Nonlinear Functional Analysis and Applications
Vol. 25, No. 1 (2020), pp. 35-54
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2020.25.01.04
http://nfaa.kyungnam.ac.kr/journal-nfaa

STABILITY OF SOME CUBIC-ADDITIVE FUNCTIONAL EQUATIONS

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Abstract. We prove the generalized Hyers-Ulam stability of the cubic-additive functional equation of the form $D_{i,n}f(x,y) = 0$, where $i \in \{1,2,3,4\}$ and n is an integer larger than 1.

1. Introduction

Throughout this paper, let V and W be real vector spaces, X a real normed space, Y a real Banach space, and let k be a nonzero real number with the condition $|k| \neq 1$. Moreover, assume that a is a real number larger than 1 and n is an integer larger than 1.

For a given mapping $f : V \to W$, we use the following abbreviations:

$$\begin{aligned} f_o(x) &:= \frac{f(x) - f(-x)}{2}, \\ f_e(x) &:= \frac{f(x) + f(-x)}{2}, \\ Af(x,y) &:= f(x+y) - f(x) - f(y), \\ Cf(x,y) &:= f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y), \end{aligned}$$

[0]Received January 29, 2019. Revised June 4, 2019.
[0]2010 Mathematics Subject Classification: 39B82, 39B52.
[0]Keywords: Generalized Hyers-Ulam stability, functional equation, generalized polynomial mapping, cubic-additive mapping.
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$$\begin{aligned} D_{1,k}f(x,y) :=&\ f(x+ky)+f(x-ky)-k^2f(x+y)-k^2f(x-y) \\ &+(k^2-1)f(x)-(k^2-1)f(-x), \\ D_{2,k}f(x,y) :=&\ f(x+ky)-f(ky-x)-k^2f(x+y)+k^2f(y-x) \\ &+2(k^2-1)f(x), \\ D_{3,k}f(x,y) :=&\ f(kx+y)-f(y-kx)-kf(x+y)+kf(y-x) \\ &-2f(kx)+2kf(x), \\ D_{4,k}f(x,y) :=&\ f(kx+y)+f(kx-y)-kf(x+y)-kf(x-y) \\ &-2f(kx)+2kf(x) \end{aligned}$$

for all $x, y \in V$.

Every solution of functional equation $Af(x,y) = 0$ is called an additive mapping, while each solution of functional equation $Cf(x,y) = 0$ is called a cubic mapping. A mapping expressed by the sum of an additive mapping and a cubic mapping is called a cubic-additive mapping. A functional equation is called a cubic-additive functional equation provided that each solution of the equation is a cubic-additive mapping and every cubic-additive mapping is a solution of that equation. The functional equations $D_{i,n}f(x,y) = 0$, for $i \in \{1,2,3,4\}$, are cubic-additive functional equations.

In 1940, Ulam [12] raised an important question concerning the stability of group homomorphisms: When does the solution of an equation differing slightly from a given one has to necessarily be close to the solution of the given equation? In the following year, Hyers [3] solved the problem for the case of Cauchy additive functional equation $Af(x,y) = 0$. After about three decades, Rassias [11] generalized the Hyers' result and then Găvruta [4] extended the Rassias' result by allowing unbounded control functions. The stability concept presented by Rassias and Găvruta is known today as the term 'generalized Hyers-Ulam stability' of functional equations.

We now consider the functional equation

$$D_{i,n}f(x,y) = 0, \tag{1.1}$$

where $i \in \{1,2,3,4\}$ and n is an integer larger than 1. One of typical examples for solutions of equation (1.1) is the mapping $f(x) = cx^2 + dx$ with real constants c and d. In 2015, Jin and Lee [6] proved the stability of equation $D_{a,b}f(x,y) = 0$ for an arbitrary real number $a \neq 0$ and $b = 1$ in Fuzzy spaces. For more detailed information on the stability of cubic-additive functional equations, we may refer to [1, 5, 7, 10, 13, 14, 15, 16].

In this paper, we will prove the generalized Hyers-Ulam stability of the functional equation (1.1), where $i \in \{1,2,3,4\}$ and n is an integer larger than 1.

2. Preliminaries

Let V and W be real vector spaces and let Y be a real Banach space. For a given mapping $f : V \to W$, we use the following abbreviations:

$$f^{(1)}(x) := \frac{2^3 f(x) - f(2x)}{2^3 - 2}, \qquad f^{(2)}(x) := -\frac{2f(x) - f(2x)}{2^3 - 2}$$

for all $x \in V$. We will first introduce the following lemmas. Since their proofs are very similar to the proofs of [8, Corollaries 2.2 and 2.3], we omit their proofs.

Lemma 2.1. *Given a real constant $a > 1$, let $\phi : V\backslash\{0\} \to [0, \infty)$ be a function satisfying either*

$$\Phi(x) := \sum_{i=0}^{\infty} \frac{1}{a^i}\phi\big(a^i x\big) < \infty \ (\textit{for all } x \in V\backslash\{0\})$$

or

$$\Phi(x) := \sum_{i=0}^{\infty} a^{3i}\phi\left(\frac{x}{a^i}\right) < \infty \ (\textit{for all } x \in V\backslash\{0\})$$

and let $f : V \to Y$ be a mapping. If there exists a mapping $F : V \to Y$ satisfying

$$\|f(x) - F(x)\| \le \Phi(x) \tag{2.1}$$

for all $x \in V\backslash\{0\}$ and

$$F^{(1)}(ax) = aF^{(1)}(x), \ F^{(2)}(ax) = a^3 F^{(2)}(x) \tag{2.2}$$

for all $x \in V$, then F is the unique mapping satisfying (2.1) *and* (2.2)*.*

Lemma 2.2. *Given a real number $a > 1$, let $\phi, \psi : V\backslash\{0\} \to [0, \infty)$ be functions satisfying each of the following conditions:*

$$\sum_{i=0}^{\infty} \frac{1}{a^{2i}}\phi\big(a^i x\big) < \infty, \qquad \sum_{i=0}^{\infty} a^i\psi\left(\frac{x}{a^i}\right) < \infty,$$

$$\tilde{\Phi}(x) := \sum_{i=0}^{\infty} a^i\phi\left(\frac{x}{a^i}\right) < \infty, \qquad \tilde{\Psi}(x) := \sum_{i=0}^{\infty} \frac{1}{a^{2i}}\psi\big(a^i x\big) < \infty$$

for all $x \in V\backslash\{0\}$ and let $f : V \to Y$ be a mapping. If there exists a mapping $F : V \to Y$ satisfying the inequality

$$\|f(x) - F(x)\| \le \tilde{\Phi}(x) + \tilde{\Psi}(x) \tag{2.3}$$

for all $x \in V\backslash\{0\}$ as well as the conditions in (2.2) *for all $x \in V$, then F is the unique mapping satisfying the conditions* (2.2) *for all $x \in V$ and the inequality* (2.3) *for all $x \in V\backslash\{0\}$.*

We define $Df : V^2 \to W$ by

$$Df(x,y) := \sum_{i=1}^{m} c_i f(a_i x + b_i y)$$

for all $x, y \in V$, where m is a positive integer and a_i, b_i, c_i are real constants.

