \( \gamma \)-pseudomonotonicity is a real generalization of pseudomonotonicity and relaxed monotonicity.

(ii) We point out that a pseudomonotone operator is relaxed \( \gamma \)-pseudomonotone operator; and a relaxed monotone operator is relaxed \( \gamma \)-pseudomonotone operator. However, the converse are not true in general.
(iii) The following implications hold:

\[(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f).\]

(iv) The reverse assertions are not true in general.

**Example 2.7.** The function \( Q : [0, +\infty) \to \mathbb{R} \) defined by

\[
Q(x) = \begin{cases} 
    x^2 - 2, & \text{if } x \geq \sqrt{3} - 1, \\
    -2x, & \text{if } 0 \leq x < \sqrt{3} - 1.
\end{cases}
\]

is relaxed \( \gamma \)-pseudomonotone on \([0, +\infty)\) with \( \gamma = 2 \), but not pseudomonotone on \([0, +\infty)\), i.e., \( Q(x) \) is not pseudomonotone at \( x = 0 \).

**Example 2.8.** The function \( Q : (-\infty, 0) \to (0, +\infty) \) defined by

\[
Q(x) = x^2
\]

is relaxed \( \gamma \)-pseudomonotone, but not relaxed monotone on \((-\infty, 0)\), since for all \( \sigma > 0 \), there exists \( x_0 < 0 \) and \( y_0 < 0 \) with \( x_0 + y_0 < -\sigma < 0 \) such that

\[
\langle Q(y_0) - Q(x_0), y_0 - x_0 \rangle < -\sigma \|y_0 - x_0\|^2.
\]

**Remark 2.9.** From above, we conclude that the relaxed \( \gamma \)-pseudomonotonicity is a real generalization of pseudomonotonicity and relaxed monotonicity.

**Example 2.10.** The function \( Q : \mathbb{R} \to \mathbb{R} \) defined by

\[
Q(x) = -x
\]

has a solution \( x = 0 \). However, it is easy to observe that \( Q(x) = -x \) is neither monotone on \( \mathbb{R} \) nor pseudomonotone on \( \mathbb{R} \). But it is relaxed monotone on \( \mathbb{R} \).

**Example 2.11.** The function \( Q : (-\infty, 2) \to \mathbb{R} \) defined by

\[
Q(x) = \begin{cases} 
    x^2, & \text{if } x \geq 1, \\
    -x + 2, & \text{if } 1 < x < 2,
\end{cases}
\]

is relaxed \( \gamma \)-pseudomonotone with \( \gamma = 1 \) on \((-\infty, 0) \cup (0, 2)\), but not relaxed \( \gamma \)-pseudomonotone on \((-\infty, 2)\).

Now we can consider the following problem of finding \( x \in \Omega \) such that

\[
\langle Q(x^*), x - x^* \rangle \geq -\gamma \|x - x^*\|^2, \quad \forall x \in \Omega.
\]

We denote by \( \Omega^* \) and \( \Omega^d \) the solutions sets of problem (2.1) and problem (2.3), respectively. Now we give the relationships between problem (2.1) and problem (2.3).
Proposition 2.12. ([17])

(i) The set $\Omega^d$ is closed and convex;
(ii) $\Omega^d \subseteq \Omega^*$ holds;
(iii) If $Q$ is pseudomonotone, then

$$\Omega^* \subseteq \Omega^d.$$  

Proposition 2.13. ([1]) Assume that $\Omega$ is a nonempty, closed and convex subset of a normed space $X$. If $Q : \Omega \to X^*$ (dual space) is hemicontinuous, then $\Omega^d \subseteq \Omega^*$. In addition, if $Q$ is relaxed $\gamma$-pseudomonotone, then

$$\Omega^d = \Omega^*.$$  

Definition 2.14. A mapping $Q : \mathcal{H} \to \mathcal{H}$ is said to be

(i) $\xi$-Lipschitz continuous if there exists $\xi > 0$ such that

$$\|Q(x) - Q(y)\| \leq \xi \|x - y\|, \ \forall x, y \in \mathcal{H};$$

(ii) weakly sequentially continuous if for each sequence $\{x_n\}$ we have

$$x_n \rightharpoonup x$$

implies

$$Q(x_n) \rightharpoonup Q(x).$$

Lemma 2.15. ([19]) Let $\Omega$ be a nonempty, closed and convex subset of $\mathcal{H}$, and $\{x_n\}$ be a sequence on $\mathcal{H}$ if

(i) $\lim_{n \to \infty} \|x_n - x\|$ exists for each $x \in \Omega$;
(ii) $\mathcal{M}_w(x_n) \subseteq \Omega$;

then $\{x_n\}$ weakly converges to a point in $\Omega$.

