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DIFFERENTIAL INCLUSIONS OF FRACTIONAL ORDER WITH IMPULSE EFFECTS IN BANACH SPACES

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Abstract. Impulsive fractional semilinear differential inclusions in Banach spaces are considered. We investigate the situation when the linear part generates a semigroup not required to be compact and the multivalued function is lower semicontinuous and nonconvex. Our result are obtained by using noncompactness Hausdorff measure (NCHM), multivalued properties and fixed point theorems. We finally present an example to lighten our results.

1. Introduction

In this paper, we investigate the existence of solutions for following impulsive differential inclusion with nonlocal condition:

$$\begin{cases} {}^cD^{\alpha}x(t) \in Ax(t) + F(t, x(t)), \ t \in J = [0, b], \ t \neq t_i, \ i = 1, ..., m, \\ x(t_i^+) = x(t_i) + I_i(x(t_i)), \\ x(0) = g(x), \end{cases} \tag{Q}$$

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where ${}^cD^\alpha$ is the Caputo derivative of order α, $A : D(A) \subseteq E \to E$ is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ on a real separable Banach space E, $F : J \times E \to 2^E$ is a multifunction, 2^E is the power set of E, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, for every $i = 1, 2, ..., m$, $I_i : E \to E$ are impulsive functions, $g : PC(J, E) \to E$ is a nonlinear function, and $x(t_i^+)$ is the right limit of $x(t)$ at the point t_i.

Impulsive differential equations and inclusions occur in many disciplines; physics, engineering, biology and et al. Because of their accuracy in modeling phenomena which change rapidly at certain moments. For more details, see [16, 21, 1, 6, 29, 8, 14, 12].

Nonlocal conditions problems were derived from physical problems, for instance see [5, 15, 11]. The basic general theory of nonlocal conditions problems was initiated by Byszewski [11]. However, compactness of the solution operator at zero remains the essential obstacle in case of nonlocal conditions problems. Various methods and techniques have been embraced by many authors in this direction. For further details, we refer to [2, 3, 23, 13, 14, 19, 28, 31, 32, 33, 12, 25, 22, 27]. Among them, Wang et al. [32] obtained existence and uniqueness results when F is a Lipschitz single-valued function or continuous function sends bounded sets into bounded sets and $\{T(t)\}_{t>0}$ is compact. Using NCHM, Li [25] gave existence results concerning nonlocal fractional differential equations, where the semigroup is equicontinuous as well as the nonlocal term is compact. Moreover, Ibrahim and Alsarori [22] established sufficient conditions which guarantee the existence of mild solutions for the problem (Q) with delay when the semigroup is compact. Recently, Lian et al. [26] discussed the existence results of mild solutions for (Q) without impulses when the operator semigroup is not necessarily compact and F is convex. Very recently, Alsarori et al. [3] investigated the problem (Q) when the semigroup is not compact and F is upper semicontinuous, convex and compact.

Motivated by the above works, we consider a case differs from previous cases. We study the existence of mild solution for (Q) in the case when the multifunction $F : J \times E \to 2^E$ is lower semicontinuous, and the convexity condition on F is relaxed in this paper. Also in our results, the C_0-semigroup $\{T(t), t \geq 0\}$ generated by the linear part of (Q) on the real separable Banach space E is equicontinuous.

After presenting some definitions and facts related to fractional calculus and the set-valued analysis in Section 2. Section 3 proceeds to prove the existence results of mild solutions for (Q) (mild solution concept as introduced in [32]). The results are derived by techniques and methods of NCHM, as fixed point theorems. An example is provided to clarify the applicability of our results in section 4.

2. Preliminaries

During this section, we state some previous known results so that we can use them later throughout this paper. By $C(J,E)$, we denote the Banach space of all continuous functions on J with the uniform norm $\|x\| = \sup\{\|x(t)\|, t \in J\}$, $L^1(J,E)$ the space of $E-$valued Bochner integrable functions on J with the norm $\|x\|_{L^1(J,E)} = \int_0^b \|x(t)\|dt$.

