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EXISTENCE THEOREMS FOR THE GENERALIZED RELAXED PSEUDOMONOTONE VARIATIONAL INEQUALITIES

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Abstract. This work aims to suggest a generalized relaxed γ-pseudomonotone variational inequalities in Hilbert spaces and show that the iterative sequence defined by an algorithm weakly converges to a solution.

1. Introduction

The variational inequality theory plays a very important role in many areas, such as optimal control, mechanics, economics, transportation equilibrium and engineering sciences *etc.* An important part of the research focuses on the existence of solutions to variational inequality. The most basic methods for the solution of variational inequalities are projection method, extra-gradient method, Tikhonov regularization method and proximal point method; *see,*

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[9, 13, 16, 21]. The well-known gradient projection method can be successfully applied for solving strongly monotone variational inequalities and inverse strongly monotone variational inequalities [4, 7, 9, 12, 20]. Korpelevich introduced the extra-gradient method [18], and this method was applied for solving monotone variational inequalities in infinite-dimensional spaces. It is a known fact [9] that the extra-gradient method can be successfully applied for solving monotone variational inequalities in infinite-dimensional Hilbert spaces [3, 5, 6, 13, 15, 22, 23, 24]. Providing that the variational inequality has solutions and the assigned mapping is monotone and Lipschitz continuous, it is proved that the iterative sequence defined by the extra-gradient method weakly converges to a solution.

The goal of this work is to define the generalized relaxed γ-pseudomonotone variational inequalities in Hilbert spaces and prove that the iterative sequence suggested by the algorithm for solving generalized relaxed γ-pseudomonotone variational inequalities weakly converges to a solution.

2. Preliminaries

As a matter of convenience, we introduce the notation first. Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and Ω be a nonempty, closed and convex subset of $\mathcal{H}$. Let $x_n \to x$ and $x_n \rightharpoonup x$ represent sequence $\{x_n\}$ converging strongly and weakly to x, *respectively*, and $\mathfrak{W}_\omega(x_n)$ denote the set of weak cluster points of sequence $\{x_n\}$.

Let $Q : \mathcal{H} \to \mathcal{H}$ be a mapping. The variational inequality $VI(\Omega, Q)$ defined by Ω and Q consists in finding a point $x^* \in \Omega$ such that

$$\langle Q(x^*), x - x^* \rangle \geq 0, \ \ \forall x \in \Omega. \tag{2.1}$$

The solution set of (2.1) is denoted by $Sol(\Omega, Q)$.

Let P_Ω be a mapping from $\mathcal{H}$ onto a nonempty, closed and convex subset Ω of $\mathcal{H}$. Then P_Ω is called the orthogonal projection from $\mathcal{H}$ onto Ω, if

$$P_\Omega(x) = \arg\min_{y \in \Omega} \|x - y\|, \ \ \forall x \in \mathcal{H},$$

Lemma 2.1. ([10]) *Let Ω be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$. For any $x, y \in \mathcal{H}$ and $z \in \Omega$, it satisfies:*

(i) $\|x - P_\Omega(x)\| \leq \|x - y\|$;
(ii) $\langle x - P_\Omega(x), z - P_\Omega(x) \rangle \leq 0$;
(iii) $\|P_\Omega(x) - P_\Omega(y)\|^2 \leq \|x - y\|$;
(iv) $\|P_\Omega(x) - z\|^2 \leq \|x - z\|^2 - \|P_\Omega(x) - x\|^2$.

Definition 2.2. ([2]) A mapping $P_\Omega : \mathcal{H} \to \Omega$ goes by the name of

(a) non-expansive if

$$\|P_\Omega(x) - P_\Omega(y)\| \leq \|x - y\|, \ \ \forall x, y \in \mathcal{H};$$

(b) firmly nonexpansive if

$$\|P_\Omega(x) - P_\Omega(y)\|^2 \leq \langle x - y, P_\Omega(x) - P_\Omega(y)\rangle, \ \ \forall x, y \in \mathcal{H}.$$

Lemma 2.3. ([10, 16]) *Let $x \in \mathcal{H}$ and $z \in \Omega$. Then $z = P_\Omega(x)$ if and only if*

$$P_\Omega(x) \in \Omega$$

and

$$\langle x - P_\Omega(x), y - P_\Omega(x)\rangle \leq 0, \ \ \forall y \in \Omega. \tag{2.2}$$

Remark 2.4. $x^* \in \mathcal{H}$ is a solution of (2.1) if and only if

$$x^* = P_\Omega(x^* - \lambda Q(x^*)), \ \lambda > 0.$$

Definition 2.5. Let Ω be a convex set in $\mathbb{R}^n$ and $Q : \Omega \to \Omega$ be a mapping. Then, Q is said to be:

