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COMMON FIXED POINT THEOREMS FOR A PAIR OF MAPPINGS SATISFYING CONTRACTIVE INEQUALITIES OF INTEGRAL TYPE

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Abstract. Five results involving the existence and uniqueness of common fixed points for mappings satisfying contractive inequalities of integral type in complete metric spaces are proved. And two examples are given. The results presented in this paper generalize or differ from a few results in the literature.

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1. Introduction

In 2002, Branciari [3] extended the famous Banach's fixed point theorem and gained the following fixed point theorem for contractive mapping of integral type.

Theorem 1.1. ([3]) *Let f be a mapping from a complete metric space (X,d) into itself satisfying for all $x,y \in X$,*

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq c \int_0^{d(x,y)} \varphi(t)dt, \quad \forall x,y \in X,$$

where $c \in (0,1)$ is a constant and $\varphi \in \Phi_1$. Then f has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} f^n x = a$ for each $x \in X$.

Afterwards, the authors [1, 2, 4–14] continued the study of Branciari and got some fixed point theorems for contractive mappings of integral type. In 2011, Liu et al. [9] established the following fixed point theorems.

Theorem 1.2. ([9]) *Let f be a mapping from a complete metric space (X,d) into itself satisfying for all $x,y \in X$,*

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq \alpha(d(x,y)) \int_0^{d(x,y)} \varphi(t)dt, \quad \forall x,y \in X,$$

where $\varphi \in \Phi_1$ and $\alpha : \mathbb{R}^+ \to [0,1)$ is a function with

$$\limsup_{s\to t} \alpha(s) < 1, \quad \forall t > 0.$$

Then f has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} f^n x = a$ for each $x \in X$.

Theorem 1.3. ([9]) *Let f be a mapping from a complete metric space (X,d) into itself satisfying for all $x,y \in X$,*

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq \alpha(d(x,y)) \int_0^{d(x,fx)} \varphi(t)dt + \beta(d(x,y)) \int_0^{d(y,fy)} \varphi(t)dt,$$

where $\varphi \in \Phi_1$ and $\alpha, \beta : \mathbb{R}^+ \to [0,1)$ are two functions with

$$\alpha(t)+\beta(t) < 1, \quad \forall t \in \mathbb{R}^+, \quad \limsup_{s\to 0^+} \beta(s) < 1, \quad \limsup_{s\to t^+} \frac{\alpha(s)}{1-\beta(s)} < 1, \ \forall t > 0.$$

Then f has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} f^n x = a$ for each $x \in X$.

The aim of this paper is to introduce five classes of contractive mappings of integral type in complete metric spaces and to study the existence and uniqueness of common fixed points for these mappings. Our results extend Theorem 1.1 and differ from Theorems 1.2 and 1.3. Two examples are given.

2. Preliminaries

Throughout this paper, we assume that $\mathbb{R} = (-\infty, +\infty), \mathbb{R}^+ = [0, +\infty)$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers. Let (X, d) be a metric space. For $f, g : X \to X$, define

$$d_n = d(fx_n, fx_{n+1}), \quad \forall n \in \mathbb{N}_0 \text{ and } \{x_n\}_{n\in\mathbb{N}_0} \subseteq X,$$

$$\begin{aligned} m_1(x,y) = \max\Big\{ & d(gx,gy), d(fx,gx), d(fy,gy), \frac{1}{2}(d(fy,gx)+d(fx,gy)), \\ & \frac{d(fx,gx)d(fy,gy)}{1+d(fx,fy)}, \frac{d(fx,gy)d(fy,gx)}{1+d(fx,fy)}, \\ & \frac{d(fx,gx)d(fy,gy)}{1+d(gx,gy)}, \frac{d(fx,gy)d(fy,gx)}{1+d(gx,gy)}\Big\}, \quad \forall x,y \in X, \end{aligned}$$

$$\begin{aligned} m_2(x,y) = \max\Big\{ & d(gx,gy), d(fx,gx), d(fy,gy), \frac{1}{2}(d(fy,gx)+d(fx,gy)), \\ & \frac{d(fx,gy)d(fx,gx)}{2[1+d(fx,fy)]}, \frac{d(fy,gx)d(fy,gy)}{2[1+d(fx,fy)]}\Big\}, \quad \forall x,y \in X, \end{aligned}$$

$$\begin{aligned} m_3(x,y) = \max\Big\{ & d(gx,gy), d(fx,gx), d(fy,gy), \frac{1}{2}(d(fy,gx)+d(fx,gy)), \\ & \frac{d(fx,gx)d(fy,gx)}{2[1+d(gx,gy)]}, \frac{d(fy,gy)d(fx,gy)}{2[1+d(gx,gy)]}\Big\}, \quad \forall x,y \in X. \end{aligned}$$

Let $\Phi_1 = \big\{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies that φ is Lebesgue integrable, summable on each compact subset of $\mathbb{R}^+$ and $\int_0^\varepsilon \varphi(t)dt > 0$ for each $\varepsilon > 0\big\}$;
$\Phi_2 = \big\{\varphi : \varphi : \mathbb{R}^+ \to [0,1)$ satisfies that $\limsup_{s\to t} \varphi(s) < 1, \forall t > 0\big\}$;
$\Phi_3 = \big\{\varphi : \varphi : \mathbb{R}^+ \to [0,1)$ satisfies that $\limsup_{s\to t} \varphi(s) < 1, \forall t \in \mathbb{R}^+\big\}$.

It is clear that $\Phi_3 \subseteq \Phi_2$.

Definition 2.1. ([5]) A pair of self mappings f and g in a metric space (X, d) is said to be *weakly compatible* if for all $t \in X$ the equality $ft = gt$ implies that $fgt = gft$.

Lemma 2.2. ([9]) *Let $\varphi \in \Phi_1$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence with $\lim_{n\to\infty} r_n = a$. Then*

$$\lim_{n\to\infty} \int_0^{r_n} \varphi(t)dt = \int_0^a \varphi(t)dt.$$

Lemma 2.3. ([9]) *Let $\varphi \in \Phi_1$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence. Then*

$$\lim_{n\to\infty}\int_0^{r_n}\varphi(t)dt = 0 \Leftrightarrow \lim_{n\to\infty} r_n = 0.$$

3. Five Common Fixed Point Theorems

Our main results are as follows.

Theorem 3.1. *Let f and g be self mappings in a complete metric space (X, d) such that*

(F1) *f and g are weakly compatible;*
(F2) *$f(X) \subseteq g(X)$;*
(F3) *$g(X)$ is complete and*

$$\int_0^{d(fx,fy)}\varphi(t)dt \le \alpha(m_1(x,y))\int_0^{m_1(x,y)}\varphi(t)dt, \qquad \forall x, y \in X, \tag{3.1}$$

where $(\varphi, \alpha) \in \Phi_1 \times \Phi_2$.

Then f and g have a unique common fixed point in X.

Proof. First, we prove that f and g have at most one common fixed point in X. Suppose that f and g have two different common fixed points $a, b \in X$. It follows from (3.1) and $(\varphi, \alpha) \in \Phi_1 \times \Phi_2$ that

$$\begin{aligned} m_1(a,b) = \max\Big\{ & d(ga,gb), d(fa,ga), d(fb,gb), \frac{1}{2}(d(fb,ga)+d(fa,gb)), \\ & \frac{d(fa,ga)d(fb,gb)}{1+d(fa,fb)}, \frac{d(fa,gb)d(fb,ga)}{1+d(fa,fb)}, \\ & \frac{d(fa,ga)d(fb,gb)}{1+d(ga,gb)}, \frac{d(fa,gb)d(fb,ga)}{1+d(ga,gb)}\Big\} \\ = \max\Big\{ & d(a,b), 0, 0, d(a,b), 0, \frac{d^2(a,b)}{1+d(a,b)}, 0, \frac{d^2(a,b)}{1+d(a,b)}\Big\} \\ = d(a,b) & \end{aligned}$$

and

$$\begin{aligned} 0 < \int_0^{d(a,b)}\varphi(t)dt &= \int_0^{d(fa,fb)}\varphi(t)dt \\ &\le \alpha(m_1(a,b))\int_0^{m_1(a,b)}\varphi(t)dt \\ &= \alpha(d(a,b))\int_0^{d(a,b)}\varphi(t)dt < \int_0^{d(a,b)}\varphi(t)dt, \end{aligned}$$

which is a contradiction.

Next, we show that f and g have a common fixed point in X. Let x_0 be an arbitrary point in X. In light of $(F2)$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}_0}$ in X satisfying $fx_n = gx_{n+1}$ for each $n \in \mathbb{N}_0$.

Assume that $d_{n_0} = 0$ for some $n_0 \in \mathbb{N}_0$. It follows from $(F1)$ that

$$fx_{n_0} = fx_{n_0+1} = gx_{n_0+1} \tag{3.2}$$

and

$$f^2x_{n_0+1} = fgx_{n_0+1} = gfx_{n_0+1} = g^2x_{n_0+1}. \tag{3.3}$$

Suppose that $fx_{n_0+1} \neq f^2x_{n_0+1}$. In view of (3.1)-(3.3) and $(\varphi, \alpha) \in \Phi_1 \times \Phi_2$, we deduce that

$$\begin{aligned}
&m_1(fx_{n_0+1}, x_{n_0+1})\\
&\quad= \max\Big\{d(gfx_{n_0+1}, gx_{n_0+1}), d(f^2x_{n_0+1}, gfx_{n_0+1}), d(fx_{n_0+1}, gx_{n_0+1}),\\
&\qquad\quad \frac{1}{2}(d(fx_{n_0+1}, gfx_{n_0+1}) + d(f^2x_{n_0+1}, gx_{n_0+1})),\\
&\qquad\quad \frac{d(f^2x_{n_0+1}, gfx_{n_0+1})d(fx_{n_0+1}, gx_{n_0+1})}{1 + d(f^2x_{n_0+1}, fx_{n_0+1})},\\
&\qquad\quad \frac{d(f^2x_{n_0+1}, gx_{n_0+1})d(fx_{n_0+1}, gfx_{n_0+1})}{1 + d(f^2x_{n_0+1}, fx_{n_0+1})},\\
&\qquad\quad \frac{d(f^2x_{n_0+1}, gfx_{n_0+1})d(fx_{n_0+1}, gx_{n_0+1})}{1 + d(gfx_{n_0+1}, gx_{n_0+1})},\\
&\qquad\quad \frac{d(f^2x_{n_0+1}, gx_{n_0+1})d(fx_{n_0+1}, gfx_{n_0+1})}{1 + d(gfx_{n_0+1}, gx_{n_0+1})}\Big\}\\
&\quad= \max\Big\{d(f^2x_{n_0+1}, fx_{n_0+1}), 0, 0, d(f^2x_{n_0+1}, fx_{n_0+1}), 0,\\
&\qquad\quad \frac{d(f^2x_{n_0+1}, fx_{n_0+1})d(fx_{n_0+1}, f^2x_{n_0+1})}{1 + d(f^2x_{n_0+1}, fx_{n_0+1})}, 0,\\
&\qquad\quad \frac{d(f^2x_{n_0+1}, fx_{n_0+1})d(fx_{n_0+1}, f^2x_{n_0+1})}{1 + d(f^2x_{n_0+1}, fx_{n_0+1})}\Big\}\\
&\quad= d(f^2x_{n_0+1}, fx_{n_0+1})
\end{aligned}$$