Let $P_b(E)$, $P_{cl}(E)$ be denote the families of all nonempty subsets of E which are bounded and closed, respectively, and $\overline{conv}(B)$ denote the closed convex hull in E of subset B.

Definition 2.1. ([24]) The noncompactness Hausdorff measure(NCHM) on E, $\chi : P_b(E) \to [0,+\infty)$ is defined by

$$\chi(B) = \inf\{\varepsilon > 0 : B \subseteq \cup_{j=1}^{n} B_j \ and\ radius(B_j) \leq \varepsilon\}.$$

Lemma 2.2. ([24]) Let χ be the noncompactness Hausdorff measure. Then we have the following statements.

(1) If $B_1, B_2 \in P_b(E), B_1 \subset B_2$, then $\chi(B_1) \leq \chi(B_2)$.
(2) $\chi(\{a\} \cup B) = \chi(B)$, for every $a \in E, B \in P_b(E)$.
(3) For any compact subset $K \subset E$ and any $B \in P_b(E), \chi(B \cup K) = \chi(B)$.
(4) $\chi(B_1 + B_2) \leq \chi(B_1) + \chi(B_2)$, for every $B_1, B_2 \in P_b(E)$.
(5) $\chi(B) = 0$ iff B is relatively compact, for every $B \in P_b(E)$.
(6) $\chi(tB) = \mid t \mid \chi(B),\ t \in \mathbb{R}, B \in P_b(E)$.
(7) $\chi(L(B)) \leq \|L\|\chi(B)$, for every $B \in P_b(E)$, where L is a bounded linear operator on E.

Let $\{t_0, t_1, \cdots, t_m, t_{m+1}\}$ be a partition on $[0, b]$. Let $J_0 = [0, t_1]$ and for each i, $J_i =]t_i, t_{i+1}]$, define

$$PC(J,E) = \{x : J \to E : x_{|_{J_i}} \in C(J_i,E),\ x(t_i^+),\ x(t_i^-)\ exist\ for\ all\ 0 \leq i \leq m\}.$$

Obviously, $PC(J,E)$ with uniform norm $\|x\|_{PC(J,E)}$ is a Banach space. Also, let us consider the map

$$\chi_{PC} : P_b(PC(J,E)) \to [0,\infty[, \chi_{PC}(B) = \max_{0\leq i\leq m} \chi_i(B_{|_{\overline{J_i}}}),\ B \in P_b(PC(J,E),$$

where χ_i is defined on $C(\overline{J_i},E)$ and

$$B_{|_{\overline{J_i}}} = \{x^* : \overline{J_i} \longrightarrow E, x^*(t) = x(t), t \in J_i, x^*(t_i) = x(t_i^+), x \in B, 0 \leq i \leq m\}.$$

Clearly, χ_{PC} is NCHM on $PC(J,E)$.

Definition 2.3. Let $f \in L^1(J, E)$. The Riemann-Liouville fractional integral of order $\alpha \in (0,1)$ of f is defined by

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided the right side is pointwise defined on J, where Γ is the Euler gamma function.

Definition 2.4. Let $f : J \to E$ be continuously differentiable function. The Caputo derivative of order $\alpha \in (0,1)$ of f is defined by

$${}^{c}D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f^{(1)}(s) ds = I^{(1-\alpha)} f^{(1)}.$$