The following lemmas are essential for establishing our main theorems.

Lemma 2.3. *Let $\mu : V \to [0,\infty)$ be a function satisfying the condition*

$$\sum_{i=0}^{\infty} \frac{\mu\big(a^i x\big)}{|a|^i} < \infty \tag{2.4}$$

for all $x \in V$ and let $\varphi : V^2 \to [0,\infty)$ be a function satisfying the condition

$$\sum_{i=0}^{\infty} \frac{\varphi\big(a^i x, a^i y\big)}{|a|^i} < \infty \tag{2.5}$$

for all $x, y \in V$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$,

$$\big\| f\big(a^2 x\big) - \big(a + a^3\big) f(ax) + a^4 f(x) \big\| \le \mu(x) \tag{2.6}$$

for all $x \in V$, and moreover

$$\|Df(x,y)\| \le \varphi(x,y) \tag{2.7}$$

for all $x, y \in V$, then there exists a unique mapping $F : V \to Y$ satisfying

$$DF(x,y) = 0 \tag{2.8}$$

for all $x, y \in V$, equalities in (2.2) for all $x \in V$, and moreover

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{a^{2i+2} + 1}{|a^3 - a||a|^{3i+3}} \mu\big(a^i x\big) \tag{2.9}$$

for all $x \in V$.

Proof. First, we define $A := \big\{ f : V \to Y \mid f(0) = 0 \big\}$ and a mapping $J_m : A \to A$ by

$$J_m f(x) := \frac{f^{(2)}(a^m x)}{a^{3m}} + \frac{f^{(1)}(a^m x)}{a^m}$$

for all $x \in V$ and all $m \in \mathbb{N}_0$, where we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. It follows from (2.6) that

$$\begin{aligned}
&\big\| J_m f(x) - J_{m+l} f(x) \big\| \\
&\leq \sum_{i=m}^{m+l-1} \| J_i f(x) - J_{i+1} f(x) \| \\
&= \sum_{i=m}^{m+l-1} \Bigg\| \frac{f\big(a^{i+1}x\big) - af\big(a^i x\big)}{\big(a^3 - a\big)a^{3i}} - \frac{f\big(a^{i+2}x\big) - af\big(a^{i+1}x\big)}{\big(a^3 - a\big)a^{3i+3}} \\
&\qquad - \frac{f\big(a^{i+1}x\big) - a^3 f\big(a^i x\big)}{\big(a^3 - a\big)a^{i}} + \frac{f\big(a^{i+2}x\big) - a^3 f\big(a^{i+1}x\big)}{\big(a^3 - a\big)a^{i+1}} \Bigg\| \qquad (2.10) \\
&\leq \sum_{i=m}^{m+l-1} \Bigg\| - \frac{f\big(a^2 a^i x\big) - \big(a + a^3\big) f\big(a^{i+1}x\big) + a^4 f\big(a^i x\big)}{\big(a^3 - a\big)a^{3i+3}} \\
&\qquad + \frac{f\big(a^2 a^i x\big) - \big(a + a^3\big) f\big(a^{i+1}x\big) + a^4 f\big(a^i x\big)}{\big(a^3 - a\big)a^{i+1}} \Bigg\| \\
&\leq \sum_{i=m}^{m+l-1} \frac{\big|a^{2i+2} + 1\big| \mu\big(a^i x\big)}{\big|a^3 - a\big| |a|^{3i+3}}
\end{aligned}$$

for all $x \in V \backslash \{0\}$.

In view of (2.4) and (2.10), the sequence $\{J_m f(x)\}$ is Cauchy for all $x \in V \backslash \{0\}$. Since Y is complete and $f(0) = 0$, the sequence $\{J_m f(x)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{m\to\infty} J_m f(x) = \lim_{m\to\infty} \left(\frac{f^{(2)}(a^m x)}{a^{3m}} + \frac{f^{(1)}(a^m x)}{a^m} \right)$$

for all $x \in V$. Moreover, if we put $m = 0$ and let $l \to \infty$ in (2.10), then we obtain the inequality (2.9). Since the remaining part of this theorem can be proved in the same way as in [9, Theorem 3.1], we omit the remaining proof. □

Lemma 2.4. *Let $\mu : V \to [0, \infty)$ be a function satisfying the condition*

$$\sum_{i=0}^{\infty} |a|^{3i} \mu\left(\frac{x}{a^i}\right) < \infty \qquad (2.11)$$

for all $x \in V$ and let $\varphi : V^2 \to [0, \infty)$ be a function satisfying the condition

$$\sum_{i=0}^{\infty} |a|^{3i} \varphi\left(\frac{x}{a^i}, \frac{y}{a^i}\right) < \infty \qquad (2.12)$$

for all $x, y \in V$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$, the inequality (2.6) for all $x \in V$, and also the inequality (2.7) for all $x, y \in V$, then there exists a unique mapping $F : V \to Y$ satisfying (2.8) for all $x, y \in V$, equalities in (2.2) for all $x \in V$, as well as

$$\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} \frac{|a^{3i} + a^i|}{|a^3 - a|} \mu\left(\frac{x}{a^{i+1}}\right) \tag{2.13}$$

for all $x \in V$.