Proposition 2.16. ([11])

(i) If $Q$ is strictly monotone, then variational inequality (2.1) has at most one solution.
(ii) If $Q$ is strongly monotone, then variational inequality (2.1) has a unique solution.

Proposition 2.17. ([8]) Let $\Omega$ be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$ and $Q : \Omega \to \mathcal{H}$ be a pseudomonotone and continuous. Then, $x^*$ is a solution of $VI(\Omega, Q)$ if and only if

$$\langle Q(x), x - x^* \rangle \geq 0, \ \forall x \in \Omega.$$
3. WEAKLY CONVERGENCE

In this section, we consider the problem $VI(\Omega, Q)$ with $\Omega$ being nonempty, closed and convex and $Q$ being relaxed $\gamma$-pseudomonotone on $H$ and Lipschitz continuous with constant $\xi > 0$. We assume that the solution set $\text{Sol}(\Omega, Q) \neq \emptyset$.

**Algorithm 3.1.** Input: $x^0 \in \Omega$ and $\{\lambda_\ell\} \in [a, b]$, where $0 < a \leq b < \frac{1}{\xi}$.

**Step 0:** Set $\ell = 0$.

**Step 1:** If $x^\ell = P_\Omega(x^\ell - \lambda_\ell Q(x^\ell))$ then stop.

**Step 2:** Otherwise, set

$$
\bar{x}^\ell = P_\Omega(x^\ell - \lambda_\ell Q(x^\ell)),
$$

$$
x^{\ell+1} = P_\Omega(x^\ell - \lambda_\ell Q(\bar{x}^\ell)).
$$

Replace $\ell$ by $\ell + 1$; go to **Step 1**.

**Remark 3.2.** Assume that $Q(x^\ell) = 0$, then

$$
x^\ell = P_\Omega(x^\ell - \lambda_\ell Q(x^\ell)),
$$

and the Algorithm terminates at step $\ell$ with a solution $x^\ell$. Again, we assume that $Q(x^\ell) \neq 0$ for all $\ell$ then algorithm generates an infinite sequence.

**Lemma 3.3.** ([15, 18]) Assume that $Q$ is pseudomonotone and $\xi$-Lipschitz continuous on $\Omega$ and $\text{Sol}(\Omega, Q)$ is nonempty. Let $x^*$ be a solution of $VI(\Omega, Q)$. Then, for every $\ell \in \mathbb{N}$, we have

$$
\|x^{\ell+1} - x^*\|^2 \leq \|x^\ell - x^*\|^2 - (1 - \lambda_\ell^2\xi^2)\|x^\ell - \bar{x}^\ell\|^2.
$$

**Theorem 3.4.** Suppose that $Q$ is relaxed $\gamma$-pseudomonotone on $H$, weakly sequentially continuous, $\xi$-Lipschitz continuous on $\Omega$ and $\text{Sol}(\Omega, Q) \neq \emptyset$. Then, the sequence $\{x^\ell\}$ defined by Algorithm 3.1 weakly converges to a solution of $VI(\Omega, Q)$.