Definition 2.5. The mild solution for (Q) is a function $x \in PC(J, E)$ such that

$$x(t) = \begin{cases} \mathcal{T}_{\alpha}(t) g(x) + \displaystyle\int_0^t (t-s)^{\alpha-1} \mathcal{S}_{\alpha}(t-s) f(s) ds, \ t \in J_0, \\ \mathcal{T}_{\alpha}(t) g(x) + \displaystyle\sum_{i=1}^{i=m} \mathcal{T}_{\alpha}(t-t_i) I_i(x(t_i^-)) \\ \quad + \displaystyle\int_0^t (t-s)^{\alpha-1} \mathcal{S}_{\alpha}(t-s) f(s) ds, \ t \in J_i, \ 0 \le i \le m, \end{cases}$$

where

$$f \in S^1_{F(\cdot, x(\cdot))}, \quad \varpi_{\alpha}(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-\alpha n - 1} \frac{\Gamma(n\alpha + 1)}{n!} \sin(n\pi\alpha),$$

$$\mathcal{T}_{\alpha}(t) = \int_0^{\infty} \xi_{\alpha}(\theta) T(t^{\alpha}\theta) d\theta, \mathcal{S}_{\alpha}(t) = \alpha \int_0^{\infty} \theta \xi_{\alpha}(\theta) T(t^{\alpha}\theta) d\theta,$$

$$\xi_{\alpha}(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \varpi_{\alpha}(\theta^{\frac{-1}{\alpha}}) \ge 0,$$

$\theta \in (0, \infty)$ and ξ is a probability density function defined on $(0, \infty)$, that is $\int_0^{\infty} \xi_{\alpha}(\theta) d\theta = 1$.

Next, we restate some results regarding of $\mathcal{T}_{\alpha}(\cdot)$ and $\mathcal{S}_{\alpha}(\cdot)$.

Lemma 2.6. ([33])

(i) $\mathcal{T}_{\alpha}(t), \mathcal{S}_{\alpha}(t)$ are linear, bounded and strongly continuous operators for any fixed $t \in [0, \infty[$.

(ii) For $\gamma \in [0,1]$, $\int_0^{\infty} \theta^{\gamma} \xi_{\alpha}(\theta) d\theta = \dfrac{\Gamma(1+\gamma)}{\Gamma(1+\alpha\gamma)}$.

(ii) If $\|T(t)\| \leq M, t \geq 0$, then for any $x \in E$, $\|\mathcal{T}_\alpha(t)x\| \leq M\|x\|$, $\|\mathcal{S}_\alpha(t)x\| \leq \frac{M}{\Gamma(\alpha)}\|x\|$.

(iv) If $\{T(t)\}_{t\geq 0}$ is equicontinuous, then so $\mathcal{T}_\alpha(t)$ and $\mathcal{S}_\alpha(t)$.

Lemma 2.7. ([12]) Assume that $(W_n)_{n\geq 1}$ is a decreasing sequence of nonempty bounded and closed subsets of E with $\chi(W_n) \to 0$ as $n \to \infty$. Then $\cap_{n=1}^{\infty} W_n$ is nonempty compact subset of E.

Lemma 2.8. ([7]) Let W be a bounded and equicontinuous subset of $C(J, E)$. Then $\chi(W(t))$ is a continuous function on J and $\chi(W) = \sup_{t\in J} \chi(W(t))$.

Lemma 2.9. ([18]) If $\{u_n\}_{n=1}^{\infty}$ is a sequence of uniformly integrable functions in $L^1(J, E)$, then $\chi(\{u_n(t)\}_{n=1}^{\infty})$ is measurable and

$$\chi(\{\int_0^t u_n(s)ds\}_{n=1}^{\infty}) \leq 2\int_0^t \chi(\{u_n(s)\}_{n=1}^{\infty})ds.$$

Lemma 2.10. ([9]) If $B \subseteq E$ is bounded, then for all $\epsilon > 0$, there exists a sequence $\{u_n\}_{n=1}^{\infty}$ in B such that $\chi(B) \leq 2\chi(\{u_n\}_{n=1}^{\infty}) + \epsilon$.