Proof. First, we define the mappings $J_m f : V \to Y$ by

$$J_m f(x) := a^{3m} f^{(2)}\left(\frac{x}{a^m}\right) + a^m f^{(1)}\left(\frac{x}{a^m}\right)$$

for all $x \in V$ and $m \in \mathbb{N}_0$. It follows from (2.6) that

$$\begin{aligned}
&\|J_m f(x) - J_{m+l} f(x)\| \\
&\leq \sum_{i=m}^{m+l-1} \|J_i f(x) - J_{i+1} f(x)\| \\
&\leq \sum_{i=m}^{m+l-1} \frac{1}{|a^3 - a|} \\
&\qquad \times \left\| a^{3i}\left(f\left(\frac{a^2 x}{a^{i+1}}\right) - (a + a^3) f\left(\frac{ax}{a^{i+1}}\right) + a^4 f\left(\frac{x}{a^{i+1}}\right)\right) \right. \\
&\qquad\quad \left. - a^i\left(f\left(\frac{a^2 x}{a^{i+1}}\right) - (a + a^3) f\left(\frac{ax}{a^{i+1}}\right) + a^4 f\left(\frac{x}{a^{i+1}}\right)\right) \right\| \\
&\leq \sum_{i=m}^{m+l-1} \frac{|a^{3i} + a^i|}{|a^3 - a|} \mu\left(\frac{x}{a^{i+1}}\right)
\end{aligned} \tag{2.14}$$

for all $x \in V \backslash \{0\}$.

On account of (2.11) and (2.14), the sequence $\{J_m f(x)\}$ is Cauchy for all $x \in V \backslash \{0\}$. Since Y is complete and $f(0) = 0$, the sequence $\{J_m f(x)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{m \to \infty} \left(a^{3m} f^{(2)}\left(\frac{x}{a^m}\right) + a^m f^{(1)}\left(\frac{x}{a^m}\right)\right)$$

for all $x \in V$. Moreover, if we put $m = 0$ and let $l \to \infty$ in (2.14), we obtain the inequality (2.13). Since the rest part of this theorem can be proved in the same way as in [9, Theorem 3.2], we omit the remaining proof. $\square$

Lemma 2.5. *Let $\mu : V \to [0,\infty)$ be a function satisfying the conditions*

$$\sum_{i=0}^{\infty} \frac{\mu(a^i x)}{|a|^{3i}} < \infty \text{ and } \sum_{i=0}^{\infty} |a|^i \mu\Big(\frac{x}{a^i}\Big) < \infty \tag{2.15}$$

for all $x \in V$ and let $\varphi : V^2 \to [0,\infty)$ be a function satisfying the conditions

$$\sum_{i=0}^{\infty} \frac{\varphi(a^i x, a^i y)}{|a|^{3i}} < \infty \text{ and } \sum_{i=0}^{\infty} |a|^i \varphi\Big(\frac{x}{a^i}, \frac{y}{a^i}\Big) < \infty \tag{2.16}$$

for all $x, y \in V$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$, inequality (2.6) *for all $x \in V$, and inequality* (2.7) *for all $x, y \in V$, then there exists a unique mapping $F : V \to Y$ satisfying equality* (2.8) *for all $x, y \in V$, equalities in* (2.2) *for all $x \in V$, and moreover*

$$\|f(x) - F(x)\| \le \frac{1}{|a^3 - a|} \sum_{i=0}^{\infty} \left(\frac{\mu(a^i x)}{|a|^{3i+3}} + |a|^i \mu\Big(\frac{x}{a^{i+1}}\Big) \right) \tag{2.17}$$

for all $x \in V$.

Proof. First, we define a set $A := \{f : V \to Y \,|\, f(0) = 0\}$ and a mapping $J_m : A \to A$ by

$$J_m f(x) := \frac{f^{(2)}(a^m x)}{a^{3m}} + a^m f^{(1)}\Big(\frac{x}{a^m}\Big)$$

for all $x \in V$ and each $m \in \mathbb{N}_0$. It follows from (2.6) that

$$\begin{aligned}
&\|J_m f(x) - J_{m+l} f(x)\| \\
&\quad \le \sum_{i=m}^{m+l-1} \|J_i f(x) - J_{i+1} f(x)\| \\
&\quad = \sum_{i=m}^{m+l-1} \left\| \frac{f^{(2)}(a^i x)}{a^{3i}} - \frac{f^{(2)}(a^{i+1} x)}{a^{3i+3}} + a^i f^{(1)}\Big(\frac{x}{a^i}\Big) - a^{i+1} f^{(1)}\Big(\frac{x}{a^{i+1}}\Big) \right\| \\
&\quad \le \frac{1}{a^3 - a} \sum_{i=m}^{m+l-1} \left\| - \frac{f(a^2 \cdot a^i x) - (a + a^3) f(a^{i+1} x) + a^4 f(a^i x)}{a^{3i+3}} \right. \\
&\qquad\qquad \left. - a^i \Big(f\Big(\frac{a^2 x}{a^{i+1}}\Big) - (a + a^3) f\Big(\frac{a x}{a^{i+1}}\Big) + a^4 f\Big(\frac{x}{a^{i+1}}\Big)\Big) \right\| \\
&\quad \le \frac{1}{|a^3 - a|} \sum_{i=m}^{m+l-1} \left(\frac{\mu(a^i x)}{|a|^{3i+3}} + |a|^i \mu\Big(\frac{x}{a^{i+1}}\Big) \right)
\end{aligned} \tag{2.18}$$

for all $x \in V \backslash \{0\}$.

In view of (2.6), (2.15) and (2.18), the sequence $\{J_m f(x)\}$ is Cauchy for all $x \in V\backslash\{0\}$. Since Y is complete and $f(0) = 0$, the sequence $\{J_m f(x)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{m\to\infty} \frac{f^{(2)}(a^m x)}{a^{3m}} + a^m f^{(1)}\left(\frac{x}{a^m}\right)$$

for all $x \in V$. Moreover, if we put $m = 0$ and let $l \to \infty$ in (2.18), we obtain the first inequality of (2.17). Since the rest part of this theorem can be proved in the same way as in [9, Theorem 3.2], we omit the remaining proof. □

3. Characterizations of cubic-additive mappings

The following theorem is a specific version of Baker's theorem [2] which is essential for establishing Theorems 3.3, 3.4 and 3.5.

Theorem 3.1. ([2, Theorem 1]) *Assume that V and W are vector spaces over $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ and α_0, $\beta_0, \ldots, \alpha_m$, β_m are scalars such that $\alpha_j\beta_\ell - \alpha_\ell\beta_j \neq 0$ whenever $0 \le j < \ell \le m$. If the mappings $f_\ell : V \to W$ satisfy*

$$\sum_{\ell=0}^{m} f_\ell(\alpha_\ell x + \beta_\ell y) = 0$$

for all $x, y \in V$, then each f_ℓ is a generalized polynomial mapping of degree at most $m - 1$.

