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BEST PROXIMITY POINTS OF GENERALIZED CYCLIC WEAK (F, ψ, φ)-CONTRACTIONS IN ORDERED METRIC SPACES

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Abstract. The purpose of this paper is to introduce a new generalized cyclic weak (F, ψ, φ)-contraction based on the generalized weak φ -contraction which is proposed in [6], where F is a C-class function. Moreover, we obtain a corresponding best proximity point theorem for this cyclic mapping under certain condition. Our results obtained in this paper improve and extend previous known results in [6], as well as other results for cyclic contractions in the existing literature.

1. Introduction and mathematical preliminaries

Let A and B be nonempty subsets of a metric space (X, d). As a non-self mapping $U : A \to B$ does not necessarily have a fixed point, it is of natural interest to find an element $x \in A$ which is as close to $Ux \in B$ as possible. In

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other words, if the fixed point equation $Ux = x$ has no exact solution, then it is contemplated to find an approximate solution $x \in A$ such that the error $d(x, Ux)$ is minimum, where d is the distance function. Indeed, best proximity point theorems investigate the existence of such optimal approximate solutions, called best proximity points, to the fixed point equation $Ux = x$ when there is no exact solution. As the distance between any element $x \in A$ and its image $Ux \in B$ is at least the distance between the sets A and B, a best proximity pair theorem achieves global minimum of $d(x, Ux)$ by stipulating an approximate solution x of the fixed point equation $Ux = x$ to satisfy the condition that $d(x, Ux) = d(A, B)$.

Best proximity problems have been developed in this direction. Besides establishing the existence of best proximity points, iterative algorithms are also furnished to determine such optimal approximate solutions. Banach fixed point theorem states that when (X, d) is a complete metric space and $U : X \to X$ is a contraction, then U has a unique fixed point in X. If U is a non-self-mapping from A to B, where A and B are nonempty subsets of X, solutions of equation $Ux = x$ may not exist, particularly when $A \cap B = \emptyset$, then we want to find a solution x^* such that $d(x^*, Ux^*) = \min\{d(x, Ux) : x \in A\}$. With the variety of fixed point problems, the best proximity point problem becomes a hot topic recently and a number of authors obtained best proximity point results in many different settings (see [1], [3], [5], [8], [9], [10], for examples). Moreover, the research on the fixed point theory for cyclic contractions have received considerable interest.

In 2003, Kirk et al. [15] stated the first result in this area. Later, other authors also obtained many important results in this area (see [8], [11], [13], [16], [17] and the references therein).

The purpose of this article is to establish best proximity point theorems for generalized cyclic weak contractive non-self mappings, yielding global optimal approximate solutions of certain fixed point equations.

First, we present the definitions of a cyclic map.

Definition 1.1. ([15]) Let A and B be nonempty subsets of a metric space (X, d). A mapping $U : A \bigcup B \to A \bigcup B$ is called a cyclic map provided that $U(A) \subseteq B$ and $U(B) \subseteq A$.

Next, we recall the definitions of several well-known cyclic maps and best proximity point results.

Definition 1.2. ([12, 16]) Let A and B be nonempty subsets of a metric space (X, d). If U is a cyclic map, we say that:

(i) U is a *cyclic contraction*, if, for any $x \in A, y \in B$ and some $\alpha \in (0,1)$,

$$d(Ux, Uy) \leq \alpha d(x, y);$$

(ii) U is a *Kannan Type cyclic contraction*, if, for any $x \in A, y \in B$ and some $\alpha \in (0, \frac{1}{2})$,

$$d(Ux, Uy) \leq \alpha[d(Ux, x) + d(Uy, y)];$$

(iii) U is a *Chatterjee Type cyclic contraction*, if, for any $x \in A, y \in B$ and some $\alpha \in (0, \frac{1}{2})$,

$$d(Ux, Uy) \leq \alpha[d(Ux, y) + d(Uy, x)];$$

(iv) U is a *Reich type cyclic contraction*, if, for any $x \in A, y \in B$ and some $\alpha \in (0, \frac{1}{3})$,

$$d(Ux, Uy) \leq \alpha M(x, y),$$

where $M(x, y) = \max\{d(x, y), d(Ux, x), d(Uy, y)\}$.