and

$$\begin{aligned}0 &< \int_0^{d(f^2x_{n_0+1},fx_{n_0+1})} \varphi(t)dt \\ &\le \alpha(m_1(fx_{n_0+1}, x_{n_0+1})) \int_0^{m_1(fx_{n_0+1},x_{n_0+1})} \varphi(t)dt \\ &= \alpha(d(f^2x_{n_0+1}, fx_{n_0+1})) \int_0^{d(f^2x_{n_0+1},fx_{n_0+1})} \varphi(t)dt \\ &< \int_0^{d(f^2x_{n_0+1},fx_{n_0+1})} \varphi(t)dt,\end{aligned}$$

which is absurd. Therefore $fx_{n_0+1} = f^2x_{n_0+1}$, which together with (3.2) and (3.3) means that fx_{n_0+1} is a common fixed point of f and g in X.

Assume that $d_n \neq 0$ for all $n \in \mathbb{N}_0$. Observe that

$$\begin{aligned}\frac{1}{2}d(fx_{n+1}, fx_{n-1}) &\le \frac{1}{2}(d_{n-1} + d_n) \\ &\le \max\{d_{n-1}, d_n\}, \quad \forall n \in \mathbb{N}.\end{aligned} \tag{3.4}$$

Using (3.4), we infer that

$$\begin{aligned}&m_1(x_n, x_{n+1}) \\ &\quad = \max\Big\{d(gx_n, gx_{n+1}), d(fx_n, gx_n), d(fx_{n+1}, gx_{n+1}), \\ &\qquad \frac{1}{2}(d(fx_{n+1}, gx_n) + d(fx_n, gx_{n+1})), \frac{d(fx_n, gx_n)d(fx_{n+1}, gx_{n+1})}{1 + d(fx_n, fx_{n+1})}, \\ &\qquad \frac{d(fx_n, gx_{n+1})d(fx_{n+1}, gx_n)}{1 + d(fx_n, fx_{n+1})}, \frac{d(fx_n, gx_n)d(fx_{n+1}, gx_{n+1})}{1 + d(gx_n, gx_{n+1})}, \\ &\qquad \frac{d(fx_n, gx_{n+1})d(fx_{n+1}, gx_n)}{1 + d(gx_n, gx_{n+1})}\Big\} \\ &\quad = \max\Big\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n-1}), d(fx_{n+1}, fx_n), \\ &\qquad \frac{1}{2}(d(fx_{n+1}, fx_{n-1}) + d(fx_n, fx_n)), \frac{d(fx_n, fx_{n-1})d(fx_{n+1}, fx_n)}{1 + d(fx_n, fx_{n+1})}, \\ &\qquad \frac{d(fx_n, fx_n)d(fx_{n+1}, fx_{n-1})}{1 + d(fx_n, fx_{n+1})}, \frac{d(fx_n, fx_{n-1})d(fx_{n+1}, fx_n)}{1 + d(fx_{n-1}, fx_n)}, \\ &\qquad \frac{d(fx_n, fx_n)d(fx_{n+1}, fx_{n-1})}{1 + d(fx_{n-1}, fx_n)}\Big\}\end{aligned}$$

$$
\begin{aligned}
&= \max\left\{d_{n-1}, d_{n-1}, d_n, \frac{1}{2}d(fx_{n+1}, fx_{n-1}), \frac{d_{n-1}d_n}{1+d_n}, 0, \frac{d_{n-1}d_n}{1+d_{n-1}}, 0\right\} \\
&= \max\{d_{n-1}, d_n\}, \quad \forall n \in \mathbb{N}.
\end{aligned} \tag{3.5}
$$

If $d_n > d_{n-1}$ for some $n \in \mathbb{N}$, it follows from (3.1), (3.4), (3.5) and $(\varphi, \alpha) \in \Phi_1 \times \Phi_2$ that

$$
\begin{aligned}
0 &< \int_0^{d_n} \varphi(t)dt = \int_0^{d(fx_n, fx_{n+1})} \varphi(t)dt \\
&\le \alpha(m_1(x_n, x_{n+1})) \int_0^{m_1(x_n, x_{n+1})} \varphi(t)dt \\
&= \alpha(d_n) \int_0^{d_n} \varphi(t)dt \le \alpha(d_n) \int_0^{d_n} \varphi(t)dt < \int_0^{d_n} \varphi(t)dt,
\end{aligned}
$$

which is a contradiction. Hence $d_n \le d_{n-1}$ for each $n \in \mathbb{N}$. Consequently, the sequence $\{d_n\}_{n\in\mathbb{N}_0}$ is nonincreasing and bounded, which means that there exists a constant c with

$$
\lim_{n\to\infty} d_n = c \ge 0. \tag{3.6}
$$

Suppose that $c > 0$. Making use of (3.1), (3.4)-(3.6), Lemma 2.2 and $(\varphi, \alpha) \in \Phi_1 \times \Phi_2$, we conclude immediately that

$$
\begin{aligned}
0 &< \int_0^{c} \varphi(t)dt \\
&= \limsup_{n\to\infty} \int_0^{d_n} \varphi(t)dt \\
&= \limsup_{n\to\infty} \int_0^{d(fx_n, fx_{n+1})} \varphi(t)dt \\
&\le \limsup_{n\to\infty} \left(\alpha(m_1(x_n, x_{n+1})) \int_0^{m_1(x_n, x_{n+1})} \varphi(t)dt\right) \\
&\le \limsup_{n\to\infty} \alpha(d_{n-1}) \cdot \limsup_{n\to\infty} \int_0^{d_{n-1}} \varphi(t)dt \\
&\le \limsup_{s\to c} \alpha(s) \cdot \int_0^{c} \varphi(t)dt \\
&< \int_0^{c} \varphi(t)dt,
\end{aligned}
$$

which is absurd. Therefore, $c = 0$, that is,

$$
\lim_{n\to\infty} d_n = 0. \tag{3.7}
$$

Now, we claim that $\{fx_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. Suppose that $\{fx_n\}_{n\in\mathbb{N}_0}$ is not a Cauchy sequence. It follows that there exist a constant $\varepsilon > 0$ and two sequences $\{m(k)\}_{k\in\mathbb{N}}$ and $\{n(k)\}_{k\in\mathbb{N}}$ in $\mathbb{N}_0$ with $m(k) < n(k) < m(k+1)$ and

$$d(fx_{m(k)}, fx_{n(k)}) \geq \varepsilon \quad and \quad d(fx_{m(k)}, fx_{n(k)-1}) < \varepsilon, \quad \forall k \in \mathbb{N}. \tag{3.8}$$

Note that

$$\begin{aligned}
&d(fx_{m(k)}, fx_{n(k)}) \leq d(fx_{m(k)}, fx_{n(k)-1}) + d_{n(k)-1}, \quad \forall k \in \mathbb{N};\\
&\left|d(fx_{m(k)}, fx_{n(k)}) - d(fx_{m(k)}, fx_{n(k)-1})\right| \leq d_{n(k)-1}, \quad \forall k \in \mathbb{N};\\
&\left|d(fx_{m(k)}, fx_{n(k)}) - d(fx_{m(k)-1}, fx_{n(k)})\right| \leq d_{m(k)-1}, \quad \forall k \in \mathbb{N};\\
&\left|d(fx_{m(k)-1}, fx_{n(k)-1}) - d(fx_{m(k)-1}, fx_{n(k)})\right| \leq d_{n(k)-1}, \quad \forall k \in \mathbb{N}.
\end{aligned} \tag{3.9}$$

On account of (3.7)-(3.9), we get that

$$\begin{aligned}
\varepsilon &= \lim_{k\to\infty} d(fx_{m(k)}, fx_{n(k)})\\
&= \lim_{k\to\infty} d(fx_{m(k)}, fx_{n(k)-1})\\
&= \lim_{k\to\infty} d(fx_{m(k)-1}, fx_{n(k)})\\
&= \lim_{k\to\infty} d(fx_{m(k)-1}, fx_{n(k)-1}).
\end{aligned} \tag{3.10}$$