Baker [2] also states that if f is a generalized polynomial mapping of degree at most $m - 1$, then f is expressed as $f(x) = x_0 + \sum_{\ell=1}^{m-1} a_\ell^*(x)$ for $x \in V$, where a_ℓ^* is a monomial mapping of degree ℓ and f has a property $f(rx) = x_0 + \sum_{\ell=1}^{m-1} r^\ell a_\ell^*(x)$ for $x \in V$ and $r \in \mathbb{Q}$. We note that a_1^*, a_2^* and a_3^* are called an additive mapping, a quadratic mapping and a cubic mapping, respectively.

The following theorem immediately follows from Theorem 3.1.

Theorem 3.2. *If a mapping $f : V \to W$ satisfies the functional equation $D_{i,k}f(x, y) = 0$ for all $x, y \in V$ and for $i \in \{1, 2, 3, 4\}$, then f is a generalized polynomial mapping of degree at most 3.*

Assume that $f, g : V \to W$ are generalized polynomial mappings of degree at most 3. In view of the above argument, it is obvious that if $f(2x) = 2f(x)$ and $g(2x) = 2^3 g(x)$ for all $x \in V$, then f and g are an additive mapping and a cubic mapping, respectively. If $f : V \to W$ is a generalized polynomial even mapping of degree at most 3 and $f(nx) = nf(x)$ for all $x \in V$, then f is an even and additive mapping, *i.e.*, $f \equiv 0$.

Theorem 3.3. *If a mapping $f : V \to W$ satisfies the functional equation $D_{i,k}f(x,y) = 0$ for all $x, y \in V$ and for $i \in \{1,2,3,4\}$, then f is a cubic-additive mapping.*

Proof. Assume that a mapping $f : V \to W$ satisfies $D_{i,k}f(x,y) = 0$ for all $x, y \in V$ and for $i \in \{1,2,4\}$. The following equalities are consequences of our long and tedious calculations:

$$\begin{aligned} &f(4x) - 10f(2x) + 16f(x) \\ &\quad = \frac{1}{k^4-k^2}\Big(\big(4k^2-3\big)D_{1,k}f_o(x,x) - 2k^2D_{1,k}f_o(2x,x) \\ &\qquad\qquad + 2k^2D_{1,k}f_o(x,2x) - 2D_{1,k}f_o\big((k+1)x,x\big) \\ &\qquad\qquad + 2D_{1,k}f_o\big((k-1)x,x\big) - k^2D_{1,k}f_o(2x,2x) \\ &\qquad\qquad + D_{1,k}f_o(x,3x) - D_{1,k}f_o\big((2k+1)x,x\big) \\ &\qquad\qquad + D_{1,k}f_o\big((2k-1)x,x\big)\Big) \\ &\qquad - \frac{1}{2(k^2-1)}\big(D_{1,k}f(4x,0) - 10D_{1,k}f(2x,0) + 16D_{1,k}f(x,0)\big), \end{aligned} \tag{3.1}$$

$$\begin{aligned} &f(4x) - 10f(2x) + 16f(x) \\ &\quad = \frac{1}{k^4-k^2}\Big(\big(4k^2-3\big)D_{2,k}f_o(x,x) - 2k^2D_{2,k}f_o(2x,x) \\ &\qquad\qquad + 2k^2D_{2,k}f_o(x,2x) - 2D_{2,k}f_o\big((k+1)x,x\big) \\ &\qquad\qquad + 2D_{2,k}f_o\big((k-1)x,x\big) - k^2D_{2,k}f_o(2x,2x) \\ &\qquad\qquad + D_{2,k}f_o(x,3x) - D_{2,k}f_o\big((2k+1)x,x\big) \\ &\qquad\qquad + D_{2,k}f_o\big((2k-1)x,x\big)\Big) \\ &\qquad + \frac{1}{2\big(k^2-1\big)}\big(D_{2,k}f(4x,0) - 10D_{2,k}f(2x,0) + 16D_{2,k}f(x,0)\big), \end{aligned} \tag{3.2}$$

$$\begin{aligned} &f(4x) - 10f(2x) + 16f(x) \\ &\quad = \frac{1}{k-k^3}\left(8D_{4,k}f_o\left(\frac{x}{2},\frac{kx}{2}\right) - 8kD_{4,k}f_o\left(\frac{x}{2},\frac{(2k+1)x}{2}\right)\right. \\ &\qquad\qquad + 8kD_{4,k}f_o\left(\frac{x}{2},\frac{(2k-1)x}{2}\right) - 8D_{4,k}f_o\left(\frac{x}{2},\frac{3kx}{2}\right) \end{aligned}$$

$$
\begin{aligned}
&+ \big(1-8k^2\big) D_{4,k} f_o(x,x) - D_{4,k} f_o(x,kx) \\
&+ 2D_{4,k} f_o\big(x,(k+1)x\big) + 2D_{4,k} f_o\big(x,(k-1)x\big) \\
&+ (k+1) D_{4,k} f_o\big(x,(2k+1)x\big) \\
&- (k-1) D_{4,k} f_o\big(x,(2k-1)x\big) \\
&+ D_{4,k} f_o(x,3kx) - 2D_{4,k} f_o(2x,x) \\
&+ k^2 D_{4,k} f_o(2x,2x) - 2D_{4,k} f_o(2x,kx) \\
&- D_{4,k} f_o(2x,2kx) - D_{4,k} f_o(3x,x)\Bigg) \\
+ \frac{1}{2-2k}&\big(D_{4,k} f(0,4x) - 10D_{4,k} f(0,2x) + 16D_{4,k} f(0,x)\big)
\end{aligned}
\tag{3.3}
$$

for all $x \in V$. From these equalities and our assumption, we get $f(4x) - 10f(2x) + 16f(x) = 0$ for all $x \in V$. Hence, we can show that $f^{(1)}(2x) = 2f^{(1)}(x)$ and $f^{(2)}(2x) = 2^3 f^{(2)}(x)$ for all $x \in V$. Since $f^{(1)}$ and $f^{(2)}$ are generalized polynomial mappings of degree at most 4 and $f = f^{(1)} + f^{(2)}$, we can conclude that $f^{(1)}$ is an additive mapping and $f^{(2)}$ is a cubic mapping, *i.e.*, f is a cubic-additive mapping.