Recall that, Kirk et al. [15] first stated and proved fixed point theorems for the cyclic contraction. In 2011, Karapinar and Erhan [12] proved fixed point theorems for the above cyclic maps.

Recently, several authors presented many results for cyclic mappings satisfying various (nonlinear) contractive conditions based on altering distance functions φ which were introduced by Khan et al. [14].

Definition 1.3. ([2]) Let A and B be nonempty subsets of a metric space (X, d). Suppose that $\varphi : [0, \infty) \to [0, \infty)$ is a strictly increasing map. A cyclic map $U : A \bigcup B \to A \bigcup B$ is called a cyclic weak φ-contraction, if for all $x \in A$ and $y \in B$

$$d(Ux, Uy) \leq d(x, y) - \varphi(d(x, y)) + \varphi(d(A, B)).$$

Let X be a nonempty set, we know that $(X, d, \preceq)$ is an ordered metric space if and only if (X, d) is a metric space and $(X, \preceq)$ is a partially ordered set. Two elements $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$.

Definition 1.4. ([18]) Let (X, d) be a metric space and A, B be two nonempty subsets of X. A point $x^* \in X$ is called a best proximity point of a cyclic map U, if

$$d(x^*, Ux^*) = d(A, B),$$

where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$.

In 2010, Rezapour et al. (see [8, 17]) stated the following best proximity point theorem for cyclic weak φ -contraction:

Theorem 1.5. *Let $(X, d, \preceq)$ be an ordered metric space, A and B be nonempty subsets of X and $U : A \bigcup B \to A \bigcup B$ be a decreasing, cyclic weak φ-contraction. Suppose there exists $x_0 \in A$ such that $x_0 \preceq U^2 x_0 \preceq U x_0$. Define $x_{n+1} = U x_n$ and $d_n := d(x_{n+1}, x_n)$ for all $n \in N$. Then $d_n \to d(A, B)$.*

Definition 1.6. Let A and B be nonempty subsets of a metric space (X, d). Suppose that $\varphi : [0, \infty) \to [0, \infty)$ is a strictly increasing map. A cyclic map $U : A \bigcup B \to A \bigcup B$ is called a Kannan type cyclic weak φ-contraction, if for all $x \in A$ and $y \in B$,

$$d(Ux, Uy) \leq \rho(x, y) - \varphi(\rho(x, y)) + \varphi(d(A, B)),$$

where $\rho(x, y) = \frac{1}{2}[d(x, fx) + d(y, fy)]$.

Recently, Cheng and Su [6] proposed a generalized cyclic weak φ-contraction and proved the best proximity point theorem of generalized Kannan type cyclic weak φ-contractions in ordered metric spaces.

Definition 1.7. ([6]) Let A and B be nonempty subsets of a metric space (X, d). Suppose that $\varphi : [0, \infty) \to [0, \infty)$ be a strictly increasing map. A cyclic map $U : A \bigcup B \to A \bigcup B$ is called a generalized cyclic weak φ-contraction, if for any $x \in A$ and $y \in B$,

$$d(Ux, Uy) \leq m(x, y) - \varphi(m(x, y)) + \varphi(d(A, B)),$$

where $m(x, y) = \max\{d(x, y), d(x, Ux), d(y, Uy), \frac{1}{2}[d(x, Uy) + d(y, Ux)]\}$.

Theorem 1.8. ([6]) *Let A and B be nonempty subsets of a metric space (X, d). Suppose that $U : A \bigcup B \to A \bigcup B$ is a generalized cyclic weak φ-contraction and there exists $y_0 \in A$. Define $y_{n+1} = U y_n$ for any $n \in N$. Then $d(y_n, y_{n+1}) \to d(A, B)$, as $n \to \infty$*

The concept of C-class functions was introduced by Ansari [4] as follows.