(a) strongly monotone on Ω with constant $\gamma > 0$, if

$$\langle Q(x) - Q(y), x - y\rangle \geq \gamma\|x - y\|^2, \ \forall x, y \in \Omega;$$

(b) strictly monotone on Ω, if

$$\langle Q(x) - Q(y), x - y\rangle > 0, \ \text{distinct } x, y \in \Omega;$$

(c) monotone on Ω, if

$$\langle Q(x) - Q(y), x - y\rangle \geq 0, \ \ \forall x, y \in \Omega;$$

(d) relaxed monotone on Ω, if

$$\langle Q(x) - Q(y), x - y\rangle \geq -\gamma\|x - y\|^2, \ \ \forall x, y \in \Omega;$$

(e) relaxed γ-pseudomonotone on Ω, if $\langle Q(y), x - y\rangle \geq 0$ then

$$\langle Q(x), x - y\rangle \geq -\gamma\|x - y\|^2, \ \ \forall x, y \in \Omega \text{ and } \gamma > 0;$$

(f) pseudo monotone on Ω, if $\langle Q(y), x - y\rangle \geq 0$ then

$$\langle Q(x), x - y\rangle \geq 0, \ \ \forall x, y \in \Omega.$$

Remark 2.6. (i) We observe that Q is relaxed γ-pseudomonotone on Ω, if Q is relaxed γ-pseudomonotone for every $x \in \Omega$. Therefore the relaxed γ-pseudomonotonicity is a real generalization of pseudomonotonicity and relaxed monotonicity.

(ii) We point out that a pseudomonotone operator is relaxed γ-pseudomonotone operator; and a relaxed monotone operator is relaxed γ-pseodomonotone operator. However, the converse are not true in general.

(iii) The following implications hold:

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f).$$

(iv) The reverse assertions are not true in general.

Example 2.7. The function $Q : [0, +\infty) \to \mathbb{R}$ defined by

$$Q(x) = \begin{cases} x^2 - 2, & \text{if } x \geq \sqrt{3} - 1, \\ -2x, & \text{if } 0 \leq x < \sqrt{3} - 1. \end{cases}$$

is relaxed γ-pseudomonotone on $[0, +\infty)$ with $\gamma = 2$, but not pseudomonotone on $[0, +\infty)$, *i.e.,* $Q(x)$ is not pseudomonotone at $x = 0$.

Example 2.8. The function $Q : (-\infty, 0) \to (0, +\infty)$ defined by

$$Q(x) = x^2$$

is relaxed γ-pseudomonotone, but not relaxed monotone on $(-\infty, 0)$, since for all $\sigma > 0$, there exists $x_0 < 0$ and $y_0 < 0$ with $x_0 + y_0 < -\sigma < 0$ such that

$$\langle Q(y_0) - Q(x_0), y_0 - x_0 \rangle < -\sigma \|y_0 - x_0\|^2.$$

Remark 2.9. From above, we conclude that the relaxed γ-pseudomonotonicity is a real generalization of pseudomonotonicity and relaxed monotonicity.

Example 2.10. The function $Q : \mathbb{R} \to \mathbb{R}$ defined by

$$Q(x) = -x$$

has a solution $x = 0$. However, it is easy to observe that $Q(x) = -x$ is neither monotone on $\mathbb{R}$ nor pseudomonotone on $\mathbb{R}$. But it is relaxed monotone on $\mathbb{R}$.

Example 2.11. The function $Q : (-\infty, 2) \to \mathbb{R}$ defined by

$$Q(x) = \begin{cases} x^2, & \text{if } x \geq 1, \\ -x + 2, & \text{if } 1 < x < 2, \end{cases}$$

is relaxed γ-pseudomonotone with $\gamma = 1$ on $(-\infty, 0) \cup (0, 2)$, but not relaxed γ-pseudomonotone on $(-\infty, 2)$.

Now we can consider the following problem of finding $x \in \Omega$ such that

$$\langle Q(x^*), x - x^* \rangle \geq -\gamma \|x - x^*\|^2, \ \ \forall x \in \Omega. \tag{2.3}$$

We denote by Ω^* and Ω^d the solutions sets of problem (2.1) and problem (2.3), respectively. Now we give the relationships between problem (2.1) and problem (2.3).