We now consider the case of $D_{3,k} f(x,y) = 0$. By a similar way, we see that $f_o^{(1)}(2x) = 2f_o^{(1)}(x)$ and $f_o^{(2)}(2x) = 2^3 f^{(2)}(x)$ for all $x \in V$ by using the following equality

$$
\begin{aligned}
&f_o(4x) - 10f_o(2x) + 16f_o(x) \\
&= \frac{1}{n-n^3}\Bigg(8D_{3,n} f_o\left(\frac{x}{2},\frac{nx}{2}\right) - 8nD_{3,n} f_o\left(\frac{x}{2},\frac{(2n+1)x}{2}\right) \\
&\qquad + 8nD_{3,n} f_o\left(\frac{x}{2},\frac{(2n-1)x}{2}\right) - 8D_{3,n} f_o\left(\frac{x}{2},\frac{3nx}{2}\right) \\
&\qquad + \big(1-8n^2\big) D_{3,n} f_o(x,x) - D_{3,n} f_o(x,nx) \\
&\qquad + 2D_{3,n} f_o\big(x,(n+1)x\big) + 2D_{3,n} f_o\big(x,(n-1)x\big) \\
&\qquad + (n+1) D_{3,n} f_o\big(x,(2n+1)x\big) - (n-1) D_{3,n} f_o\big(x,(2n-1)x\big) \\
&\qquad + D_{3,n} f_o(x,3nx) - 2D_{3,n} f_o(2x,x) + n^2 D_{3,n} f_o(2x,2x) \\
&\qquad - 2D_{3,n} f_o(2x,nx) - D_{3,n} f_o(2x,2nx) - D_{3,n} f_o(3x,x)\Bigg)
\end{aligned}
\tag{3.4}
$$

for all $x \in V$. Since $f_o^{(1)}$ and $f_o^{(2)}$ are generalized polynomial mappings of degree at most 4 and $f_o = f_o^{(1)} + f_o^{(2)}$, we may conclude that $f_o^{(1)}$ is an additive mapping and $f_o^{(2)}$ is a cubic mapping, *i.e.*, f_o is a cubic-additive mapping.

From the equality $f_e(nx) - nf_e(x) = \frac{1}{2}D_{3,n}f(0,x)$ for all $x \in V$, we know that $f_e(nx) = nf_e(x)$ for all $x \in V$ and $f_e \equiv 0$, *i.e.*, $f = f_o$ is a cubic-additive mapping. □

Theorem 3.4. *If $f : V \to W$ is an additive mapping, then f satisfies the functional equation $D_{i,n}f(x,y) = 0$ for all $x, y \in V$ and for $i \in \{1,2,3,4\}$.*

Proof. We notice that f satisfies the equalities $f(nx) = nf(x)$ and $f(x) = -f(-x)$ for all $x \in V$ and $n \in \mathbb{N}$. Thus, we get the equalities $D_i f(x,y) = 0$ for $i \in \{5,6\}$ and $D_{i,n}f(x,y) = 0$ for $i \in \{1,2,3,4\}$ by using the following equalities

$$\begin{aligned}
D_{1,n}f(x,y) &= -Af(x+ny, x-ny) + n^2 Af(x+y, x-y),\\
D_{2,n}f(x,y) &= -Af(x+ny, x-ny) + n^2 Af(x+y, x-y),\\
D_{3,n}f(x,y) &= -Af(nx+y, nx-y) + nAf(x+y, x-y),\\
D_{4,n}f(x,y) &= -Af(nx+y, nx-y) + nAf(x+y, x-y)
\end{aligned}$$

for all $x, y \in V$. □

Theorem 3.5. *If $f : V \to W$ is a cubic mapping, then f satisfies the functional equation $D_{i,n}f(x,y) = 0$ for all $x, y \in V$ and for $i \in \{1,2,3,4\}$.*

Proof. We remark that $f(nx) = n^3 f(x)$ and $f(x) = -f(-x)$ for all $x \in V$ and $n \in \mathbb{N}$ provided f is a cubic mapping.

First, we will verify that $D_{1,n}f(x,y) = 0$ and $D_{2,n}f(x,y) = 0$. It is easy to see that the equalities $D_{i,2}f(x,y) = 0$ and $D_{i,3}f(x,y) = 0$ for $i \in \{1,2\}$ follow from the equalities

$$\begin{aligned}
&D_{i,2}f(x,y) = Cf(x,y) + Cf(x,-y),\\
&D_{i,3}f(x,y) = D_{i,2}f(x+y,y) + D_{i,2}f(x-y,y) + 4D_{i,2}f(x,y)
\end{aligned}$$

for all $x, y \in V$. If $D_{i,j}f(x,y) = 0$ for all $j \in \mathbb{N}$ with $2 \le j \le n-1$, then the equality $D_{i,n}f(x,y) = 0$ for $i = 1$ or 2 follows from the equality

$$\begin{aligned}
D_{i,n}f(x,y) = D_{i,n-1}f(x+y,y) &+ D_{i,n-1}f(x-y,y) - D_{i,n-2}f(x,y)\\
&+ (n-1)^2 D_{i,2}f(x,y)
\end{aligned}$$

for all $x, y \in V$. By applying mathematical induction, we conclude that $D_{i,n}f(x,y) = 0$ for all $x, y \in V$, $i \in \{1,2\}$, and for all integers $n > 1$.

Secondly, we prove that $D_{3,n}f(x,y) = 0$ and $D_{4,n}f(x,y) = 0$. The equalities $D_{i,2}f(x,y) = 0$ and $D_{i,3}f(x,y) = 0$ for $i \in \{3,4\}$ follow from the following equalities

$$D_{i,2}f(x,y) = Cf(y,x) + Cf(y-x,x), \; D_{i,3}f(x,y) = Cf(y-x,2x)$$

for all $x, y \in V$. If $D_{i,j}f(x,y) = 0$ for all $j \in \mathbb{N}$ with $2 \le j \le n-1$, then the equality $D_{i,k}f(x,y) = 0$ for $i = 3$ or 4 follows from the equality

$$\begin{aligned} D_{i,n}f(x,y) = {} & D_{i,n-1}f(x+y,y) + D_{i,n-1}f(x-y,y) - D_{i,n-2}f(x,y) \\ & + (n-1)D_{i,2}f(x,y) \end{aligned}$$

for all $x, y \in V$. Hence, we conclude that $D_{i,n}f(x,y) = 0$ for all $x, y \in V$, $i \in \{3,4\}$, and for all integers $n > 1$. □

4. Stability of cubic-additive functional equations

In this section, by using Lemmas 2.3 to 2.5 from the previous section, we will prove our main theorems concerning the generalized Hyers-Ulam stability of cubic-additive functional equations. First, we will apply Lemma 2.3 to the proof of the following theorem.