Definition 1.9. ([4]) A continuous function $F : [0, \infty)^2 \to \mathbb{R}$ is called a C-class function, if for any $s, t \in [0, \infty)$, the following conditions hold:

(1) $F(s, t) \leq s$;
(2) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$.

An extra condition on F that $F(0, 0) = 0$ could be imposed in some cases if required.

Remark 1.10. We denote the class of all C-class functions as $\mathcal{C}$.

Example 1.11. ([4]) Following examples show that the class $\mathcal{C}$ of C-class functions is nonempty:

(1) $F(s,t) = s - t$.
(2) $F(s,t) = ms, \ \ 0 < m < 1$.
(3) $F(s,t) = \frac{s}{(1+t)^r}$ for some $r \in (0,\infty)$.
(4) $F(s,t) = \log(t + a^s)/(1+t)$, for some $a > 1$.
(5) $F(s,t) = \ln(1+a^s)/2$, for $a > e$. Indeed $F(s,1) = s$ implies that $s = 0$.
(6) $F(s,t) = (s+l)^{(1/(1+t)^r)} - l$, $l > 1$, for $r \in (0,\infty)$.
(7) $F(s,t) = s \log_{t+a} a$, for $a > 1$.
(8) $F(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$.
(9) $F(s,t) = s\beta(s)$, where $\beta : [0,\infty) \to [0,1)$.
(10) $F(s,t) = s - \frac{t}{k+t}$.
(11) $F(s,t) = s - \varphi(s)$, where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$.
(12) $F(s,t) = sh(s,t)$, where $h : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function such that $h(t,s) < 1$ for all $t, s > 0$.
(13) $F(s,t) = s - (\frac{2+t}{1+t})t$.
(14) $F(s,t) = \sqrt[n]{\ln(1+s^n)}$.
(15) $F(s,t) = \phi(s)$, where $\phi : [0,\infty) \to [0,\infty)$ is an upper semicontinuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$.
(16) $F(s,t) = \frac{s}{(1+s)^r}$; $r \in (0,\infty)$.

Definition 1.12. ([14]) A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function, if the following properties are satisfied:

(i) ψ is non-decreasing and continuous,
(ii) $\psi(t) = 0$ if and only if $t = 0$.

Remark 1.13. We denote the class of altering distance functions as Ψ.

Definition 1.14. ([4]) An ultra altering distance function is a continuous, non-decreasing mapping $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) > 0$, $t > 0$ and $\varphi(0) \geq 0$.

Remark 1.15. We denote the class of ultra altering distance functions as Ψ_u.

Lemma 1.16. ([7]) *Suppose (X,d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exists an $\varepsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that*

$$d(x_{m(k)}, x_{n(k)}) \geq \varepsilon,$$
$$d(x_{m(k)-1}, x_{n(k)}) < \varepsilon$$

and

(i) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \varepsilon$;
(ii) $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon$;
(iii) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \varepsilon$.

Remark 1.17.

$$\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \varepsilon$$

and

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)-1}) = \varepsilon.$$

Motivated by the above mentioned results, the purpose of this paper is to introduce a new generalized cyclic weak (F, ψ, φ)-contraction based on the generalized weak φ-contraction which is proposed in [6], where F is a C-class function. Moreover, we obtain a corresponding best proximity point theorem for this cyclic mapping under certain conditions. Our results extend and improve the results obtained in [6].

2. Main Results

In this section, we first introduce the definition of a generalized cyclic weak (F, ψ, φ)-contraction, then we prove a new best proximity point theorem for this cyclic mapping under certain condition.