Definition 2.11. ([18], [24]) If X, Y are topological spaces. A multifunction $F : X \to P(Y)$ is called:

(1) upper semicontinuous (u.s.c) if $F^{-1}(V)$ is an open subset of X for every open $V \subseteq Y$.
(2) lower semicontinuous (l.s.c) when $F^{+1}(V) = \{x \in X : F(x) \cap V \neq \varnothing\}$ is an open for every open subset V of Y.
(3) closed in case when its graph is closed in the topological space $X \times Y$.
(4) F is said to have a fixed point if there is $x \in X$ such that $x \in F(x)$.

Remark 2.12.

(1) If $U \subset X$ and $F(U)$ are closed and $\overline{F(U)}$ is compact, then F is u.s.c. iff F is closed.
(2) If $F : X \to P(Y) - \{\varnothing\}$ is a multifunction. Then $d(y, F(\cdot))$ is u.s.c. iff F is l.s.c. for every $y \in Y$, where X, Y are Banach spaces.

Definition 2.13. If B is a nonempty subset of $L^1(J, E)$, we call B is decomposable if for every $f, g \in B$ and for all Lebesgue measurable set $M \subset J$, $f\beta_M + g\beta_{(J-M)} \in B$, where β is the characteristic function of M.

Lemma 2.14. ([10, Theorem 3]) If $F : J \times X \to P(L^1(J, X))$ is a multifunction with closed decomposable values, then F has a continuous selection, where X is a separable metric space.

Lemma 2.15. ([4, Lemma 4]) Let $\{f_n\}_{n\geq 1}$ be a sequence of integrable bounded functions in $L^P(J,E)$, $P \in [1,\infty[$ with $\chi\{f_n : n \geq 1\} \leq \varrho(t)$, *a.e.* $t \in J$, $\varrho \in L^1(J,\mathbb{R}^+)$. Then, for all $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq E$, a measurable set $J_\varepsilon \subset J$ with measure less than ε and a sequence $\{\zeta_n\} \subset L^P(J,E)$ such that $\{\zeta_n(t) : n \geq 1\} \subseteq K_\varepsilon$ for all $t \in J$ and

$$\|f_n(t) - \zeta_n(t)\| < 2\varrho(t) + \varepsilon, \ \forall\, t \in J - J_\varepsilon.$$

Theorem 2.16. ([17]) Let E be a Banach space, W be a nonempty, convex, closed and bounded subset of E and $G : W \to W$ be a continuous function. If either G or W is compact, then G has a fixed point.

3. Main Results

By using NCHM and fixed point theorems, we will prove the following theorem:

Theorem 3.1. *Suppose the following hypotheses:*

(HA) *The C_0-semigroup $\{T(t) : t \geq 0\}$ is equicontinuous and for some positive constant M, $\sup_{t\in J}\|T(t)\| \leq M$.*

(HF) *Let $F : J \times E \to P_{cl}(E)$ be a multifunction such that:*
   1. *$(t,x) \to F(t,x)$ is graph measurable.*
   2. *$x \to F(t,x)$ is l.s.c. for a.e. $t \in J$.*
   3. *If $q \in (0,\alpha)$, there exists $\varsigma \in L^{\frac{1}{q}}(J,\mathbb{R}^+)$, with for any $x \in E$, $\|F(t,x)\| \leq \varsigma(t)$ for a.e. $t \in J$.*
   4. *If $q \in (0,\alpha)$, then there exists $\mu \in L^{\frac{1}{q}}(J,\mathbb{R}^+)$, $4L\|\mu\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} < 1$ with for any bounded subset B of E, we have $\chi(F(t,B)) \leq \mu(t)\chi(B)$ for a.e. $t \in J$, where $L = \dfrac{Mb^{\alpha-q}}{\Gamma(\alpha)(\omega+1)^{1-q}}$, $\omega = \dfrac{\alpha-1}{1-q}$.*

(Hg) *Let $g : PC(J,E) \to E$ be continuous, compact function and satisfies $\|g(x)\| \leq N$ on $PC(J,E)$, where N is a positive constant.*

(HI) *For every $i = 1,2\cdots,m$, let $I_i : E \to E$ be continuous compact functions with $\|I_i(x)\| \leq h_i\|x\|$ for all $x \in E$, where h_i is a positive constant.*

(Hr) *There is a positive constant r such that*

$$MN + M\sum_{i=1}^{m} h_i r + \frac{Mb^{(1+\varpi)(1-q)}}{\Gamma(1+\alpha)(1+\varpi)^{(1-q)}}\|\varsigma\|_{L^{\frac{1}{q}}_{([0,t],\mathbb{R}^+)}} \leq r. \tag{3.1}$$

Then the problem (Q) has a mild solution on J.