Proposition 2.12. ([17])

(i) *The set* Ω^d *is closed and convex;*
(ii) $\Omega^d \subseteq \Omega^*$ *holds;*
(iii) *If* Q *is pseudomonotone, then*

$$\Omega^* \subseteq \Omega^d.$$

Proposition 2.13. ([1]) *Assume that* Ω *is a nonempty, closed and convex subset of a normed space* X*. If* $Q : \Omega \to X^*$ *(dual space) is hemicontinuous, then* $\Omega^d \subset \Omega^*$*. In addition, if* Q *is relaxed* γ*-pseudomonotone, then*

$$\Omega^d = \Omega^*.$$

Definition 2.14. A mapping $Q : \mathcal{H} \to \mathcal{H}$ is said to be

(i) ξ-Lipschitz continuous if there exists $\xi > 0$ such that

$$\|Q(x) - Q(y)\| \leq \xi\|x - y\|, \ \ \forall x, y \in \mathcal{H};$$

(ii) weakly sequentially continuous if for each sequence $\{x_n\}$ we have

$$x_n \rightharpoonup x$$

implies

$$Q(x_n) \rightharpoonup Q(x).$$

Lemma 2.15. ([19]) *Let* Ω *be a nonempty, closed and convex subset of* $\mathcal{H}$*, and* $\{x_n\}$ *be a sequence on* $\mathcal{H}$ *if*

(i) $\lim_{n\to\infty} \|x_n - x\|$ *exists for each* $x \in \Omega$*;*
(ii) $\mathfrak{W}_\omega(x_n) \subseteq \Omega$*;*

then $\{x_n\}$ *weakly converges to a point in* Ω*.*

Proposition 2.16. ([11])

(i) *If* Q *is strictly monotone, then variational inequality* (2.1) *has at most one solution.*
(ii) *If* Q *is strongly monotone, then variational inequality* (2.1) *has a unique solution.*

Proposition 2.17. ([8]) *Let* Ω *be a nonempty, closed and convex subset of a real Hilbert space* $\mathcal{H}$ *and* $Q : \Omega \to \mathcal{H}$ *be a pseudomonotone and continuous. Then,* x^* *is a solution of* $VI(\Omega, Q)$ *if and only if*

$$\langle Q(x), x - x^* \rangle \geq 0, \ \forall x \in \Omega.$$

3. Weakly Convergence

In this section, we consider the problem $VI(\Omega, Q)$ with Ω being nonempty, closed and convex and Q being relaxed γ-pseudomonotone on $\mathcal{H}$ and Lipschitz continuous with constant $\xi > 0$. We assume that the solution set $Sol(\Omega, Q) \neq \emptyset$.

Algorithm 3.1. ***Input:*** *$x^0 \in \Omega$ and $\{\lambda_\ell\} \in [a, b]$, where $0 < a \leq b < \dfrac{1}{\xi}$.*
Step 0: *Set $\ell = 0$.*
Step 1: *If $x^\ell = P_\Omega(x^\ell - \lambda_\ell Q(x^\ell))$ then stop.*
Step 2: *Otherwise, set*

$$\bar{x}^\ell = P_\Omega(x^\ell - \lambda_\ell Q(x^\ell)),$$
$$x^{\ell+1} = P_\Omega(x^\ell - \lambda_\ell Q(\bar{x}^\ell)).$$

*Replace ℓ by $\ell + 1$; go to **Step 1**.*

Remark 3.2. Assume that $Q(x^\ell) = 0$, then

$$x^\ell = P_\Omega(x^\ell - \lambda_\ell Q(x^\ell)),$$

and the Algorithm terminates at step ℓ with a solution x^ℓ. Again, we assume that $Q(x^\ell) \neq 0$ for all ℓ then algorithm generates an infinite sequence.

Lemma 3.3. ([15, 18]) *Assume that Q is pseudomonotone and ξ-Lipschitz continuous on Ω and $Sol(\Omega, Q)$ is nonempty. Let x^* be a solution of $VI(\Omega, Q)$. Then, for every $\ell \in N$, we have*

$$\|x^{\ell+1} - x^*\|^2 \leq \|x^\ell - x^*\|^2 - (1 - \lambda_\ell^2 \xi^2)\|x^\ell - \bar{x}^\ell\|^2. \tag{3.1}$$

Theorem 3.4. *Suppose that Q is relaxed γ-pseudomonotone on $\mathcal{H}$, weakly sequentially continuous, ξ-Lipschitz continuous on Ω and $Sol(\Omega, Q) \neq \emptyset$. Then, the sequence $\{x^\ell\}$ defined by Algorithm 3.1 weakly converges to a solution of $VI(\Omega, Q)$.*