Definition 2.1. Let A and B be nonempty subsets of a metric space (X, d). Suppose that $\varphi, \psi : [0,\infty) \to [0,\infty)$ and φ is a strictly increasing map. A cyclic map $U : A\bigcup B \to A\bigcup B$ is called a generalized cyclic weak (F, ψ, φ)-contraction, if for any $x \in A$ and $y \in B$,

$$\begin{aligned}\psi(d(Ux, Uy)) \quad &\leq \quad F\big(\psi(\mathcal{M}(x, y)) - \psi(d(A, B)), \varphi(\mathcal{M}(x, y)) - \varphi(d(A, B))\big) \\ &\qquad + \psi(d(A, B)), \end{aligned} \tag{2.1}$$

where $F \in \mathcal{C}, \psi \in \Psi$ with $\psi(s + t) \leq \psi(s) + \psi(t)$, $\varphi \in \Psi_u$ and

$$\mathcal{M}(x, y) = \max\{d(x, y), d(x, Ux), d(y, Uy), \frac{1}{2}[d(x, Uy) + d(y, Ux)]\}.$$

Theorem 2.2. *Let A and B be nonempty subsets of a metric space (X, d). Suppose that $U : A\bigcup B \to A\bigcup B$ is a generalized cyclic weak (F, ψ, φ)-contraction and there exists $y_0 \in A$. Define $y_{n+1} = Uy_n$ for any $n \in N$. Then $d(y_n, y_{n+1}) \to d(A, B)$, as $n \to \infty$.*

Proof. Let $d_n = d(y_n, y_{n+1})$. First we claim that the sequence $\{d_n\}$ is non-increasing. By the assumption, we have

$$\begin{aligned}\psi(d_{n+1}) &= \psi(d(y_{n+1}, y_{n+2}))\\ &= \psi(d(Uy_n, Uy_{n+1}))\\ &\le F\big(\psi(\mathcal{M}(y_n, y_{n+1})) - \psi(d(A,B)),\\ &\qquad \varphi(\mathcal{M}(y_n, y_{n+1})) - \varphi(d(A,B))\big)\\ &\quad + \psi(d(A,B)),\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}\mathcal{M}(y_n, y_{n+1}) &= \max\{d(y_n, y_{n+1}), d(y_n, Uy_n), d(y_{n+1}, Uy_{n+1}),\\ &\qquad \frac{1}{2}[d(y_n, Uy_{n+1}) + d(y_{n+1}, Uy_n)]\}\\ &= \max\{d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2})\}\end{aligned}$$

Assume that there exists $n_0 \in N$ such that $\mathcal{M}(y_{n_0}, y_{n_0+1}) = d(y_{n_0+1}, y_{n_0+2})$. From $d(y_{n_0+1}, y_{n_0+2}) > d(y_{n_0}, y_{n_0+1})$, we have

$$\begin{aligned}\psi(d(y_{n_0+1}, y_{n_0+2})) &\le F\big(\psi(d(y_{n_0+1}, y_{n_0+2})) - \psi(d(A,B)),\\ &\qquad \varphi(d(y_{n_0+1}, y_{n_0+2})) - \varphi(d(A,B))\big) + \psi(d(A,B))\\ &\le \psi(d(y_{n_0+1}, y_{n_0+2})).\end{aligned}$$

This implies that

$$\psi(d(y_{n_0+1}, y_{n_0+2})) - \psi(d(A,B)) = 0$$

or

$$\varphi(d(y_{n_0+1}, y_{n_0+2})) - \varphi(d(A,B)) = 0,$$

which is a contradiction. Hence, for all $n \in N$

$$\mathcal{M}(y_n, y_{n+1}) = d(y_n, y_{n+1}).$$

Then the expression (2.2) turns into

$$\begin{aligned}\psi(d(y_{n+1}, y_{n+2})) &\le F\big(\psi(d(y_n, y_{n+1})) - \psi(d(A,B)), \varphi(d(y_n, y_{n+1}))\\ &\qquad - \varphi(d(A,B))\big) + \psi(d(A,B))\\ &\le \psi(d(y_n, y_{n+1})).\end{aligned} \tag{2.3}$$

Therefore,

$$d(y_{n+1}, y_{n+2}) \le d(y_n, y_{n+1}).$$

That is, the sequence $\{d_n\}$ is non-increasing and bounded below, it is obvious that $\lim_{n\to\infty} d_n$ exists.