Theorem 4.1. *Let the function $\varphi : V^2 \to [0,\infty)$ satisfy the condition*

$$\sum_{i=0}^{\infty} \frac{\varphi\big(2^i x, 2^i y\big)}{2^i} < \infty \tag{4.1}$$

for all $x, y \in V$. Given an integer $n > 1$, if a mapping $f : V \to Y$ satisfies $f(0) = 0$ and

$$\|D_{1,n}f(x,y)\| \le \varphi(x,y) \tag{4.2}$$

for all $x, y \in V$, then there exists a unique mapping $F : V \to Y$ such that

$$D_{1,n}F(x,y) = 0 \tag{4.3}$$

for all $x, y \in V$ and

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{2^{2i+2}+1}{3 \cdot 2^{3i+4}} \mu_1\big(2^i x\big) \tag{4.4}$$

for all $x \in V$, where $\varphi_e : V^2 \to [0,\infty)$ and $\mu_1 : V \to [0,\infty)$ are defined by

$$\begin{aligned} \varphi_e(x,y) := {} & \frac{\varphi(x,y) + \varphi(-x,-y)}{2}, \\ \mu_1(x) := {} & \frac{1}{n^4 - n^2}\Big((4n^2-3)\varphi_e(x,x) + 2n^2\varphi_e(2x,x) + 2n^2\varphi_e(x,2x) \\ & + 2\varphi_e\big((n+1)x,x\big) + 2\varphi_e\big((n-1)x,x\big) + n^2\varphi_e(2x,2x) \\ & + \varphi_e(x,3x) + \varphi_e\big((2n+1)x,x\big) + \varphi_e\big((2n-1)x,x\big)\Big) \\ & + \frac{1}{2\big(n^2-1\big)}\big(\varphi(4x,0) + 5n^2\varphi(2x,0) + 8n^2\varphi(x,0)\big) \end{aligned}$$

for all $x, y \in V$.

Proof. By (3.1) and (4.2), we get

$$\big\|f\big(2^2x\big) - \big(2+2^3\big)f(2x) + 2^4 f(x)\big\| \le \mu_1(x)$$

for all $x \in V$. If we put $a = 2$ and replace μ with μ_1 in (2.4), then μ_1 satisfies the condition (2.4) instead of μ and the mapping f satisfies inequality (2.6) for all $x \in V$ in view of the last inequality.

By Lemma 2.3, there exists a mapping $F : V \to Y$ satisfying equality (4.3) for all $x, y \in V$, equalities in (2.2) with $a = 2$ for all $x \in V$, and inequality (4.4) for all $x \in V$. Since the equalities in (2.2) follow from the equality (4.3), we can conclude that $F : V \to Y$ is the unique mapping satisfying (4.3) and (4.4). □

We can prove the following theorem in the same way as we did in the proof of Theorem 4.1 by applying Lemma 2.4 instead of Lemma 2.3. Hence, we omit the proof.

Theorem 4.2. *Let the function* $\varphi : V^2 \to [0, \infty)$ *satisfy the condition*

$$\sum_{i=0}^{\infty} 2^{3i}\varphi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty \tag{4.5}$$

for all $x, y \in V$. *Given an integer* $n > 1$, *if a mapping* $f : V \to Y$ *satisfies* $f(0) = 0$ *and* (4.2) *for all* $x, y \in V$, *then there exists a unique mapping* $F : V \to Y$ *satisfying* (4.3) *for all* $x, y \in V$ *and*

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{2^{3i}+2^i}{6}\mu_1\left(\frac{x}{2^{i+1}}\right)$$

for all $x \in V$.

The following theorem can be proved in the same way as we did in the proof of Theorem 4.1 by applying Lemma 2.5 instead of Lemma 2.3. Hence, we omit the proof.

Theorem 4.3. *Let the function* $\varphi : V^2 \to [0, \infty)$ *satisfy the conditions*

$$\sum_{i=0}^{\infty} \frac{\varphi\big(2^ix, 2^iy\big)}{2^{3i}} < \infty \text{ and } \sum_{i=0}^{\infty} 2^i\varphi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty \tag{4.6}$$

for all $x, y \in V$. *Given an integer* $n > 1$, *if a mapping* $f : V \to Y$ *satisfies* $f(0) = 0$ *and* (4.2) *for all* $x, y \in V$, *then there exists a unique mapping* $F :$

$V \to Y$ satisfying equality (4.3) for all $x, y \in V$ and

$$\|f(x) - F(x)\| \leq \frac{1}{6} \sum_{i=0}^{\infty} \left(\frac{\mu_1(2^i x)}{2^{3i+3}} + 2^i \mu_1\left(\frac{x}{2^{i+1}}\right) \right)$$

for all $x \in V$.

In the following theorems, we deal with the stability problems of cubic-additive functional equations $D_{2,n}f(x,y) = 0$ by applying Lemmas 2.3, 2.4 and 2.5 in order.

Theorem 4.4. *Let $\varphi : V^2 \to [0, \infty)$ be a function satisfying the condition (4.1) for all $x, y \in V$. Given an integer $n > 1$, if a mapping $f : V \to Y$ satisfies $f(0) = 0$ and*

$$\|D_{2,n}f(x,y)\| \leq \varphi(x,y) \tag{4.7}$$

for all $x, y \in V$, then there exists a unique mapping $F : V \to Y$ such that $D_{2,n}F(x,y) = 0$ for all $x, y \in V$ and

$$\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} \frac{2^{2i+2} + 1}{3 \cdot 2^{3i+4}} \mu_2(2^i x) \tag{4.8}$$

for all $x \in V$, where $\mu_2 : V \to [0, \infty)$ is defined by

$$\begin{aligned} \mu_2(x) := \frac{1}{n^4 - n^2} \Big(&(4n^2 - 3)\varphi_e(x,x) + 2n^2\varphi_e(2x,x) + 2n^2\varphi_e(x,2x) \\ &+ 2\varphi_e\big((n+1)x, x\big) + 2\varphi_e\big((n-1)x, x\big) + n^2\varphi_e(2x,2x) \\ &+ \varphi_e(x,3x) + \varphi_e\big((2n+1)x, x\big) + \varphi_e\big((2n-1)x, x\big) \Big) \\ + \frac{1}{2(n^2-1)} &\big(\varphi(4x,0) + 5n^2\varphi(2x,0) + 8n^2\varphi(x,0)\big) \end{aligned}$$

for all $x, y \in V$.