Combining (3.1), (3.7), (3.10), Lemma 2.2 and $(\varphi, \alpha) \in \Phi_1 \times \Phi_2$, we know that

$$\begin{aligned}
&\lim_{k\to\infty} m_1(x_{m(k)}, x_{n(k)})\\
&\quad = \lim_{k\to\infty} \max\Bigg\{ d(gx_{m(k)}, gx_{n(k)}), d(fx_{m(k)}, gx_{m(k)}), d(fx_{n(k)}, gx_{n(k)}),\\
&\qquad \frac{1}{2}(d(fx_{n(k)}, gx_{m(k)}) + d(fx_{m(k)}, gx_{n(k)})),\\
&\qquad \frac{d(fx_{m(k)}, gx_{m(k)})d(fx_{n(k)}, gx_{n(k)})}{1 + d(fx_{m(k)}, fx_{n(k)})},\\
&\qquad \frac{d(fx_{m(k)}, gx_{n(k)})d(fx_{n(k)}, gx_{m(k)})}{1 + d(fx_{m(k)}, fx_{n(k)})},\\
&\qquad \frac{d(fx_{m(k)}, gx_{m(k)})d(fx_{n(k)}, gx_{n(k)})}{1 + d(gx_{m(k)}, gx_{n(k)})},\\
&\qquad \frac{d(fx_{m(k)}, gx_{n(k)})d(fx_{n(k)}, gx_{m(k)})}{1 + d(gx_{m(k)}, gx_{n(k)})} \Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{k\to\infty} \max \Big\{ d(fx_{m(k)-1}, fx_{n(k)-1}), d(fx_{m(k)}, fx_{m(k)-1}), d(fx_{n(k)}, fx_{n(k)-1}), \\
&\qquad \frac{1}{2}(d(fx_{n(k)}, fx_{m(k)-1}) + d(fx_{m(k)}, fx_{n(k)-1})), \\
&\qquad \frac{d(fx_{m(k)}, fx_{m(k)-1}) d(fx_{n(k)}, fx_{n(k)-1})}{1 + d(fx_{m(k)}, fx_{n(k)})}, \\
&\qquad \frac{d(fx_{m(k)}, fx_{n(k)-1}) d(fx_{n(k)}, fx_{m(k)-1})}{1 + d(fx_{m(k)}, fx_{n(k)})}, \\
&\qquad \frac{d(fx_{m(k)}, fx_{m(k)-1}) d(fx_{n(k)}, fx_{n(k)-1})}{1 + d(fx_{m(k)-1}, fx_{n(k)-1})}, \\
&\qquad \frac{d(fx_{m(k)}, fx_{n(k)-1}) d(fx_{n(k)}, fx_{m(k)-1})}{1 + d(fx_{m(k)-1}, fx_{n(k)-1})} \Big\} \\
&= \max \Big\{ \varepsilon, 0, 0, \varepsilon, 0, \frac{\varepsilon^2}{1+\varepsilon}, 0, \frac{\varepsilon^2}{1+\varepsilon} \Big\} \\
&= \varepsilon
\end{aligned}
$$

and

$$
\begin{aligned}
0 &< \int_0^{\varepsilon} \varphi(t)dt \\
&= \limsup_{k\to\infty} \int_0^{d(fx_{m(k)}, fx_{n(k)})} \varphi(t)dt \\
&\leq \limsup_{k\to\infty} \left(\alpha(m_1(x_{m(k)}, x_{n(k)})) \int_0^{m_1(x_{m(k)}, x_{n(k)})} \varphi(t)dt \right) \\
&\leq \limsup_{k\to\infty} \alpha(m_1(x_{m(k)}, x_{n(k)})) \cdot \limsup_{k\to\infty} \int_0^{m_1(x_{m(k)}, x_{n(k)})} \varphi(t)dt \\
&\leq \limsup_{s\to\varepsilon} \alpha(s) \cdot \int_0^{\varepsilon} \varphi(t)dt \\
&< \int_0^{\varepsilon} \varphi(t)dt,
\end{aligned}
$$

which is a contradiction. Thus $\{fx_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. It follows from $(F3)$ that there exists $(a, b) \in g(X) \times X$ with

$$
\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = a = gb. \tag{3.11}
$$

Suppose that $fb \neq a$. In view of (3.1), (3.11), Lemma 2.2 and $(\varphi, \alpha) \in \Phi_1 \times \Phi_2$, we deduce that

$$\begin{aligned}
&\lim_{n\to\infty} m_1(b, x_n) \\
&\quad = \lim_{n\to\infty} \max\Big\{ d(gb, gx_n), d(fb, gb), d(fx_n, gx_n), \\
&\qquad\qquad \frac{1}{2}(d(fx_n, gb) + d(fb, gx_n)), \\
&\qquad\qquad \frac{d(fb, gb)d(fx_n, gx_n)}{1 + d(fb, fx_n)}, \frac{d(fb, gx_n)d(fx_n, gb)}{1 + d(fb, fx_n)}, \\
&\qquad\qquad \frac{d(fb, gb)d(fx_n, gx_n)}{1 + d(gb, gx_n)}, \frac{d(fb, gx_n)d(fx_n, gb)}{1 + d(gb, gx_n)} \Big\} \\
&\quad = \max\Big\{ d(a, a), d(fb, a), d(a, a), \frac{1}{2}(d(a, a) + d(fb, a)), \frac{d(fb, a)d(a, a)}{1 + d(fb, a)}, \\
&\qquad\qquad \frac{d(fb, a)d(a, a)}{1 + d(fb, a)}, \frac{d(fb, a)d(a, a)}{1 + d(a, a)}, \frac{d(fb, a)d(a, a)}{1 + d(a, a)} \Big\} \\
&\quad = \max\Big\{ 0, d(fb, a), 0, \frac{1}{2}d(fb, a), 0, 0, 0, 0 \Big\} \\
&\quad = d(fb, a)
\end{aligned}$$

and

$$\begin{aligned}
0 &< \int_0^{d(fb,a)} \varphi(t)dt = \limsup_{n\to\infty} \int_0^{d(fb, fx_n)} \varphi(t)dt \\
&\le \limsup_{n\to\infty} \left(\alpha(m_1(b, x_n)) \int_0^{m_1(b,x_n)} \varphi(t)dt \right) \\
&\le \limsup_{n\to\infty} \alpha(m_1(b, x_n)) \cdot \limsup_{n\to\infty} \int_0^{m_1(b,x_n)} \varphi(t)dt \\
&\le \limsup_{s\to d(fb,a)} \alpha(s) \cdot \int_0^{d(fb,a)} \varphi(t)dt \\
&< \int_0^{d(fb,a)} \varphi(t)dt,
\end{aligned}$$

which is absurd. That is, $a = fb = gb$. It follows from $(F1)$ that

$$fa = f^2b = fgb = gfb = g^2b = ga. \tag{3.12}$$

Suppose that $a \neq fa$. On account of (3.1), (3.12) and $(\varphi, \alpha) \in \Phi_1 \times \Phi_2$, we get that

$$\begin{aligned}
&m_1(b, gb) \\
&= \max\Big\{d(gb, g^2b), d(fb, gb), d(fgb, g^2b), \frac{1}{2}(d(fgb, gb) + d(fb, g^2b)), \\
&\qquad \frac{d(fb, gb)d(fgb, g^2b)}{1 + d(fb, fgb)}, \frac{d(fb, g^2b)d(fgb, gb)}{1 + d(fb, fgb)}, \\
&\qquad \frac{d(fb, gb)d(fgb, g^2b)}{1 + d(gb, g^2b)}, \frac{d(fb, g^2b)d(fgb, gb)}{1 + d(gb, g^2b)}\Big\} \\
&= \max\Big\{d(a, fa), 0, 0, \frac{1}{2}(d(fa, a) + d(a, fa)), 0, \frac{d^2(a, fa)}{1 + d(a, fa)}, 0, \frac{d^2(a, fa)}{1 + d(a, fa)}\Big\} \\
&= d(a, fa)
\end{aligned}$$

and

$$\begin{aligned}
0 &< \int_0^{d(a,fa)} \varphi(t)dt = \int_0^{d(fb,fgb)} \varphi(t)dt \\
&\leq \alpha(m_1(b, gb)) \int_0^{m_1(b,gb)} \varphi(t)dt \\
&= \alpha(d(a, fa)) \int_0^{d(a,fa)} \varphi(t)dt < \int_0^{d(a,fa)} \varphi(t)dt,
\end{aligned}$$

which is impossible. Therefore $a = fa$. It follows from (3.12) that f and g have a common fixed point $a \in X$. This completes the proof. □

Similar to the proof of Theorem 3.1, we have the following results and omit their proofs.

Theorem 3.2. *Let f and g be self mappings in a complete metric space (X, d) satisfying $(F1)$-$(F3)$ and*

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq \alpha(d(fx, fy)) \int_0^{m_1(x,y)} \varphi(t)dt, \qquad \forall x, y \in X, \tag{3.13}$$

where $(\varphi, \alpha) \in \Phi_1 \times \Phi_2$. Then f and g have a unique common fixed point in X.

Theorem 3.3. *Let f and g be self mappings in a complete metric space (X, d) satisfying $(F1)$-$(F3)$ and*

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq \alpha(d(gx, gy)) \int_0^{m_1(x,y)} \varphi(t)dt, \qquad \forall x, y \in X, \tag{3.14}$$

where $(\varphi, \alpha) \in \Phi_1 \times \Phi_3$. *Then* f *and* g *have a unique common fixed point in* X.

Theorem 3.4. *Let* f *and* g *be self mappings in a complete metric space* (X, d) *satisfying* $(F1)$-$(F3)$ *and*

$$\begin{aligned}\int_0^{d(fx,fy)} \varphi(t)dt \le \alpha(d(fx,fy)) \int_0^{m_2(x,y)} \varphi(t)dt \\ + \beta(d(fx,fy)) \int_0^{m_3(x,y)} \varphi(t)dt, \quad \forall x, y \in X,\end{aligned} \tag{3.15}$$

where $\varphi \in \Phi_1$ *and* $\alpha, \beta : \mathbb{R}^+ \to [0, 1)$ *are two functions with*

$$\alpha(t) + \beta(t) < 1, \quad \limsup_{s \to t} (\alpha(s) + \beta(s)) < 1, \quad \forall t > 0. \tag{3.16}$$

Then f *and* g *have a unique common fixed point in* X.

Proof. First, we prove that f and g have at most one common fixed point in X. Suppose that f and g have two different common fixed points $a, b \in X$. It follows from (3.15), (3.16) and $\varphi \in \Phi_1$ that

$$\begin{aligned} m_2(a,b) &= \max \Big\{ d(ga,gb), d(fa,ga), d(fb,gb), \\ &\qquad \frac{1}{2}(d(fb,ga) + d(fa,gb)), \\ &\qquad \frac{d(fa,gb)d(fa,ga)}{2[1 + d(fa,fb)]}, \frac{d(fb,ga)d(fb,gb)}{2[1 + d(fa,fb)]} \Big\} \\ &= \max \{ d(a,b), 0, 0, d(a,b), 0, 0 \} \\ &= d(a,b), \end{aligned} \tag{3.17}$$

$$\begin{aligned} m_3(a,b) &= \max \Big\{ d(ga,gb), d(fa,ga), d(fb,gb), \\ &\qquad \frac{1}{2}(d(fb,ga) + d(fa,gb)), \\ &\qquad \frac{d(fa,ga)d(fb,ga)}{2[1 + d(ga,gb)]}, \frac{d(fb,gb)d(fa,gb)}{2[1 + d(ga,gb)]} \Big\} \\ &= \max \{ d(a,b), 0, 0, d(a,b), 0, 0 \} \\ &= d(a,b) \end{aligned} \tag{3.18}$$

and

$$
\begin{aligned}
0 &< \int_0^{d(a,b)} \varphi(t)dt \\
&= \int_0^{d(fa,fb)} \varphi(t)dt \\
&\le \alpha(d(fa,fb)) \int_0^{m_2(a,b)} \varphi(t)dt + \beta(d(fa,fb)) \int_0^{m_3(a,b)} \varphi(t)dt \\
&= \alpha(d(a,b)) \int_0^{d(a,b)} \varphi(t)dt + \beta(d(a,b)) \int_0^{d(a,b)} \varphi(t)dt \\
&= (\alpha(d(a,b)) + \beta(d(a,b))) \int_0^{d(a,b)} \varphi(t)dt \\
&< \int_0^{d(a,b)} \varphi(t)dt,
\end{aligned}
$$

which is a contradiction.