If $d_{n_0} = 0$, for some $n_0 \in N$, obviously, $d_n \to 0$ and $d(A,B) = 0$, that is, $d_n \to d(A,B)$.

If $d_n \neq 0$, for all $n \in N$, put $d_n \to \gamma$, thus $\gamma \geq d(A,B)$. Since φ is a strictly increasing map, we have $\varphi(\gamma) \geq \varphi(d(A,B))$. From the expression (2.3), we get that

$$\begin{aligned}\psi(d(y_n, y_{n+1})) \leq F\big(&\psi(d(y_{n-1}, y_n)) - \psi(d(A,B)), \varphi(d(y_{n-1}, y_n)) - \varphi(d(A,B))\big)\\ &+ \psi(d(A,B)),\end{aligned}$$

from which it follows that

$$\begin{aligned}\psi(\gamma) &\leq F\big(\psi(\gamma) - \psi(d(A,B)), \varphi(\gamma) - \varphi(d(A,B))\big) + \psi(d(A,B))\\ &\leq \psi(\gamma).\end{aligned}$$

This implies that

$$\psi(\gamma) - \psi(d(A,B)) = 0$$

or

$$\varphi(\gamma) - \varphi(d(A,B)) = 0.$$

Therefore, $\gamma = d(A,B)$. That is, $d_n \to d(A,B)$. This completes the proof. □

Example 2.3. Consider the Euclidean ordered space $X = \mathbb{R}$ with the usual metric. Suppose $A = [-3,-1], B = [1,3]$, $\varphi(t) = \frac{1}{3}t$, $\psi(t) = t$, for all $t \geq 0$ and $F(s,t) = s - t$, $s,t \geq 0$. Define $V : A \bigcup B \to A \bigcup B$ by

$$V(x) = \begin{cases} -\frac{1}{3}x + \frac{2}{3}, & if \quad x \in A,\\ \\ -\frac{1}{3}x - \frac{2}{3}, & if \quad x \in B.\end{cases}$$

Clearly $d(A,B) = 2$, and V is a cyclic map. And

$$d(Vx, Vy) = \frac{1}{3}|y - x| + \frac{4}{3}.$$

We can see that

$$\begin{aligned}&F\big(\psi(\mathcal{M}(x,y)) - \psi(d(A,B)), \varphi(\mathcal{M}(x,y)) - \varphi(d(A,B))\big)\\ &\quad + \psi(d(A,B)) - \psi(d(Vx,Vy))\\ &= \psi(\mathcal{M}(x,y)) - \varphi(\mathcal{M}(x,y)) - \varphi(d(A,B))\big) - d(Vx,Vy)\\ &= \mathcal{M}(x,y) - \frac{1}{3}\mathcal{M}(x,y) - \frac{2}{3} - \frac{1}{3}|y-x| - \frac{4}{3}\\ &= \mathcal{M}(x,y) - \frac{1}{3}\mathcal{M}(x,y) - \frac{1}{3}|y-x| - 2 \geq 0,\end{aligned}$$

for all $x \in A, y \in B$, where

$$\mathcal{M}(x,y) = \max\{d(x,y), d(x,Vx), d(y,Vy), \frac{1}{2}[d(x,Vy) + d(y,Vx)]\}.$$

Therefore, V satisfies (2.1). That is, V is a generalized cyclic weak (F,ψ,φ)-contraction. All the conditions of Theorem 2.2 hold true, and U has a best proximity point. Here $x^* = -1$ is the best proximity point of V.