**Proof.** Since $0 < a \leq \lambda_\ell \leq b < \frac{1}{\xi}$, it satisfies that

$$
0 < 1 - b^2\xi^2 \leq 1 - \lambda_\ell^2\xi^2 \leq 1 - a^2\xi^2 < 1.
$$

From Lemma 3.3, the sequence $\{x^\ell\}$ is bounded and

$$
\lim_{\ell \to \infty} \|x^\ell - \bar{x}^\ell\| = 0.
$$

By the $\xi$-Lipschitz continuity of $Q$ in $\Omega$, we have

$$
\|Q(x^\ell) - Q(\bar{x}^\ell)\| \leq \xi\|x^\ell - \bar{x}^\ell\|.
$$
Therefore
\[
\lim_{\ell \to \infty} \|Q(x^\ell) - Q(\bar{x}^\ell)\| = 0.
\]
Since \(\{x^\ell\}\) is a bounded sequence, there exists a subsequence \(\{x^\ell_i\}\) of \(\{x^\ell\}\) weakly converging to \(\hat{x} \in \Omega\). Since
\[
\lim_{\ell \to \infty} \|x^\ell - \bar{x}^\ell\| = 0,
\]
we have
\[
x^\ell_i \rightharpoonup \hat{x}.
\]
We show that \(\hat{x} \in \text{Sol}(\Omega, Q)\). Since
\[
\bar{x}^\ell = P_{\Omega}(x^\ell - \lambda^\ell Q(x^\ell))
\]
by the projection characterization (2.2), it holds
\[
\langle x^\ell_i - \lambda^\ell_i Q(x^\ell_i) - \bar{x}^\ell_i, y - \bar{x}^\ell_i \rangle \leq 0, \quad \forall y \in \Omega,
\]
it implies that
\[
\frac{1}{\lambda^\ell_i} \langle x^\ell_i - \bar{x}^\ell_i, y - \bar{x}^\ell_i \rangle \leq \langle Q(x^\ell_i), y - \bar{x}^\ell_i \rangle, \quad \forall y \in \Omega.
\]
Hence we have
\[
\frac{1}{\lambda^\ell_i} \langle x^\ell_i - \bar{x}^\ell_i, y - \bar{x}^\ell_i \rangle + \langle Q(x^\ell_i), \bar{x}^\ell_i - x^\ell_i \rangle \leq \langle Q(x^\ell_i), y - x^\ell_i \rangle, \forall y \in \Omega. \quad (3.2)
\]
Letting \(i \to +\infty\) in the last inequality, we obtain
\[
\lim_{\ell \to \infty} \|x^\ell - \bar{x}^\ell\| = 0, \text{ fixed } y \in \Omega,
\]
and \(\lambda^\ell \in [a, b] \subset ]0, \frac{1}{\xi[}\) for all \(\ell\), we have
\[
\lim \inf_{i \to \infty} \langle Q(x^\ell_i), y - x^\ell_i \rangle \geq 0. \quad (3.3)
\]
Now we choose a sequence \(\{\epsilon_i\}\) of positive numbers decreasing and tending to 0. For each \(\epsilon_i\), the \(n_i\) is a smallest positive integer such that
\[
\langle Q(x^\ell_j), y - x^\ell_j \rangle + \epsilon_i > 0, \quad \forall j \geq n_i, \quad (3.4)
\]
where the existence of \(n_i\) follows from (3.3). Since \(\{\epsilon_i\}\) is decreasing, it is easy to see that the sequence \(\{n_i\}\) is increasing. Furthermore, for each \(i\), \(Q(x^{\ell_{n_i}}) \neq 0\) and, setting
\[
y^{\ell_{n_i}} = \frac{Q(x^{\ell_{n_i}})}{\|Q(x^{\ell_{n_i}})\|^2},
\]
we have
\[
\langle Q(x^{\ell_{n_i}}), y^{\ell_{n_i}} \rangle = 1 \text{ for each } i.
\]
From (3.4), for each $i$
\[ \langle Q(x^{\ell_{ni}}), y + \epsilon_i y^{\ell_{ni}} - x^{\ell_{ni}} \rangle \geq 0. \]
Since $Q$ is relaxed $\gamma$-pseudomonotone, that is
\[ \langle Q(y + \epsilon_i y^{\ell_{ni}}), y + \epsilon_i y^{\ell_{ni}} - x^{\ell_{ni}} \rangle \geq -\gamma \| y + \epsilon_i y^{\ell_{ni}} - x^{\ell_{ni}} \|^2 = 0. \]
Hence we have
\[ \langle Q(y + \epsilon_i y^{\ell_{ni}}), y + \epsilon_i y^{\ell_{ni}} - x^{\ell_{ni}} \rangle \geq 0. \] (3.5)
On the other hand, when $i \to \infty$ we have
\[ x^{\ell_i} \rightharpoonup \hat{x}. \]
Since $Q$ is weakly sequentially continuous on $\Omega$, we have
\[ Q(x^{\ell_i}) \rightharpoonup Q(\hat{x}). \]
We can suppose that $Q(\hat{x}) \neq 0$ (otherwise, $\hat{x}$ is a solution). Since the norm mapping is weakly sequentially lower semicontinuous, we have
\[ \|Q(\hat{x})\| \leq \liminf_{i \to \infty} \|Q(x^{\ell_i})\|. \]
Since $\{x^{\ell_{ni}}\} \subset \{x^{\ell_i}\}$ and $\epsilon_i \to 0$ as $i \to 0$, we obtain
\[ 0 \leq \lim_{i \to \infty} \| \epsilon_i y^{\ell_{ni}} \| = \lim_{i \to \infty} \frac{\epsilon_i}{\|Q(x^{\ell_{ni}})\|} \leq \frac{0}{\|Q(\hat{x})\|} = 0. \]
Hence, when $i \to \infty$ in (3.5), we get
\[ \langle Q(y), y - \hat{x} \rangle \geq 0. \]
It show that $\hat{x} \in \text{Sol}(\Omega, Q)$.