Next, we show that f and g have a common fixed point in X. Let x_0 be an arbitrary point in X. By means of $(F2)$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}_0}$ in X satisfying $fx_n = gx_{n+1}$ for each $n \in \mathbb{N}_0$.

Assume that $d_{n_0} = 0$ for some $n_0 \in \mathbb{N}_0$. It follows from $(F1)$ that (3.2) and (3.3) hold.

Suppose that $fx_{n_0+1} \ne f^2x_{n_0+1}$. In light of (3.2), (3.3), (3.15), (3.16) and $\varphi \in \Phi_1$, we infer that

$$
\begin{aligned}
&m_2(fx_{n_0+1}, x_{n_0+1}) \\
&\quad= \max\Big\{d(gfx_{n_0+1}, gx_{n_0+1}), d(f^2x_{n_0+1}, gfx_{n_0+1}), \\
&\qquad\quad d(fx_{n_0+1}, gx_{n_0+1}), \\
&\qquad\quad \frac{1}{2}(d(fx_{n_0+1}, gfx_{n_0+1}) + d(f^2x_{n_0+1}, gx_{n_0+1})), \\
&\qquad\quad \frac{d(f^2x_{n_0+1}, gx_{n_0+1})d(f^2x_{n_0+1}, gfx_{n_0+1})}{2[1+d(f^2x_{n_0+1}, fx_{n_0+1})]}, \\
&\qquad\quad \frac{d(fx_{n_0+1}, gfx_{n_0+1})d(fx_{n_0+1}, gx_{n_0+1})}{2[1+d(f^2x_{n_0+1}, fx_{n_0+1})]}\Big\} \\
&\quad= \max\{d(f^2x_{n_0+1}, fx_{n_0+1}), 0, 0, d(f^2x_{n_0+1}, fx_{n_0+1}), 0, 0\} \\
&\quad= d(f^2x_{n_0+1}, fx_{n_0+1}),
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
& m_3(fx_{n_0+1}, x_{n_0+1}) \\
&\quad = \max \Big\{ d(gfx_{n_0+1}, gx_{n_0+1}), d(f^2x_{n_0+1}, gfx_{n_0+1}), \\
&\qquad\quad d(fx_{n_0+1}, gx_{n_0+1}), \\
&\qquad\quad \frac{1}{2}(d(fx_{n_0+1}, gfx_{n_0+1}) + d(f^2x_{n_0+1}, gx_{n_0+1})), \\
&\qquad\quad \frac{d(f^2x_{n_0+1}, gfx_{n_0+1})d(fx_{n_0+1}, gfx_{n_0+1})}{2[1 + d(gfx_{n_0+1}, gx_{n_0+1})]}, \\
&\qquad\quad \frac{d(fx_{n_0+1}, gx_{n_0+1})d(f^2x_{n_0+1}, gx_{n_0+1})}{2[1 + d(gfx_{n_0+1}, gx_{n_0+1})]} \Big\} \\
&\quad = \max \{ d(f^2x_{n_0+1}, fx_{n_0+1}), 0, 0, d(f^2x_{n_0+1}, fx_{n_0+1}), 0, 0 \} \\
&\quad = d(f^2x_{n_0+1}, fx_{n_0+1})
\end{aligned}
\tag{3.20}
$$

and

$$
\begin{aligned}
0 &< \int_0^{d(f^2x_{n_0+1}, fx_{n_0+1})} \varphi(t)dt \\
&\le \alpha(d(f^2x_{n_0+1}, fx_{n_0+1})) \int_0^{m_2(fx_{n_0+1}, x_{n_0+1})} \varphi(t)dt \\
&\quad + \beta(d(f^2x_{n_0+1}, fx_{n_0+1})) \int_0^{m_3(fx_{n_0+1}, x_{n_0+1})} \varphi(t)dt \\
&= \alpha(d(f^2x_{n_0+1}, fx_{n_0+1})) \int_0^{d(f^2x_{n_0+1}, fx_{n_0+1})} \varphi(t)dt \\
&\quad + \beta(d(f^2x_{n_0+1}, fx_{n_0+1})) \int_0^{d(f^2x_{n_0+1}, fx_{n_0+1})} \varphi(t)dt \\
&= (\alpha(d(f^2x_{n_0+1}, fx_{n_0+1})) + \beta(d(f^2x_{n_0+1}, fx_{n_0+1}))) \\
&\qquad \times \int_0^{d(f^2x_{n_0+1}, fx_{n_0+1})} \varphi(t)dt \\
&< \int_0^{d(f^2x_{n_0+1}, fx_{n_0+1})} \varphi(t)dt,
\end{aligned}
$$

which is absurd. Therefore $fx_{n_0+1} = f^2x_{n_0+1}$, which together with (3.2) and (3.3) means that fx_{n_0+1} is a common fixed point of f and g in X.

Assume that $d_n \neq 0$ for all $n \in \mathbb{N}_0$. Using (3.4), we infer that

$$\begin{aligned}
&m_2(x_n, x_{n+1})\\
&\quad = \max\Big\{d(gx_n, gx_{n+1}), d(fx_n, gx_n), d(fx_{n+1}, gx_{n+1}),\\
&\qquad \frac{1}{2}(d(fx_{n+1}, gx_n) + d(fx_n, gx_{n+1})), \frac{d(fx_n, gx_{n+1})d(fx_n, gx_n)}{2[1 + d(fx_n, fx_{n+1})]},\\
&\qquad \frac{d(fx_{n+1}, gx_n)d(fx_{n+1}, gx_{n+1})}{2[1 + d(fx_n, fx_{n+1})]}\Big\}\\
&\quad = \max\Big\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n-1}), d(fx_{n+1}, fx_n),\\
&\qquad \frac{1}{2}(d(fx_{n+1}, fx_{n-1}) + d(fx_n, fx_n)), \frac{d(fx_n, fx_n)d(fx_n, fx_{n-1})}{2[1 + d(fx_n, fx_{n+1})]},\\
&\qquad \frac{d(fx_{n+1}, fx_{n-1})d(fx_{n+1}, fx_n)}{2[1 + d(fx_n, fx_{n+1})]}\Big\}\\
&\quad = \max\Big\{d_{n-1}, d_{n-1}, d_n, \frac{1}{2}d(fx_{n+1}, fx_{n-1}), 0, \frac{d(fx_{n+1}, fx_{n-1})d_n}{2(1 + d_n)}\Big\}\\
&\quad = \max\{d_{n-1}, d_n\}, \quad \forall n \in \mathbb{N},
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
&m_3(x_n, x_{n+1})\\
&\quad = \max\Big\{d(gx_n, gx_{n+1}), d(fx_n, gx_n), d(fx_{n+1}, gx_{n+1}),\\
&\qquad \frac{1}{2}(d(fx_{n+1}, gx_n) + d(fx_n, gx_{n+1})), \frac{d(fx_n, gx_n)d(fx_{n+1}, gx_n)}{2[1 + d(gx_n, gx_{n+1})]},\\
&\qquad \frac{d(fx_{n+1}, gx_{n+1})d(fx_n, gx_{n+1})}{2[1 + d(gx_n, gx_{n+1})]}\Big\}\\
&\quad = \max\Big\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n-1}), d(fx_{n+1}, fx_n),\\
&\qquad \frac{1}{2}(d(fx_{n+1}, fx_{n-1}) + d(fx_n, fx_n)), \frac{d(fx_n, fx_{n-1})d(fx_{n+1}, fx_{n-1})}{2[1 + d(fx_{n-1}, fx_n)]},\\
&\qquad \frac{d(fx_{n+1}, fx_n)d(fx_n, fx_n)}{2[1 + d(fx_{n-1}, fx_n)]}\Big\}\\
&\quad = \max\Big\{d_{n-1}, d_{n-1}, d_n, \frac{1}{2}d(fx_{n+1}, fx_{n-1}), \frac{d_{n-1}d(fx_{n+1}, fx_{n-1})}{2(1 + d_{n-1})}, 0\Big\}\\
&\quad = \max\{d_{n-1}, d_n\}, \quad \forall n \in \mathbb{N}.
\end{aligned} \tag{3.22}$$

If $d_n > d_{n-1}$ for some $n \in \mathbb{N}$, it follows from (3.15), (3.16), (3.21), (3.22) that

$$\begin{aligned} 0 &< \int_0^{d_n} \varphi(t)dt = \int_0^{d(fx_n, fx_{n+1})} \varphi(t)dt \\ &\le \alpha(d(fx_n, fx_{n+1})) \int_0^{m_2(x_n, x_{n+1})} \varphi(t)dt \\ &\quad + \beta(d(fx_n, fx_{n+1})) \int_0^{m_3(x_n, x_{n+1})} \varphi(t)dt \\ &= \alpha(d_n) \int_0^{d_n} \varphi(t)dt + \beta(d_n) \int_0^{d_n} \varphi(t)dt \\ &< \int_0^{d_n} \varphi(t)dt, \end{aligned}$$

which is impossible. Hence the sequence $\{d_n\}_{n\in\mathbb{N}_0}$ satisfies (3.6).