Proof. First, it follows from equality (3.2) and inequality (4.7) that

$$\left\| f(2^2 x) - (2 + 2^3) f(2x) + 2^4 f(x) \right\| \leq \mu_2(x)$$

for all $x \in V$. If we put $a = 2$ and replace μ with μ_2 in (2.4), then μ_2 satisfies (2.4) and the mapping f satisfies inequality (2.6) for all $x \in V$. According to Lemma 2.3, there exists the unique mapping $F : V \to Y$ satisfying equality $D_{2,n}F(x,y) = 0$ for all $x, y \in V$ and inequality (4.8) for all $x \in V$. □

Theorem 4.5. *Let $\varphi : V^2 \to [0,\infty)$ be a function satisfying the condition* (4.5) *for all $x, y \in V$. Given an integer $n > 1$, if a mapping $f : V \to Y$ satisfies $f(0) = 0$ and* (4.7) *for all $x, y \in V$, then there exists a unique mapping $F : V \to Y$ satisfying $D_{2,n}F(x,y) = 0$ for all $x, y \in V$ and*

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{2^{3i} + 2^i}{6} \mu_2\left(\frac{x}{2^{i+1}}\right)$$

for all $x \in V$.

Theorem 4.6. *Let $\varphi : V^2 \to [0,\infty)$ be a function satisfying the conditions in* (4.6) *for all $x, y \in V$. Given an integer $n > 1$, if a mapping $f : V \to Y$ satisfies $f(0) = 0$ and* (4.7) *for all $x, y \in V$, then there exists a unique mapping $F : V \to Y$ satisfying equality $D_{2,n}F(x,y) = 0$ for all $x, y \in V$ and*

$$\|f(x) - F(x)\| \le \frac{1}{6} \sum_{i=0}^{\infty} \left(\frac{\mu_2(2^i x)}{2^{3i+3}} + 2^i \mu_2\left(\frac{x}{2^{i+1}}\right) \right)$$

for all $x \in V$.

Now, the following three theorems deal with the stability problems of cubic-additive functional equations $D_{3,n}f(x,y) = 0$ by using Lemmas 2.3, 2.4 and 2.5 in order.

Theorem 4.7. *For a fixed integer $n > 1$, let $\varphi : V^2 \to [0,\infty)$ be a function satisfying the conditions* (4.1) *and*

$$\sum_{i=0}^{\infty} \frac{\varphi(n^i x, n^i y)}{n^i} < \infty \tag{4.9}$$

for all $x, y \in V$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$ and

$$\|D_{3,n}f(x,y)\| \le \varphi(x,y) \tag{4.10}$$

for all $x, y \in V$, then there exists a mapping $F : V \to Y$ such that

$$D_{3,n}F(x,y) = 0 \tag{4.11}$$

for all $x, y \in V$ and

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{2^{2i+2} + 1}{3 \cdot 2^{3i+4}} \mu_3(2^i x) + \sum_{i=0}^{\infty} \frac{\varphi(n^i x, 0)}{2n^{i+1}} \tag{4.12}$$

for all $x \in V$, *where*

$$\begin{aligned}\mu_3(x) := \frac{1}{n^3-n}\bigg(&8\varphi_e\Big(\frac{x}{2},\frac{nx}{2}\Big)+8n\varphi_e\Big(\frac{x}{2},\frac{(2n+1)x}{2}\Big)\\ &+8n\varphi_e\Big(\frac{x}{2},\frac{(2n-1)x}{2}\Big)+8\varphi_e\Big(\frac{x}{2},\frac{3nx}{2}\Big)\\ &+\big(8n^2-1\big)\varphi_e(x,x)+\varphi_e(x,nx)+2\varphi_e\big(x,(n+1)x\big)\\ &+2\varphi_e\big(x,(n-1)x\big)+(n+1)\varphi_e\big(x,(2n+1)x\big)\\ &+(n-1)\varphi_e\big(x,(2n-1)x\big)+\varphi_e(x,3nx)+2\varphi_e(2x,x)\\ &+n^2\varphi_e(2x,2x)+2\varphi_e(2x,nx)+\varphi_e(2x,2nx)+\varphi_e(3x,x)\bigg).\end{aligned}$$

Proof. From equality (3.4) and inequality (4.10), it follows that

$$\big\|f_o\big(2^2x\big)-\big(2+2^3\big)f_o(2x)+2^4f_o(x)\big\|\le\mu_3(x)$$

for all $x \in V$. If we put $a = n = 2$ temporarily and replace μ by μ_3 in (2.4), then μ_3 satisfies the condition (2.4) instead of μ and the mapping f_o satisfies inequality (2.6) for all $x \in V$.

Due to Lemma 2.3, there exists a mapping $F : V \to Y$ satisfying equality $D_{3,n}F(x, y) = 0$ for all $x, y \in V$ and inequality

$$\|f_o(x)-F(x)\|\le\sum_{i=0}^{\infty}\frac{2^{2i+2}+1}{3\cdot 2^{3i+4}}\mu_3\big(2^ix\big)$$

for all $x \in V$.

In view of the equality $f_e(nx) - nf_e(x) = -\frac{1}{2}D_{3,n}f(x, 0)$ for all $x \in V$, we have

$$\|f_e(nx)-nf_e(x)\|\le\frac{\varphi(x,0)}{2}$$

for all $x \in V$. Using the above inequality and (4.9), we can define $F'(x) = \lim_{i\to\infty}\frac{1}{n^i}f_e(n^ix) \in Y$ for all $x \in V$ such that $F' : V \to Y$ satisfies inequality

$$\|f_e(x)-F'(x)\|\le\sum_{i=0}^{\infty}\frac{\varphi\big(n^ix,0\big)}{2n^{i+1}}$$

and $D_{3,n}F'(x) = 0$ for all $x \in V$.

By the definition of F' and the equality $D_{3,n}F'(x) = 0$ for all $x \in V$, we see that F' is an even mapping and a cubic-additive mapping, *i.e.*, $F' \equiv 0$.