Finally, we show that $x^{\ell} \rightharpoonup \hat{x}$. To do this, it is sufficient to show that $\{x^{\ell}\}$ can not have two distinct weak sequential limit points in $\text{Sol}(\Omega, Q)$. Let $\{x^{\ell_j}\}$ be another subsequence of $\{x^{\ell}\}$ converging weakly to $\bar{x}$. We have to prove that $\hat{x} = \bar{x}$ and $\bar{x} \in \text{Sol}(\Omega, Q)$. From Lemma 3.3, the sequences $\{\|x^{\ell} - \hat{x}\|\}$ and $\{\|x^{\ell} - \bar{x}\|\}$ are monotonically decreasing and therefore converge. Since for all $\ell \in N$,
\[ 2\langle x^{\ell}, \bar{x} - \hat{x} \rangle = \|x^{\ell} - \hat{x}\|^2 - \|x^{\ell} - \bar{x}\|^2 + \|\bar{x}\|^2 - \|\hat{x}\|^2. \]
We deduce that the sequence $\{\langle x^{\ell}, \bar{x} - \hat{x} \rangle\}$ also converges. Setting
\[ \Im = \lim_{\ell \to \infty} \langle x^{\ell}, \bar{x} - \hat{x} \rangle, \]
and passing to the limit along $\{x^{\ell_i}\}$ and $\{x^{\ell_j}\}$ yields,
\[ \Im = \langle \hat{x}, \bar{x} - \hat{x} \rangle = \langle \bar{x}, \bar{x} - \hat{x} \rangle. \]
This implies that $\|\hat{x} - \bar{x}\|^2 = 0$ and therefore $\hat{x} = \bar{x}$. \qed
Remark 3.5. (i) In [14], author discussed the extragradient method for solving strongly pseudomonotone variational inequalities with choice of the step sizes:

\[ \sum_{\ell=0}^{\infty} \lambda_\ell = \infty, \quad \lim_{\ell \to \infty} \lambda_\ell = 0, \]

and proved that the iterative sequence defined by the extragradient method converges strongly to a solution.

(ii) If \( Q \) is monotone function but it is not necessary to weakly sequentially continuous on \( Q \). Then, from the monotonicity of \( Q \) and (3.2) we have

\[
\frac{1}{\lambda_{\ell_i}} \langle x^{\ell_i} - \bar{x}^{\ell_i}, y - \bar{x}^{\ell_i} \rangle + \langle Q(x^{\ell_i}), \bar{x}^{\ell_i} - x^{\ell_i} \rangle \leq \langle Q(x^{\ell_i}), y - x^{\ell_i} \rangle \\
\leq \langle Q(y), y - x^{\ell_i} \rangle, \quad \forall y \in \Omega.
\]

From last inequality letting \( i \to +\infty \), we have

\[
\lim_{\ell \to \infty} \| x^{\ell} - \bar{x}^{\ell} \| = 0
\]

and \( \lambda_\ell \in [a, b] \subset (0, \frac{1}{\xi}) \) for all \( \ell \), we obtain

\[
\langle Q(y), y - \hat{x} \rangle \geq 0, \quad \forall y \in \Omega.
\]

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References