Suppose that $c > 0$. Using (3.6), (3.15), (3.16), (3.21), (3.22), Lemma 2.2 and $\varphi \in \Phi_1$, we conclude that

$$\begin{aligned} 0 &< \int_0^{c} \varphi(t)dt = \limsup_{n\to\infty} \int_0^{d_n} \varphi(t)dt \\ &= \limsup_{n\to\infty} \int_0^{d(fx_n, fx_{n+1})} \varphi(t)dt \\ &\le \limsup_{n\to\infty} \Bigg(\alpha(d(fx_n, fx_{n+1})) \int_0^{m_2(x_n, x_{n+1})} \varphi(t)dt \\ &\quad + \beta(d(fx_n, fx_{n+1})) \int_0^{m_3(x_n, x_{n+1})} \varphi(t)dt \Bigg) \\ &= \limsup_{n\to\infty} \left(\alpha(d_n) \int_0^{d_{n-1}} \varphi(t)dt + \beta(d_n) \int_0^{d_{n-1}} \varphi(t)dt \right) \\ &\le \limsup_{n\to\infty} (\alpha(d_n) + \beta(d_n)) \cdot \limsup_{n\to\infty} \int_0^{d_{n-1}} \varphi(t)dt \\ &\le \limsup_{s\to c^+} (\alpha(s) + \beta(s)) \cdot \int_0^{c} \varphi(t)dt \\ &< \int_0^{c} \varphi(t)dt, \end{aligned}$$

which is absurd. Therefore, $c = 0$, that is, (3.7) holds.

Now, we claim that $\{fx_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. Suppose that $\{fx_n\}_{n\in\mathbb{N}_0}$ is not a Cauchy sequence. It follows that there exist a constant $\varepsilon > 0$ and two sequences $\{m(k)\}_{k\in\mathbb{N}}$ and $\{n(k)\}_{k\in\mathbb{N}}$ in $\mathbb{N}_0$ with (3.8)-(3.10). It follows from

(3.10), (3.15), (3.16), Lemma 2.2 and $\varphi \in \Phi_1$ that

$$
\begin{aligned}
&\lim_{k\to\infty} m_2(x_{m(k)}, x_{n(k)}) \\
&\quad = \lim_{k\to\infty} \max\Big\{ d(gx_{m(k)}, gx_{n(k)}), d(fx_{m(k)}, gx_{m(k)}), \\
&\qquad d(fx_{n(k)}, gx_{n(k)}), \\
&\qquad \frac{1}{2}(d(fx_{n(k)}, gx_{m(k)}) + d(fx_{m(k)}, gx_{n(k)})), \\
&\qquad \frac{d(fx_{m(k)}, gx_{n(k)}) d(fx_{m(k)}, gx_{m(k)})}{2[1 + d(fx_{m(k)}, fx_{n(k)})]}, \\
&\qquad \frac{d(fx_{n(k)}, gx_{m(k)}) d(fx_{n(k)}, gx_{n(k)})}{2[1 + d(fx_{m(k)}, fx_{n(k)})]} \Big\} \\
&\quad = \lim_{k\to\infty} \max\Big\{ d(fx_{m(k)-1}, fx_{n(k)-1}), d(fx_{m(k)}, fx_{m(k)-1}), \\
&\qquad d(fx_{n(k)}, fx_{n(k)-1}), \\
&\qquad \frac{1}{2}(d(fx_{n(k)}, fx_{m(k)-1}) + d(fx_{m(k)}, fx_{n(k)-1})), \\
&\qquad \frac{d(fx_{m(k)}, fx_{n(k)-1}) d(fx_{m(k)}, fx_{m(k)-1})}{2[1 + d(fx_{m(k)}, fx_{n(k)})]}, \\
&\qquad \frac{d(fx_{n(k)}, fx_{m(k)-1}) d(fx_{n(k)}, fx_{n(k)-1})}{2[1 + d(fx_{m(k)}, fx_{n(k)})]} \Big\} \\
&\quad = \max\{\varepsilon, 0, 0, \varepsilon, 0, 0\} \\
&\quad = \varepsilon,
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
&\lim_{k\to\infty} m_3(x_{m(k)}, x_{n(k)}) \\
&\quad = \lim_{k\to\infty} \max\Big\{ d(gx_{m(k)}, gx_{n(k)}), d(fx_{m(k)}, gx_{m(k)}), \\
&\qquad d(fx_{n(k)}, gx_{n(k)}), \\
&\qquad \frac{1}{2}(d(fx_{n(k)}, gx_{m(k)}) + d(fx_{m(k)}, gx_{n(k)})), \\
&\qquad \frac{d(fx_{m(k)}, gx_{m(k)}) d(fx_{n(k)}, gx_{m(k)})}{2[1 + d(gx_{m(k)}, gx_{n(k)})]}, \\
&\qquad \frac{d(fx_{n(k)}, gx_{n(k)}) d(fx_{m(k)}, gx_{n(k)})}{2[1 + d(gx_{m(k)}, gx_{n(k)})]} \Big\}
\end{aligned}
$$

$$\begin{aligned}
&= \lim_{k\to\infty} \max\Big\{ d(fx_{m(k)-1}, fx_{n(k)-1}), d(fx_{m(k)}, fx_{m(k)-1}), \\
&\qquad d(fx_{n(k)}, fx_{n(k)-1}), \\
&\qquad \frac{1}{2}(d(fx_{n(k)}, fx_{m(k)-1}) + d(fx_{m(k)}, fx_{n(k)-1})), \\
&\qquad \frac{d(fx_{m(k)}, fx_{m(k)-1}) d(fx_{n(k)}, fx_{m(k)-1})}{2[1 + d(fx_{m(k)-1}, fx_{n(k)-1})]}, \\
&\qquad \frac{d(fx_{n(k)}, fx_{n(k)-1}) d(fx_{m(k)}, fx_{n(k)-1})}{2[1 + d(fx_{m(k)-1}, fx_{n(k)-1})]} \Big\} \\
&= \max\{\varepsilon, 0, 0, \varepsilon, 0, 0\} \\
&= \varepsilon
\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}
0 &< \int_0^{\varepsilon} \varphi(t)dt = \limsup_{k\to\infty} \int_0^{d(fx_{m(k)}, fx_{n(k)})} \varphi(t)dt \\
&\le \limsup_{k\to\infty} \Big(\alpha(d(fx_{m(k)}, fx_{n(k)})) \int_0^{m_2(x_{m(k)}, x_{n(k)})} \varphi(t)dt \\
&\qquad + \beta(d(fx_{m(k)}, fx_{n(k)})) \int_0^{m_3(x_{m(k)}, x_{n(k)})} \varphi(t)dt \Big) \\
&\le \limsup_{k\to\infty} (\alpha(d(fx_{m(k)}, fx_{n(k)})) + \beta(d(fx_{m(k)}, fx_{n(k)}))) \\
&\qquad \times \limsup_{k\to\infty} \max\Big\{ \int_0^{m_2(x_{m(k)}, x_{n(k)})} \varphi(t)dt, \int_0^{m_3(x_{m(k)}, x_{n(k)})} \varphi(t)dt \Big\} \\
&\le \limsup_{s\to\varepsilon^+} (\alpha(s) + \beta(s)) \cdot \max\Big\{ \int_0^{\varepsilon} \varphi(t)dt, \int_0^{\varepsilon} \varphi(t)dt \Big\} \\
&< \int_0^{\varepsilon} \varphi(t)dt,
\end{aligned}$$

which is a contradiction. Thus $\{fx_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. It follows from $(F3)$ that there exists $(a, b) \in g(X) \times X$ satisfying (3.11).

Suppose that $fb \neq a$. In light of (3.11), (3.15), (3.16), $\varphi \in \Phi_1$, Lemmas 2.2 and 2.3, we deduce that

$$\begin{aligned}
\lim_{n\to\infty} m_2(b, x_n) = \lim_{n\to\infty} \max\Big\{ & d(gb, gx_n), d(fb, gb), d(fx_n, gx_n), \\
& \frac{1}{2}(d(fx_n, gb) + d(fb, gx_n)), \\
& \frac{d(fb, gx_n) d(fb, gb)}{2[1 + d(fb, fx_n)]}, \frac{d(fx_n, gb) d(fx_n, gx_n)}{2[1 + d(fb, fx_n)]} \Big\}
\end{aligned}$$

$$\begin{aligned}
&= \max\left\{d(a,a), d(fb,a), d(a,a), \frac{1}{2}(d(a,a)+d(fb,a)),\right.\\
&\qquad\left.\frac{d(fb,a)d(fb,a)}{2[1+d(fb,a)]}, \frac{d(a,a)d(a,a)}{2[1+d(fb,a)]}\right\}\\
&= \max\left\{0, d(fb,a), 0, \frac{1}{2}d(fb,a), \frac{d^2(fb,a)}{2[1+d(fb,a)]}, 0\right\}\\
&= d(fb,a),
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
\lim_{n\to\infty} m_3(b,x_n) &= \lim_{n\to\infty} \max\left\{d(gb,gx_n), d(fb,gb), d(fx_n,gx_n),\right.\\
&\qquad \frac{1}{2}(d(fx_n,gb)+d(fb,gx_n)),\\
&\qquad\left.\frac{d(fb,gb)d(fx_n,gb)}{2[1+d(gb,gx_n)]}, \frac{d(fx_n,gx_n)d(fb,gx_n)}{2[1+d(gb,gx_n)]}\right\}\\
&= \max\left\{d(a,a), d(fb,a), d(a,a), \frac{1}{2}(d(a,a)+d(fb,a)),\right.\\
&\qquad\left.\frac{d(fb,a)d(a,a)}{2[1+d(a,a)]}, \frac{d(a,a)d(fb,a)}{2[1+d(a,a)]}\right\}\\
&= \max\left\{0, d(fb,a), 0, \frac{1}{2}d(fb,a), 0, 0\right\}\\
&= d(fb,a)
\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}
0 &< \int_0^{d(fb,a)} \varphi(t)dt = \limsup_{n\to\infty} \int_0^{d(fb,fx_n)} \varphi(t)dt\\
&\le \limsup_{n\to\infty}\left(\alpha(d(fb,fx_n)) \int_0^{m_2(b,x_n)} \varphi(t)dt + \beta(d(fb,fx_n)) \int_0^{m_3(b,x_n)} \varphi(t)dt\right)\\
&\le \limsup_{n\to\infty}(\alpha(d(fb,fx_n)) + \beta(d(fb,fx_n)))\\
&\qquad \times \limsup_{n\to\infty} \max\left\{\int_0^{m_2(b,x_n)} \varphi(t)dt, \int_0^{m_3(b,x_n)} \varphi(t)dt\right\}\\
&\le \limsup_{s\to d(fb,a)}(\alpha(s)+\beta(s)) \cdot \int_0^{d(fb,a)} \varphi(t)dt\\
&< \int_0^{d(fb,a)} \varphi(t)dt,
\end{aligned}$$

which is absurd. That is, $a = fb = gb$. It follows from $(F1)$ that (3.12) holds.