Hence, we obtain

$$\|f_e(x)\| \le \sum_{i=0}^{\infty} \frac{\varphi(n^i x, 0)}{2n^{i+1}}$$

for all $x \in V$. Thus,

$$\begin{aligned}\|f(x) - F(x)\| &\le \|f_e(x)\| + \|f_o(x) - F(x)\| \\ &\le \sum_{i=0}^{\infty} \frac{2^{2i+2}+1}{3 \cdot 2^{3i+4}} \mu_3(2^i x) + \sum_{i=0}^{\infty} \frac{\varphi(n^i x, 0)}{2n^{i+1}}\end{aligned}$$

for all $x \in V$. □

Theorem 4.8. *For a fixed integer $n > 1$, let $\varphi : V^2 \to [0, \infty)$ be a function satisfying the conditions* (4.5) *and*

$$\sum_{i=0}^{\infty} n^i \varphi\left(\frac{x}{n^i}, \frac{y}{n^i}\right) < \infty$$

for all $x, y \in V$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$ and (4.10) *for all $x, y \in V$, then there exists a mapping $F : V \to Y$ satisfying* (4.11) *for all $x, y \in V$ and*

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{2^{3i}+2^i}{6} \mu_3\left(\frac{x}{2^{i+1}}\right) + \sum_{i=0}^{\infty} \frac{n^i}{2} \varphi\left(\frac{x}{n^{i+1}}, 0\right)$$

for all $x \in V$.

Theorem 4.9. *For a fixed integer $n > 1$, let $\varphi : V^2 \to [0, \infty)$ be a function satisfying the conditions* (4.6) *and*

$$\sum_{i=0}^{\infty} n^i \varphi\left(\frac{x}{n^i}, \frac{y}{n^i}\right) < \infty$$

for all $x, y \in V$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$ and (4.10) *for all $x, y \in V$, then there exists a mapping $F : V \to Y$ satisfying equality* (4.11) *for all $x, y \in V$ and*

$$\|f(x) - F(x)\| \le \frac{1}{6} \sum_{i=0}^{\infty} \left(\frac{\mu_3(2^i x)}{2^{3i+3}} + 2^i \mu_3\left(\frac{x}{2^{i+1}}\right) \right) + \sum_{i=0}^{\infty} \frac{n^i}{2} \varphi\left(\frac{x}{n^{i+1}}, 0\right)$$

for all $x \in V$.

Analogously, we deal with the generalized Hyers-Ulam stability problems of cubic-additive functional equations $D_{4,n} f(x, y) = 0$ in the following three theorems.

Theorem 4.10. *Let $\varphi : V^2 \to [0,\infty)$ be a function satisfying the condition (4.1) for all $x, y \in V$. Given an integer $n > 1$, if a mapping $f : V \to Y$ satisfies $f(0) = 0$ and*

$$\|D_{4,n}f(x,y)\| \le \varphi(x,y) \tag{4.13}$$

for all $x, y \in V$, then there exists a unique mapping $F : V \to Y$ such that $D_{4,n}F(x,y) = 0$ for all $x, y \in V$ and

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{2^{2i+2}+1}{3\cdot 2^{3i+4}}\mu_4\big(2^i x\big) \tag{4.14}$$

for all $x \in V$, where

$$\begin{aligned}\mu_4(x) := \frac{1}{n^3-n}\Bigg(&8\varphi_e\left(\frac{x}{2},\frac{nx}{2}\right) + 8n\varphi_e\left(\frac{x}{2},\frac{(2n+1)x}{2}\right)\\ &+ 8n\varphi_e\left(\frac{x}{2},\frac{(2n-1)x}{2}\right) + 8\varphi_e\left(\frac{x}{2},\frac{3nx}{2}\right)\\ &+ \big(8n^2-1\big)\varphi_e(x,x) + \varphi_e(x,nx) + 2\varphi_e\big(x,(n+1)x\big)\\ &+ 2\varphi_e\big(x,(n-1)x\big) + (n+1)\varphi_e\big(x,(2n+1)x\big)\\ &+ (n-1)\varphi_e\big(x,(2n-1)x\big) + \varphi_e(x,3nx) + 2\varphi_e(2x,x)\\ &+ n^2\varphi_e(2x,2x) + 2\varphi_e(2x,nx) + \varphi_e(2x,2nx) + \varphi_e(3x,x)\Bigg)\\ + \frac{1}{2n-2}&\big(\varphi(0,4x) + 10\varphi(0,2x) + 16\varphi(0,x)\big).\end{aligned}$$

Proof. The inequality

$$\big\|f\big(2^2x\big) - \big(2+2^3\big)f(2x) + 2^4 f(x)\big\| \le \mu_4(x)$$

follows from (3.3) and (4.13) for all $x \in V$. If we put $a = 2$, then μ_4 satisfies the condition (2.4) and the mapping f satisfies inequality (2.6) for all $x \in V$. By Lemma 2.3, there exists a unique mapping $F : V \to Y$ satisfying equality $D_{4,n}F(x,y) = 0$ for all $x, y \in V$ and inequality (4.14) for all $x \in V$. □

Theorem 4.11. *Let $\varphi : V^2 \to [0,\infty)$ be a function satisfying the condition (4.5) for all $x, y \in V$. Given an integer $n > 1$, if a mapping $f : V \to Y$ satisfies $f(0) = 0$ and (4.13) for all $x, y \in V$, then there exists a unique mapping $F : V \to Y$ satisfying $D_{4,n}F(x,y) = 0$ for all $x, y \in V$ and*

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} \frac{2^{3i}+2^i}{6}\mu_4\left(\frac{x}{2^{i+1}}\right)$$

for all $x \in V$.

Theorem 4.12. *Let $\varphi : V^2 \to [0, \infty)$ be a function satisfying the conditions (4.6) for all $x, y \in V$. Given an integer $n > 1$, if a mapping $f : V \to Y$ satisfies $f(0) = 0$ and (4.13) for all $x, y \in V$, then there exists a unique mapping $F : V \to Y$ satisfying equality $D_{4,n}F(x, y) = 0$ for all $x, y \in V$ and*

$$\|f(x) - F(x)\| \leq \frac{1}{6} \sum_{i=0}^{\infty} \left(\frac{\mu_4(2^i x)}{2^{3i+3}} + 2^i \mu_4\left(\frac{x}{2^{i+1}}\right) \right)$$

for all $x \in V$.

Acknowledgments: Soon-Mo Jung was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2016R1D1A1B03931061).

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