Remark 2.4. By taking $F(s,t) = s-t$ and $\psi(t) = t$ in Theorem 2.2 we obtain the corresponding result in [6].

Theorem 2.5. *Let (X,d) be a complete metric space. Suppose that $U : X \to X$ is a generalized cyclic weak (F,ψ,φ)-contraction with $d(A,B) = 0$ and there exists $y_0 \in A$. Define $y_{n+1} = Uy_n$ for any $n \in \mathbb{N}$. Then there exists a unique fixed point $y \in X$, that is, $Uy = y$.*

Proof. By the assumption and Theorem 2.2, we have $d(y_{n+1}, y_n) \to 0$. Next, we show that $\{y_n\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{y_n\}$ is not a Cauchy sequence. By Lemma 1.16, there exists $\varepsilon > 0$ such that, for each even integer k, we can find subsequences $\{y_{n_k}\}$ and $\{y_{m_k}\}$ of $\{y_n\}$ with $m_k > n_k > k$ such that

$$\begin{aligned}\lim_{k\to\infty} d(y_{m(k)}, y_{n(k)}) &= \lim_{k\to\infty} d(y_{m(k)}, y_{n(k)-1})\\ &= \lim_{k\to\infty} d(y_{m(k)+1}, y_{n(k)-1}) = \epsilon.\end{aligned}$$

Consider

$$\begin{aligned}\psi(d(y_{m(k)+1}, y_{n(k)})) &= \psi(d(Uy_{m(k)}, Uy_{n(k)-1}))\\ &\le F\big(\psi(\mathcal{M}(y_{m(k)}, y_{n(k)-1})), \varphi(\mathcal{M}(y_{m(k)}, y_{n(k)-1}))\big)\\ &\le \psi(\mathcal{M}(y_{m(k)}, y_{n(k)-1})),\end{aligned}$$

where

$$\begin{aligned}\mathcal{M}(y_{m(k)}, y_{n(k)-1}) = \max\{&d(y_{m(k)}, y_{n(k)-1}), d(y_{m(k)}, y_{m(k)+1}),\\ &d(y_{n(k)-1}, y_{n(k)}), \frac{1}{2}[d(y_{m(k)}, y_{n(k)})\\ &+ d(y_{m(k)+1}, y_{n(k)-1})]\}.\end{aligned}$$

Letting $k \to \infty$ and considering the continuity of F, ψ, φ, we have

$$\begin{aligned}\psi(\epsilon) &\le F\big(\psi(\epsilon), \varphi(\epsilon)\big)\\ &\le \psi(\epsilon).\end{aligned}$$

Thus, $\psi(\epsilon) = 0$ or $\varphi(\epsilon) = 0$. Therefore, $\epsilon = 0$. This is a contradiction. Thus $\{y_n\}$ is a Cauchy sequence in (X,d). Since X is complete, there exists y such that $y_n \to y$. We have

$$d(y, Uy) \le d(y, y_{n+1}) + d(y_{n+1}, Uy) = d(y, y_{n+1}) + d(Uy_n, Uy), \qquad (2.4)$$

and

$$\begin{aligned}\psi(d(Uy_n, Uy)) &\leq F(\psi(\mathcal{M}(y_n, y)), \varphi(\mathcal{M}(y_n, y)))\\ &\leq \psi(\mathcal{M}(y_n, y)).\end{aligned} \tag{2.5}$$

$$\begin{aligned}\mathcal{M}(y_n, y) &= \max\{d(y_n, y), d(y_n, Uy_n), d(y, Uy), \frac{1}{2}[d(y_n, Uy) + d(y, Uy_n)]\}\\ &= \max\{d(y_n, y), d(y_n, y_{n+1}), d(y, Uy), \frac{1}{2}[d(y_n, Uy) + d(y, y_{n+1})]\}.\end{aligned} \tag{2.6}$$

Substituting (2.6) and (2.5) into (2.4), taking the limit as $n \to \infty$, we get that

$$\begin{aligned}\psi(d(y, Uy)) &\leq F(\psi(d(y, Uy)), \varphi(d(y, Uy)))\\ &\leq \psi(d(y, Uy)).\end{aligned}$$

If $d(y, Uy) \neq 0$, then $\psi(d(y, Uy)) = 0$ or $\varphi(d(y, Uy)) = 0$. Therefore $d(y, Uy) = 0$, which is a contradiction. So, $d(y, Uy) = 0$, that is, $Uy = y$.