Suppose that $a \neq fa$. On account of (3.12), (3.15), (3.16) and $\varphi \in \Phi_1$, we deduce that

$$\begin{aligned} m_2(b, gb) &= \max\Big\{ d(gb, g^2b), d(fb, gb), d(fgb, g^2b), \\ &\qquad \frac{1}{2}(d(fgb, gb) + d(fb, g^2b)), \\ &\qquad \frac{d(fb, g^2b)d(fb, gb)}{2[1 + d(fb, fgb)]}, \frac{d(fgb, gb)d(fgb, g^2b)}{2[1 + d(fb, fgb)]} \Big\} \\ &= \max\Big\{ d(a, fa), 0, 0, \frac{1}{2}(d(fa, a) + d(a, fa)), 0, 0 \Big\} \\ &= d(a, fa), \end{aligned} \tag{3.27}$$

$$\begin{aligned} m_3(b, gb) &= \max\Big\{ d(gb, g^2b), d(fb, gb), d(fgb, g^2b), \\ &\qquad \frac{1}{2}(d(fgb, gb) + d(fb, g^2b)), \\ &\qquad \frac{d(fb, gb)d(fgb, gb)}{2[1 + d(gb, g^2b)]}, \frac{d(fgb, g^2b)d(fb, g^2b)}{2[1 + d(gb, g^2b)]} \Big\} \\ &= \max\Big\{ d(a, fa), 0, 0, \frac{1}{2}(d(fa, a) + d(a, fa)), 0, 0 \Big\} \\ &= d(a, fa) \end{aligned} \tag{3.28}$$

and

$$\begin{aligned} 0 &< \int_0^{d(a,fa)} \varphi(t)dt = \int_0^{d(fb,fgb)} \varphi(t)dt \\ &\leq \alpha(d(fb, fgb)) \int_0^{m_2(b,gb)} \varphi(t)dt + \beta(d(fb, fgb)) \int_0^{m_3(b,gb)} \varphi(t)dt \\ &= \alpha(d(a, fa)) \int_0^{d(a,fa)} \varphi(t)dt + \beta(d(a, fa)) \int_0^{d(a,fa)} \varphi(t)dt \\ &= (\alpha(d(a, fa)) + \beta(d(a, fa))) \int_0^{d(a,fa)} \varphi(t)dt \\ &< \int_0^{d(a,fa)} \varphi(t)dt, \end{aligned}$$

which is impossible. Therefore $a = fa = ga$. Consequently, f and g have a common fixed point $a \in X$. This completes the proof. □

Theorem 3.5. *Let f and g be self mappings in a complete metric space (X, d) satisfying $(F1)$-$(F3)$ and*

$$\int_0^{d(fx,fy)} \varphi(t)dt \le \alpha(d(gx,gy)) \int_0^{m_2(x,y)} \varphi(t)dt + \beta(d(gx,gy)) \int_0^{m_3(x,y)} \varphi(t)dt, \quad \forall x, y \in X, \tag{3.29}$$

where $\varphi \in \Phi_1$ and $\alpha, \beta : \mathbb{R}^+ \to [0, 1)$ are two functions with

$$\alpha(t) + \beta(t) < 1, \quad \forall t > 0, \quad \limsup_{s \to t}(\alpha(s) + \beta(s)) < 1, \quad \forall t \in \mathbb{R}^+. \tag{3.30}$$

Then f and g have a unique common fixed point in X.

Proof. First, we prove that f and g have at most one common fixed point in X. Suppose that f and g have two different common fixed points $a, b \in X$. It follows from (3.17), (3.18), (3.29), (3.30) and $\varphi \in \Phi_1$ that

$$\begin{aligned} 0 &< \int_0^{d(a,b)} \varphi(t)dt = \int_0^{d(fa,fb)} \varphi(t)dt \\ &\le \alpha(d(ga,gb)) \int_0^{m_2(a,b)} \varphi(t)dt + \beta(d(ga,gb)) \int_0^{m_3(a,b)} \varphi(t)dt \\ &= \alpha(d(a,b)) \int_0^{d(a,b)} \varphi(t)dt + \beta(d(a,b)) \int_0^{d(a,b)} \varphi(t)dt \\ &= (\alpha(d(a,b)) + \beta(d(a,b))) \int_0^{d(a,b)} \varphi(t)dt \\ &< \int_0^{d(a,b)} \varphi(t)dt, \end{aligned}$$

which is a contradiction.

Next, we show that f and g have a common fixed point in X. Let x_0 be an arbitrary point in X. By means of $(F2)$, there exists a sequence $\{x_n\}_{n \in \mathbb{N}_0}$ in X satisfying $fx_n = gx_{n+1}$ for each $n \in \mathbb{N}_0$.

Assume that $d_{n_0} = 0$ for some $n_0 \in \mathbb{N}_0$. It follows from $(F1)$ that (3.2) and (3.3) hold. Suppose that $fx_{n_0+1} \neq f^2 x_{n_0+1}$. In light of (3.2), (3.3), (3.19), (3.20), (3.29), (3.30) and $\varphi \in \Phi_1$, we infer that

$$
\begin{aligned}
0 &< \int_0^{d(f^2x_{n_0+1},fx_{n_0+1})} \varphi(t)dt \\
&\le \alpha(d(gfx_{n_0+1}, gx_{n_0+1})) \int_0^{m_2(fx_{n_0+1},x_{n_0+1})} \varphi(t)dt \\
&\quad + \beta(d(gfx_{n_0+1}, gx_{n_0+1})) \int_0^{m_3(fx_{n_0+1},x_{n_0+1})} \varphi(t)dt \\
&= \alpha(d(f^2x_{n_0+1}, fx_{n_0+1})) \int_0^{d(f^2x_{n_0+1},fx_{n_0+1})} \varphi(t)dt \\
&\quad + \beta(d(f^2x_{n_0+1}, fx_{n_0+1})) \int_0^{d(f^2x_{n_0+1},fx_{n_0+1})} \varphi(t)dt \\
&= (\alpha(d(f^2x_{n_0+1}, fx_{n_0+1})) + \beta(d(f^2x_{n_0+1}, fx_{n_0+1}))) \int_0^{d(f^2x_{n_0+1},fx_{n_0+1})} \varphi(t)dt \\
&< \int_0^{d(f^2x_{n_0+1},fx_{n_0+1})} \varphi(t)dt,
\end{aligned}
$$

which is absurd. Therefore $fx_{n_0+1} = f^2x_{n_0+1}$, which together with (3.2) and (3.3) means that fx_{n_0+1} is a common fixed point of f and g in X.

Assume that $d_n \neq 0$ for all $n \in \mathbb{N}_0$. If $d_n > d_{n-1}$ for some $n \in \mathbb{N}$, it follows from (3.21), (3.22), (3.29), (3.30) that

$$
\begin{aligned}
0 &< \int_0^{d_n} \varphi(t)dt = \int_0^{d(fx_n,fx_{n+1})} \varphi(t)dt \\
&\le \alpha(d(gx_n, gx_{n+1})) \int_0^{m_2(x_n,x_{n+1})} \varphi(t)dt \\
&\quad + \beta(d(gx_n, gx_{n+1})) \int_0^{m_3(x_n,x_{n+1})} \varphi(t)dt \\
&= \alpha(d(fx_{n-1}, fx_n)) \int_0^{d_n} \varphi(t)dt + \beta(d(fx_{n-1}, fx_n)) \int_0^{d_n} \varphi(t)dt \\
&= (\alpha(d_{n-1}) + \beta(d_{n-1})) \int_0^{d_n} \varphi(t)dt \\
&< \int_0^{d_n} \varphi(t)dt,
\end{aligned}
$$

which is impossible. Hence the sequence $\{d_n\}_{n\in\mathbb{N}_0}$ satisfies (3.6).

Suppose that $c > 0$. Using (3.6), (3.21), (3.22), (3.29), (3.30), Lemma 2.2 and $\varphi \in \Phi_1$, we conclude that

$$\begin{aligned}
0 &< \int_0^c \varphi(t)dt = \limsup_{n\to\infty} \int_0^{d_n} \varphi(t)dt \\
&= \limsup_{n\to\infty} \int_0^{d(fx_n, fx_{n+1})} \varphi(t)dt \\
&\le \limsup_{n\to\infty} \bigg(\alpha(d(gx_n, gx_{n+1})) \int_0^{m_2(x_n,x_{n+1})} \varphi(t)dt \\
&\qquad\qquad + \beta(d(gx_n, gx_{n+1})) \int_0^{m_3(x_n,x_{n+1})} \varphi(t)dt \bigg) \\
&= \limsup_{n\to\infty} \bigg(\alpha(d(fx_{n-1}, fx_n)) \int_0^{d_{n-1}} \varphi(t)dt + \beta(d(fx_{n-1}, fx_n)) \int_0^{d_{n-1}} \varphi(t)dt \bigg) \\
&\le \limsup_{n\to\infty} (\alpha(d_{n-1}) + \beta(d_{n-1})) \cdot \limsup_{n\to\infty} \int_0^{d_{n-1}} \varphi(t)dt \\
&\le \limsup_{s\to c^+} (\alpha(s) + \beta(s)) \int_0^{c} \varphi(t)dt \\
&< \int_0^c \varphi(t)dt,
\end{aligned}$$

which is absurd. Therefore, $c = 0$, that is, (3.7) holds. Now we claim that $\{fx_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. Suppose that $\{fx_n\}_{n\in\mathbb{N}_0}$ is not a Cauchy sequence. It follows that there exist a constant $\varepsilon > 0$ and two sequences $\{m(k)\}_{k\in\mathbb{N}}$ and $\{n(k)\}_{k\in\mathbb{N}}$ in $\mathbb{N}_0$ with (3.8)-(3.10). It follows from (3.10), (3.23), (3.24), (3.29), (3.30), Lemma 2.2 and $\varphi \in \Phi_1$ that

$$\begin{aligned}
0 &< \int_0^\varepsilon \varphi(t)dt = \limsup_{k\to\infty} \int_0^{d(fx_{m(k)}, fx_{n(k)})} \varphi(t)dt \\
&\le \limsup_{k\to\infty} \bigg(\alpha(d(gx_{m(k)}, gx_{n(k)})) \int_0^{m_2(x_{m(k)},x_{n(k)})} \varphi(t)dt \\
&\qquad\qquad + \beta(d(gx_{m(k)}, gx_{n(k)})) \int_0^{m_3(x_{m(k)},x_{n(k)})} \varphi(t)dt \bigg) \\
&= \limsup_{k\to\infty} \bigg(\alpha(d(fx_{m(k)-1}, gx_{n(k)-1})) \int_0^{m_2(x_{m(k)},x_{n(k)})} \varphi(t)dt \\
&\qquad\qquad + \beta(d(fx_{m(k)-1}, fx_{n(k)-1})) \int_0^{m_3(x_{m(k)},x_{n(k)})} \varphi(t)dt \bigg)
\end{aligned}$$

$$\begin{aligned}
&\leq \limsup_{k\to\infty}(\alpha(d(fx_{m(k)-1}, fx_{n(k)-1})) + \beta(d(fx_{m(k)-1}, fx_{n(k)-1})))\\
&\qquad \times \limsup_{k\to\infty}\max\left\{\int_0^{m_2(x_{m(k)},x_{n(k)})}\varphi(t)dt, \int_0^{m_3(x_{m(k)},x_{n(k)})}\varphi(t)dt\right\}\\
&\leq \limsup_{s\to\varepsilon}(\alpha(s)+\beta(s))\cdot\max\left\{\int_0^{\varepsilon}\varphi(t)dt, \int_0^{\varepsilon}\varphi(t)dt\right\}\\
&< \int_0^{\varepsilon}\varphi(t)dt,
\end{aligned}$$

which is a contradiction. Thus $\{fx_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. It follows from $(F3)$ that there exists $(a,b)\in g(X)\times X$ satisfying (3.11).