Proof. Since $0 < a \leq \lambda_\ell \leq b < \dfrac{1}{\xi}$, it satisfies that

$$0 < 1 - b^2\xi^2 \leq 1 - \lambda_\ell^2 \xi^2 \leq 1 - a^2\xi^2 < 1.$$

From Lemma 3.3, the sequence $\{x^\ell\}$ is bounded and

$$\lim_{\ell \to \infty} \|x^\ell - \bar{x}^\ell\| = 0.$$

By the ξ-Lipschitz continuity of Q in Ω, we have

$$\|Q(x^\ell) - Q(\bar{x}^\ell)\| \leq \xi \|x^\ell - \bar{x}^\ell\|.$$

Therefore

$$\lim_{\ell\to\infty} \|Q(x^\ell) - Q(\bar{x}^\ell)\| = 0.$$

Since $\{x^\ell\}$ is a bounded sequence, there exists a subsequence $\{x^{\ell_i}\}$ of $\{x^\ell\}$ weakly converging to $\hat{x} \in \Omega$. Since

$$\lim_{\ell\to\infty} \|x^\ell - \bar{x}^\ell\| = 0,$$

we have

$$\bar{x}^{\ell_i} \rightharpoonup \hat{x}.$$

We show that $\hat{x} \in Sol(\Omega, Q)$. Since

$$\bar{x}^\ell = P_\Omega(x^\ell - \lambda_\ell Q(x^\ell)),$$

by the projection characterization (2.2), it holds

$$\langle x^{\ell_i} - \lambda_{\ell_i} Q(x^{\ell_i}) - \bar{x}^{\ell_i}, y - \bar{x}^{\ell_i}\rangle \le 0,\ \forall y \in \Omega,$$

it implies that

$$\frac{1}{\lambda_{\ell_i}}\langle x^{\ell_i} - \bar{x}^{\ell_i}, y - \bar{x}^{\ell_i}\rangle \le \langle Q(x^{\ell_i}), y - \bar{x}^{\ell_i}\rangle,\ \forall y \in \Omega.$$

Hence we have

$$\frac{1}{\lambda_{\ell_i}}\langle x^{\ell_i} - \bar{x}^{\ell_i}, y - \bar{x}^{\ell_i}\rangle + \langle Q(x^{\ell_i}), \bar{x}^{\ell_i} - x^{\ell_i}\rangle \le \langle Q(x^{\ell_i}), y - x^{\ell_i}\rangle, \forall y \in \Omega. \quad (3.2)$$

Letting $i \to +\infty$ in the last inequality, we obtain

$$\lim_{\ell\to\infty} \|x^\ell - \bar{x}^\ell\| = 0,\ \text{fixed } y \in \Omega,$$

and $\lambda_\ell \in [a, b] \subset]0, \frac{1}{\xi}[$ for all ℓ, we have

$$\liminf_{i\to\infty} \langle Q(x^{\ell_i}), y - x^{\ell_i}\rangle \ge 0. \quad (3.3)$$

Now we choose a sequence $\{\epsilon_i\}_i$ of positive numbers decreasing and tending to 0. For each ϵ_i, the n_i is a smallest positive integer such that

$$\langle Q(x^{\ell_j}), y - x^{\ell_j}\rangle + \epsilon_i > 0,\ \forall j \ge n_i, \quad (3.4)$$

where the existence of n_i follows from (3.3). Since $\{\epsilon_i\}$ is decreasing, it is easy to see that the sequence $\{n_i\}$ is increasing. Furthermore, for each i, $Q(x^{\ell_{n_i}}) \neq 0$ and, setting

$$y^{\ell_{n_i}} = \frac{Q(x^{\ell_{n_i}})}{\|Q(x^{\ell_{n_i}})\|^2},$$

we have

$$\langle Q(x^{\ell_{n_i}}), y^{\ell_{n_i}}\rangle = 1 \text{ for each } i.$$

From (3.4), for each i

$$\langle Q(x^{\ell_{n_i}}), y + \epsilon_i \; y^{\ell_{n_i}} - x^{\ell_{n_i}} \rangle \geq 0.$$

Since Q is relaxed γ-pseudomonotone, that is

$$\langle Q(y + \epsilon_i \; y^{\ell_{n_i}}), y + \epsilon_i \; y^{\ell_{n_i}} - x^{\ell_{n_i}} \rangle \geq -\gamma \|y + \epsilon_i y^{\ell_{n_i}} - x^{\ell_{n_i}}\|^2 = 0.$$