To prove the uniqueness of y, we suppose that there exists $y^* \in X$ such that $Uy^* = y^*$ and $y^* \neq y$. Since U is a generalized cyclic weak (F, ψ, φ)-contraction, we have

$$\begin{aligned}\psi(d(y, y^*)) &= \psi(d(Uy, Uy^*))\\ &\leq F\big(\psi(\mathcal{M}(y, y^*)), \varphi(\mathcal{M}(y, y^*))\big)\\ &\leq \psi(\mathcal{M}(y, y^*)),\end{aligned}$$

where

$$\mathcal{M}(y, y^*) = \max\{d(y, y^*), d(y, y), d(y^*, y^*), \frac{1}{2}[d(y, y^*) + d(y, y^*)]\} = d(y, y^*).$$

Thus,

$$\begin{aligned}\psi(d(y, y^*)) &\leq F(\psi(d(y, y^*)), \varphi(d(y, y^*)))\\ &\leq \psi(d(y, y^*)),\end{aligned}$$

which means that $\psi(d(y, y^*)) = 0$ or $\varphi(d(y, y^*)) = 0$. This implies $d(y, y^*) = 0$. Hence $y^* = y$. This completes the proof. □

Example 2.6. Let $X := \mathbb{R}$ with the metric $d(x, y) = |x - y|$. Suppose that $A = B = [0, 1]$, $\varphi(t) = \frac{3}{13}t$, $\psi(t) = t$, for all $t \geq 0$ and $F(s, t) = s - t$, $s, t \geq 0$. Define $U : A \bigcup B \to A \bigcup B$ by

$$U(x) = \begin{cases} \frac{1}{8}, & \text{if } x = 1, \\ \frac{1}{2}x + \frac{1}{8}, & \text{if } x \in [0, 1). \end{cases}$$

Clearly, $d(A, B) = 0$. If $x = 1, y = 1$, then we have

$$\psi(d(Ux, Uy)) = \psi(0) = 0$$

and

$$\begin{aligned}
&F\Big(\psi(\mathcal{M}(x,y)) - \psi(d(A,B)), \varphi(\mathcal{M}(x,y)) - \varphi(d(A,B))\Big)\\
&\quad + \psi(d(A,B)) - \psi(d(Ux,Uy))\\
&= \psi(\mathcal{M}(x,y)) - \varphi(\mathcal{M}(x,y)) - d(Ux,Uy)\\
&= \mathcal{M}(x,y) - \frac{3}{13}\mathcal{M}(x,y) \geq 0.
\end{aligned}$$

If $x = 1, y \in [0,1)$, then we have

$$\psi(d(Ux,Uy)) = \psi(d(\frac{1}{8}, \frac{1}{2}y + \frac{1}{8})) = \psi(\frac{1}{2}y) = \frac{1}{2}y.$$