Suppose that $fb \neq a$. In light of (3.11), (3.25), (3.26), (3.29), (3.30), $\varphi\in\Phi_1$, Lemmas 2.2 and 2.3, we deduce that

$$\begin{aligned}
0 &< \int_0^{d(fb,a)}\varphi(t)dt = \limsup_{n\to\infty}\int_0^{d(fb,fx_n)}\varphi(t)dt\\
&\leq \limsup_{n\to\infty}\left(\alpha(d(gb,gx_n))\int_0^{m_2(b,x_n)}\varphi(t)dt + \beta(d(gb,gx_n))\int_0^{m_3(b,x_n)}\varphi(t)dt\right)\\
&\leq \limsup_{n\to\infty}(\alpha(d(gb,gx_n)) + \beta(d(gb,gx_n)))\\
&\qquad \times \limsup_{n\to\infty}\max\left\{\int_0^{m_2(b,x_n)}\varphi(t)dt, \int_0^{m_3(b,x_n)}\varphi(t)dt\right\}\\
&\leq \limsup_{s\to 0^+}(\alpha(s)+\beta(s))\cdot\int_0^{d(fb,a)}\varphi(t)dt < \int_0^{d(fb,a)}\varphi(t)dt,
\end{aligned}$$

which is absurd. That is, $a = fb = gb$. It follows from $(F1)$ that (3.12) holds.

Suppose that $a \neq fa$. On account of (3.12), (3.27)-(3.30) and $\varphi\in\Phi_1$, we deduce that

$$\begin{aligned}
0 &< \int_0^{d(a,fa)}\varphi(t)dt = \int_0^{d(fb,fgb)}\varphi(t)dt\\
&\leq \alpha(d(gb,g^2b))\int_0^{m_2(b,gb)}\varphi(t)dt + \beta(d(gb,g^2b))\int_0^{m_3(b,gb)}\varphi(t)dt\\
&= \alpha(d(a,fa))\int_0^{d(a,fa)}\varphi(t)dt + \beta(d(a,fa))\int_0^{d(a,fa)}\varphi(t)dt\\
&= (\alpha(d(a,fa)) + \beta(d(a,fa)))\int_0^{d(a,fa)}\varphi(t)dt\\
&< \int_0^{d(a,fa)}\varphi(t)dt,
\end{aligned}$$

which is impossible. Therefore $a = fa = ga$. Consequently, f and g have a common fixed point $a \in X$. This completes the proof. $\square$

Remark 3.6. In case $gx = x$ for all $x \in X$ and $\alpha(t) = c$ for all $t \in \mathbb{R}^+$, where $c \in (0,1)$ is a constant, then Theorems 3.1-3.3 reduce to results, which generalize Theorem 1.1. Example 3.7 shows that Theorems 3.1-3.3 extend indeed Theorem 1.1 and differ from Theorem 1.2.

Example 3.7. Let $X = \mathbb{R}$ be endowed with the Euclidean metric $d(x,y) = |x-y|$ for all $x, y \in X$ and $f : X \to X$ be defined by

$$f(x) = \begin{cases} 1, & \forall x \in X \setminus \{3\}, \\ 2, & x = 3. \end{cases}$$

First, we prove that Theorems 1.1 and 1.2 cannot be used to prove the existence of fixed points of the mapping f in X. Suppose that there exist $\varphi \in \Phi_1$ and $c \in (0,1)$ satisfying the conditions of Theorem 1.1. It follows that

$$\int_0^1 \varphi(t)dt = \int_0^{d(f2,f3)} \varphi(t)dt \le c \int^{d(2,3)} \varphi(t)dt = c \int_0^1 \varphi(t)dt < \int_0^1 \varphi(t)dt,$$

which is impossible.

Suppose that there exist $\varphi \in \Phi_1$ and $\alpha \in \Phi_2$ satisfying the conditions of Theorem 1.2. It follows that

$$\begin{aligned} \int_0^1 \varphi(t)dt &= \int_0^{d(f2,f3)} \varphi(t)dt \le \alpha(d(2,3)) \int_0^{d(2,3)} \varphi(t)dt \\ &= \alpha(1) \int_0^1 \varphi(t)dt < \int_0^1 \varphi(t)dt, \end{aligned}$$

which is absurd.

Next,, we prove the existence of fixed points of the mapping f in X by using Theorems 3.1- 3.3, respectively. Define $g : X \to X$, $\alpha : \mathbb{R}^+ \to [0,1)$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$g(x) = \begin{cases} 1, & \forall x \in X \setminus \{3,6\}, \\ 6, & x = 3, \\ 2, & x = 6, \end{cases}$$

$$\alpha(t) = \frac{1+t}{2+t} \quad \text{and} \quad \varphi(t) = 2t, \quad \forall t \in \mathbb{R}^+.$$

Obviously, $(\varphi, \alpha) \in \Phi_1 \times \Phi_3$, $f(X) = \{1,2\} \subseteq \{1,2,6\} = g(X)$, $g(X)$ is complete, f and g are weakly compatible in X and α is increasing in $(0, +\infty)$.

Let $x, y \in X$ with $x < y$. In order to verify (3.1), we have to consider four possible cases as follows:

Case 1. $x, y \in X \setminus \{3\}$. It is clear that

$$\int_0^{d(fx,fy)} \varphi(t)dt = \int_0^{d(1,1)} \varphi(t)dt = 0 \le \alpha(m_1(x,y)) \int_0^{m_1(x,y)} \varphi(t)dt;$$

Case 2. $x = 3$ and $y = 6$. Note that

$$m_1(x,y) \ge d(gx,gy) = d(6,2) = 4 \tag{3.31}$$

and

$$\begin{aligned}\int_0^{d(fx,fy)} \varphi(t)dt &= \int_0^{d(2,1)} \varphi(t)dt = \int_0^{1} \varphi(t)dt = 1 < \frac{40}{3} \\ &= \alpha(4) \int_0^{4} \varphi(t)dt \le \alpha(m_1(x,y)) \int_0^{m_1(x,y)} \varphi(t)dt;\end{aligned}$$

Case 3. $x = 3$ and $y \in (3,6) \cup (6,+\infty)$. It follows that

$$m_1(x,y) \ge d(gx,gy) = d(6,1) = 5 \tag{3.32}$$

and

$$\begin{aligned}\int_0^{d(fx,fy)} \varphi(t)dt &= \int_0^{d(2,1)} \varphi(t)dt = \int_0^{1} \varphi(t)dt = 1 < \frac{150}{7} \\ &= \alpha(5) \int_0^{5} \varphi(t)dt \le \alpha(m_1(x,y)) \int_0^{m_1(x,y)} \varphi(t)dt;\end{aligned}$$

Case 4. $x < 3$ and $y = 3$. It is easy to see that

$$m_1(x,y) \ge d(gx,gy) = d(1,6) = 5 \tag{3.33}$$

and

$$\begin{aligned}\int_0^{d(fx,fy)} \varphi(t)dt &= \int_0^{d(1,2)} \varphi(t)dt = \int_0^{1} \varphi(t)dt = 1 < \frac{150}{7} \\ &= \alpha(5) \int_0^{5} \varphi(t)dt \le \alpha(m_1(x,y)) \int_0^{m_1(x,y)} \varphi(t)dt.\end{aligned}$$

Hence (3.1) holds. It follows from Theorem 3.1 that f and g have a unique common fixed point $1 \in X$, that is, f has a fixed point $1 \in X$.

In order to verify (3.13), we have to consider four possible cases as follows:

Case 1. $x, y \in X \setminus \{3\}$. It is clear that

$$\int_0^{d(fx,fy)} \varphi(t)dt = \int_0^{d(1,1)} \varphi(t)dt = 0 \le \alpha(d(fx,fy)) \int_0^{m_1(x,y)} \varphi(t)dt;$$

Case 2. $x = 3$ and $y = 6$. It follows from (3.31) that

$$\int_0^{d(fx,fy)} \varphi(t)dt = \int_0^{d(2,1)} \varphi(t)dt = \int_0^1 \varphi(t)dt = 1 < \frac{32}{3}$$
$$= \alpha(1)\int_0^4 \varphi(t)dt \le \alpha(d(fx,fy))\int_0^{m_1(x,y)} \varphi(t)dt;$$

Case 3. $x = 3$ and $y \in (3,6) \cup (6,+\infty)$. In light of (3.32), we deduce that

$$\int_0^{d(fx,fy)} \varphi(t)dt = \int_0^{d(2,1)} \varphi(t)dt = \int_0^1 \varphi(t)dt = 1 < \frac{50}{3}$$
$$= \alpha(1)\int_0^5 \varphi(t)dt \le \alpha(d(fx,fy))\int_0^{m_1(x,y)} \varphi(t)dt;$$

Case 4. $x < 3$ and $y = 3$. On account of (3.33), we have

$$\int_0^{d(fx,fy)} \varphi(t)dt = \int_0^{d(1,2)} \varphi(t)dt = \int_0^1 \varphi(t)dt = 1 < \frac{50}{3}$$
$$= \alpha(1)\int_0^5 \varphi(t)dt \le \alpha(d(fx,fy))\int_0^{m_1(x,y)} \varphi(t)dt.$$

Hence (3.13) holds. It follows from Theorem 3.2 that f and g have a unique common fixed point $1 \in X$, that is, f has a fixed point $1 \in X$.