Hence we have

$$\langle Q(y + \epsilon_i \; y^{\ell_{n_i}}), y + \epsilon_i \; y^{\ell_{n_i}} - x^{\ell_{n_i}} \rangle \geq 0. \tag{3.5}$$

On the other hand, when $i \to \infty$ we have

$$x^{\ell_i} \rightharpoonup \hat{x}.$$

Since Q is weakly sequentially continuous on Ω, , we have

$$Q(x^{\ell_i}) \rightharpoonup Q(\hat{x}).$$

We can suppose that $Q(\hat{x}) \neq 0$ (otherwise, $\hat{x}$ is a solution). Since the norm mapping is weakly sequentially lower semicontinuous, we have

$$\|Q(\hat{x})\| \leq \lim \inf_{i \to \infty} \|Q(x^{\ell_i})\|.$$

Since $\{x^{\ell_{n_i}}\} \subset \{x^{\ell_i}\}$ and $\epsilon_i \to 0$ as $i \to 0$, we obtain

$$0 \leq \lim_{i \to \infty} \|\epsilon_i \; y^{\ell_{n_i}}\| = \lim_{i \to \infty} \frac{\epsilon_i}{\|Q(x^{\ell_{n_i}})\|} \leq \frac{0}{\|Q(\hat{x})\|} = 0.$$

Hence, when $i \to \infty$ in (3.5), we get

$$\langle Q(y), y - \hat{x} \rangle \geq 0.$$

It show that $\hat{x} \in Sol(\Omega, Q)$.

Finally, we show that $x^{\ell} \rightharpoonup \hat{x}$. To do this, it is sufficient to show that $\{x^{\ell}\}$ can not have two distinct weak sequential limit points in $Sol(\Omega, Q)$. Let $\{x^{\ell_j}\}$ be another subsequence of $\{x^{\ell}\}$ converging weakly to $\bar{x}$. We have to prove that $\hat{x} = \bar{x}$ and $\bar{x} \in Sol(\Omega, Q)$. From Lemma 3.3, the sequences $\{\|x^{\ell} - \hat{x}\|\}$ and $\{\|x^{\ell} - \bar{x}\|\}$ are monotonically decreasing and therefore converge. Since for all $\ell \in N$,

$$2\langle x^{\ell}, \bar{x} - \hat{x} \rangle = \|x^{\ell} - \hat{x}\|^2 - \|x^{\ell} - \bar{x}\|^2 + \|\bar{x}\|^2 - \|\hat{x}\|^2.$$

We deduce that the sequence $\{\langle x^{\ell}, \bar{x} - \hat{x} \rangle\}$ also converges. Setting

$$\Im = \lim_{\ell \to \infty} \langle x^{\ell}, \bar{x} - \hat{x} \rangle,$$

and passing to the limit along $\{x^{\ell_i}\}$ and $\{x^{\ell_j}\}$ yields,

$$\Im = \langle \hat{x}, \bar{x} - \hat{x} \rangle = \langle \bar{x}, \bar{x} - \hat{x} \rangle.$$

This implies that $\|\hat{x} - \bar{x}\|^2 = 0$ and therefore $\hat{x} = \bar{x}$. $\square$

Remark 3.5. (i) In [14], author discussed the extragradient method for solving strongly pseudomonotone variational inequalities with choice of the step sizes:

$$\sum_{\ell=0}^{\infty} \lambda_\ell = \infty, \ \lim_{\ell\to\infty} \lambda_\ell = 0,$$

and proved that the iterative sequence defined by the extragradient method converges strongly to a solution.

(ii) If Q is monotone function but it is not necessary to weakly sequentially continuous on Q. Then, from the monotonicity of Q and (3.2) we have

$$\begin{aligned}\frac{1}{\lambda_{\ell_i}}\langle x^{\ell_i} - \bar{x}^{\ell_i}, y - \bar{x}^{\ell_i}\rangle + \langle Q(x^{\ell_i}), \bar{x}^{\ell_i} - x^{\ell_i}\rangle &\leq \langle Q(x^{\ell_i}), y - x^{\ell_i}\rangle \\ &\leq \langle Q(y), y - x^{\ell_i}\rangle, \ \ \forall y \in \Omega.\end{aligned}$$

From last inequality letting $i \to +\infty$, we have

$$\lim_{\ell\to\infty} \|x^\ell - \bar{x}^\ell\| = 0$$

and $\lambda_\ell \in [a, b] \subset (0, \frac{1}{\xi})$ for all ℓ, we obtain

$$\langle Q(y), y - \hat{x}\rangle \geq 0, \ \ \forall y \in \Omega.$$

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