And also, we have

$$\begin{aligned}
&d(x,y) = 1 - y,\ d(x,Ux) = \frac{7}{8},\ \ d(x,Uy) = \frac{7}{8} - \frac{1}{2}y,\ d(y,Ux) = \frac{1}{2}y,\\
&\qquad d(y,Uy) = |\frac{1}{2}y - \frac{1}{8}|
\end{aligned}$$

and

$$\frac{1}{2}[d(x,Uy) + d(y,Ux)] = \frac{7}{8}.$$

Therefore, we have $\mathcal{M}(x,y) = \max\{1 - y, \frac{7}{8}\}$. If $y \in [0, \frac{1}{8}]$, then $\mathcal{M}(x,y) = 1 - y$, we get

$$\begin{aligned}
&F\Big(\psi(\mathcal{M}(x,y)) - \psi(d(A,B)),\ \varphi(\mathcal{M}(x,y)) - \varphi(d(A,B))\Big)\\
&\quad + \psi(d(A,B)) - \psi(d(Ux,Uy))\\
&= \psi(\mathcal{M}(x,y)) - \varphi(\mathcal{M}(x,y)) - d(Ux,Uy)\\
&= \mathcal{M}(x,y) - \frac{3}{13}\mathcal{M}(x,y) - \frac{1}{2}y\\
&= (1 - y) - \frac{3}{13}(1 - y) - \frac{1}{2}y > 0,
\end{aligned}$$

for all $y \in [0, \frac{1}{8}]$. If $y \in [\frac{1}{8}, 1)$, then $\mathcal{M}(x;y) = \frac{7}{8}$, we have

$$\begin{aligned}
&F\Big(\psi(\mathcal{M}(x,y)) - \psi(d(A,B)),\ \varphi(\mathcal{M}(x,y)) - \varphi(d(A,B))\Big)\\
&\quad + \psi(d(A,B)) - \psi(d(Ux,Uy))\\
&= \psi(\mathcal{M}(x,y)) - \varphi(\mathcal{M}(x,y)) - d(Ux,Uy)\\
&= \frac{7}{8} - (\frac{3}{13})(\frac{7}{8}) - \frac{1}{2}y > 0,
\end{aligned}$$

for all $y \in [\frac{1}{8}, 1)$. Therefore, U obviously satisfies (2.1).

If $x, y \in [0, 1)$, we have

$$d(x,y) = |x-y|, d(x,Ux) = |\frac{1}{2}x - \frac{1}{8}|, d(x,Uy) = |x - \frac{1}{2}y - \frac{1}{8}|,$$
$$d(y,Ux) = |y - \frac{1}{2}x - \frac{1}{8}|, d(y,Uy) = |\frac{1}{2}y - \frac{1}{8}|,$$

and $d(Ux,Uy) = \frac{1}{2}|x-y|$. If $\mathcal{M}(x,y) = d(x,y)$, then U obviously satisfies (2.1). If $d(x,y) < \mathcal{M}(x,y) \neq d(x;y)$, we have

$$\begin{aligned}
&F\Big(\psi(\mathcal{M}(x,y)) - \psi(d(A,B)),\ \varphi(\mathcal{M}(x,y)) - \varphi(d(A,B))\Big) \\
&\quad + \psi(d(A,B)) - \psi(d(Ux,Uy)) \\
&= \psi(\mathcal{M}(x,y)) - \varphi(\mathcal{M}(x,y)) - d(Ux,Uy) \\
&= \mathcal{M}(x,y) - \frac{3}{13}\mathcal{M}(x,y) - \frac{1}{2}|x-y| > 0,
\end{aligned}$$

for all $x, y \in [0, 1)$. Thus, from all cases above, for $x, y \in [0, 1]$, we have

$$\begin{aligned}
\psi(d(Ux,Uy)) \leq F\Big(&\psi(\mathcal{M}(x,y)) - \psi(d(A,B)), \varphi(\mathcal{M}(x,y)) - \varphi(d(A,B))\Big) \\
&+ \psi(d(A,B)).
\end{aligned}$$

Therefore U is a generalized cyclic weak (F, ψ, φ)-contraction. All the conditions of Theorem 2.5 hold, and U has a fixed point. Here $x^* = \frac{1}{4}$ is the unique fixed point of U.

Remark 2.7. By taking $F(s,t) = s - t$ and $\psi(t) = t$ in Theorem 2.5 we obtain the result in [6].

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