In order to verify (3.14), we have to consider four possible cases as follows:

Case 1. $x, y \in X \setminus \{3\}$. It is clear that

$$\int_0^{d(fx,fy)} \varphi(t)dt = \int_0^{d(1,1)} \varphi(t)dt = 0 \le \alpha(d(gx,gy))\int_0^{m_1(x,y)} \varphi(t)dt;$$

Case 2. $x = 3$ and $y = 6$. It follows from (3.31) that

$$\int_0^{d(fx,fy)} \varphi(t)dt = \int_0^{d(2,1)} \varphi(t)dt = \int_0^1 \varphi(t)dt = 1 < \frac{40}{3}$$
$$= \alpha(4)\int_0^4 \varphi(t)dt \le \alpha(d(gx,gy))\int_0^{m_1(x,y)} \varphi(t)dt;$$

Case 3. $x = 3$ and $y \in (3,6) \cup (6,+\infty)$. In light of (3.32), we deduce that

$$\int_0^{d(fx,fy)} \varphi(t)dt = \int_0^{d(2,1)} \varphi(t)dt = \int_0^1 \varphi(t)dt = 1 < \frac{150}{7}$$
$$= \alpha(5)\int_0^5 \varphi(t)dt \le \alpha(d(gx,gy))\int_0^{m_1(x,y)} \varphi(t)dt;$$

Case 4. $x < 3$ and $y = 3$. On account of (3.33), we have

$$\int_0^{d(fx,fy)} \varphi(t)dt = \int_0^{d(1,2)} \varphi(t)dt = \int_0^1 \varphi(t)dt = 1 < \frac{150}{7}$$
$$= \alpha(5)\int_0^5 \varphi(t)dt \le \alpha(d(gx,gy))\int_0^{m_1(x,y)} \varphi(t)dt.$$

Hence (3.14) holds. It follows from Theorem 3.3 that f and g have a unique common fixed point $1 \in X$, that is, f has a fixed point $1 \in X$.

Remark 3.8. Example 3.9 shows that Theorems 3.4 and 3.5 differ from Theorem 1.3.

Example 3.9. Let $X = \mathbb{R}$ be endowed with the Euclidean metric $d(x,y) = |x-y|$ for all $x, y \in X$ and $f : X \to X$ be defined by

$$f(x) = \begin{cases} 4, & \forall x \in X \setminus \{6\}, \\ 5, & x = 6. \end{cases}$$

Now we prove that Theorem 1.3 cannot be used to prove the existence of fixed points of the mapping f in X. Suppose that there exist $\varphi \in \Phi_1$ and $\alpha, \beta : \mathbb{R}^+ \to [0,1)$ satisfying the conditions of Theorem 1.3. It follows that

$$\int_0^1 \varphi(t)dt = \int_0^{d(f5,f6)} \varphi(t)dt$$
$$\le \alpha(d(5,6))\int_0^{d(5,f5)} \varphi(t)dt + \beta(d(5,6))\int_0^{d(6,f6)} \varphi(t)dt$$
$$= \alpha(1)\int_0^1 \varphi(t)dt + \beta(1)\int_0^1 \varphi(t)dt < \int_0^1 \varphi(t)dt,$$

which is a contradiction.

Next, we prove the existence of fixed points of the mapping f in X by using Theorems 3.4 and 3.5, respectively. Define $g : X \to X$, $\alpha, \beta : \mathbb{R}^+ \to [0,1)$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$g(x) = \begin{cases} 4, & \forall x \in X \setminus \{6, 8\}, \\ 10, & x = 6, \\ 5, & x = 8, \end{cases}$$

$$\alpha(t) = \frac{1}{2+t}, \quad \beta(t) = \frac{t}{1+t} \quad \text{and} \quad \varphi(t) = t, \quad \forall t \in \mathbb{R}^+.$$

Obviously, $(\varphi, \alpha) \in \Phi_1 \times \Phi_2$, $f(X) = \{4,5\} \subseteq \{4,5,10\} = g(X)$, $g(X)$ is complete, f and g are weakly compatible in X, (3.16) and (3.30) hold.

Let $x, y \in X$ with $x < y$. In order to verify (3.15), we have to consider four possible cases as follows:

Case 1. $x, y \in X \setminus \{6\}$. It is clear that

$$\begin{aligned}
&\int_0^{d(fx,fy)} \varphi(t)dt \\
&= \int_0^{d(4,4)} \varphi(t)dt = 0 \\
&\leq \alpha(d(fx,fy)) \int_0^{m_2(x,y)} \varphi(t)dt + \beta(d(fx,fy)) \int_0^{m_3(x,y)} \varphi(t)dt;
\end{aligned}$$

Case 2. $x = 6$ and $y = 8$. Note that

$$\begin{aligned}
m_2(x,y) &\geq d(gx,gy) = d(10,5) = 5, \\
m_3(x,y) &\geq d(gx,gy) = d(10,5) = 5
\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}
&\int_0^{d(fx,fy)} \varphi(t)dt \\
&= \int_0^{d(5,4)} \varphi(t)dt = \int_0^1 \varphi(t)dt = \frac{1}{2} < \frac{125}{12} \\
&= \alpha(1) \int_0^5 \varphi(t)dt + \beta(1) \int_0^5 \varphi(t)dt \\
&\leq \alpha(d(fx,fy)) \int_0^{m_2(x,y)} \varphi(t)dt + \beta(d(fx,fy)) \int_0^{m_3(x,y)} \varphi(t)dt;
\end{aligned}$$

Case 3. $x = 6$ and $y \in (6,8) \cup (8,+\infty)$. It follows that

$$\begin{aligned}
m_2(x,y) &\geq d(gx,gy) = d(10,4) = 6, \\
m_3(x,y) &\geq d(gx,gy) = d(10,4) = 6
\end{aligned} \tag{3.35}$$

and

$$\begin{aligned}
&\int_0^{d(fx,fy)} \varphi(t)dt \\
&= \int_0^{d(5,4)} \varphi(t)dt = \int_0^1 \varphi(t)dt = \frac{1}{2} < 15 \\
&= \alpha(1) \int_0^6 \varphi(t)dt + \beta(1) \int_0^6 \varphi(t)dt \\
&\leq \alpha(d(fx,fy)) \int_0^{m_2(x,y)} \varphi(t)dt + \beta(d(fx,fy)) \int_0^{m_3(x,y)} \varphi(t)dt;
\end{aligned}$$

Case 4. $x < 6$ and $y = 6$. It is easy to see that

$$\begin{aligned}
m_2(x,y) &\geq d(gx,gy) = d(4,10) = 6, \\
m_3(x,y) &\geq d(gx,gy) = d(4,10) = 6
\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}
&\int_0^{d(fx,fy)} \varphi(t)dt \\
&= \int_0^{d(4,5)} \varphi(t)dt \\
&= \int_0^{1} \varphi(t)dt = \frac{1}{2} < 15 \\
&= \alpha(1)\int_0^{6} \varphi(t)dt + \beta(1)\int_0^{6} \varphi(t)dt \\
&\le \alpha(d(fx,fy))\int_0^{m_2(x,y)} \varphi(t)dt + \beta(d(fx,fy))\int_0^{m_3(x,y)} \varphi(t)dt.
\end{aligned}$$

Hence (3.15) holds. It follows from Theorem 3.4 that f and g have a unique common fixed point $4 \in X$, that is, f has a fixed point $4 \in X$.

In order to verify (3.29), we have to consider four possible cases as follows:

Case 1. $x, y \in X \setminus \{6\}$. It is clear that

$$\begin{aligned}
&\int_0^{d(fx,fy)} \varphi(t)dt \\
&= \int_0^{d(4,4)} \varphi(t)dt = 0 \\
&\le \alpha(d(gx,gy))\int_0^{m_2(x,y)} \varphi(t)dt + \beta(d(gx,gy))\int_0^{m_3(x,y)} \varphi(t)dt;
\end{aligned}$$

Case 2. $x = 6$ and $y = 8$. It follows from (3.34) that

$$\begin{aligned}
&\int_0^{d(fx,fy)} \varphi(t)dt \\
&= \int_0^{d(5,4)} \varphi(t)dt = \int_0^{1} \varphi(t)dt = \frac{1}{2} < \frac{1025}{84} \\
&= \alpha(5)\int_0^{5} \varphi(t)dt + \beta(5)\int_0^{5} \varphi(t)dt \\
&\le \alpha(d(gx,gy))\int_0^{m_2(x,y)} \varphi(t)dt + \beta(d(gx,gy))\int_0^{m_3(x,y)} \varphi(t)dt;
\end{aligned}$$

Case 3. $x = 6$ and $y \in (6,8) \cup (8,+\infty)$. In light of (3.35), we deduce that

$$\begin{aligned}
&\int_0^{d(fx,fy)} \varphi(t)dt \\
&= \int_0^{d(5,4)} \varphi(t)dt = \int_0^1 \varphi(t)dt = \frac{1}{2} < \frac{495}{28} \\
&= \alpha(6)\int_0^6 \varphi(t)dt + \beta(6)\int_0^6 \varphi(t)dt \\
&\le \alpha(d(gx,gy))\int_0^{m_2(x,y)} \varphi(t)dt + \beta(d(gx,gy))\int_0^{m_3(x,y)} \varphi(t)dt;
\end{aligned}$$

Case 4. $x < 6$ and $y = 6$. On account of (3.36), we have

$$\begin{aligned}
&\int_0^{d(fx,fy)} \varphi(t)dt \\
&= \int_0^{d(4,5)} \varphi(t)dt = \int_0^1 \varphi(t)dt = \frac{1}{2} < \frac{495}{28} \\
&= \alpha(6)\int_0^6 \varphi(t)dt + \beta(6)\int_0^6 \varphi(t)dt \\
&\le \alpha(d(gx,gy))\int_0^{m_2(x,y)} \varphi(t)dt + \beta(d(gx,gy))\int_0^{m_3(x,y)} \varphi(t)dt.
\end{aligned}$$

Hence (3.29) holds. It follows from Theorem 3.5 that f and g have a unique common fixed point $4 \in X$, that is, f has a fixed point $4 \in X$.

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