Proof. Let $\Delta : PC(J,E) \to 2^{L^1(J,E)}$ be defined by:

$$\Delta(x) = S^1_{F(\cdot,x(\cdot))} = \{f \in L^1(J,E) : f(t) \in F(t,x(t)),\ a.e.\ t \in J\}.$$

We show that Δ has a nonempty closed, lower semicontinuous and decomposable values. S^1_F is closed because F has closed value. From (HF)(3), F is integrably bounded so S^1_F is nonempty ([20, Theorem 3.2]). One can easily check that S^1_F is decomposable. Now, we will prove that Δ is l.s.c.. To do this, we need to prove that $x \to d(u,\Delta(x))$ is u.s.c. for every $u \in L^1(J,E)$. From ([20, Theorem 2.2]),

$$\begin{aligned} d(u,\Delta(x)) &= \inf_{f\in\Delta(x)} \|u-f\|_{L^1} \\ &= \inf_{f(t)\in F(t,x(t))} \int_0^b \|u(t)-f(t)\|dt \\ &= \int_0^b \inf_{f(t)\in F(t,x(t))} \|u(t)-f(t)\|dt \\ &= \int_0^b d(u(t),F(t,x(t))dt. \end{aligned} \tag{3.2}$$

Now, we will prove that for every $\delta \geq 0$ the set $u_\delta = \{x \in PC(J,E) : d(u,\Delta(x)) \geq \delta\}$ is closed. To prove that, let $\{x_n\}_{n\geq 1} \subseteq u_\delta$, $x_n \to x$ in $PC(J,E)$. So, for every $t \in J$, $x_n(t) \to x(t)$ in E. By (HF) (2) and Remark 2.12, we have $z \to d(u(t),F(t,z))$ is u.s.c.. Therefore, by Fatou Lemma and (3.2),

$$\begin{aligned} \delta &\leq \lim_{n\to\infty}\sup d(u,\Delta(x_n)) \\ &= \lim_{n\to\infty}\sup \int_0^b d(u(t),F(t,x_n(t))dt \\ &\leq \int_0^b \lim_{n\to\infty}\sup d(u(t),F(t,x_n(t))dt \\ &\leq \int_0^b d(u(t),F(t,x(t))dt \\ &= d(u,\Delta(x)). \end{aligned}$$

Then, $x \in u_\delta$. This means that $d(u,\Delta(x))$ is u.s.c. and hence by Remark 2.12, Δ is l.s.c.. Now, we apply Lemma 2.14 which follows that Δ has a continuous selection $f : PC(J,E) \to L^1(J,E)$ with $f(x) \in \Delta(x)$, for all $x \in PC(J,E)$. That concludes $f(x)(t) \in F(t,x(t))$, $a.e.$ $t \in J$. Further, let $G : PC(J,E) \to$

$PC(J,E)$ defined as follows:

$$G(x)(t)=\begin{cases}\mathcal{T}_\alpha(t)g(x)+\displaystyle\int_0^t(t-s)^{\alpha-1}\mathcal{S}_\alpha(t-s)f(x)(s)ds,\ t\in J_0,\\ \mathcal{T}_\alpha(t)g(x)+\displaystyle\sum_{k=1}^{k=i}\mathcal{T}_\alpha(t-t_k)I_k(x(t_k^-))\\ \quad+\displaystyle\int_0^t(t-s)^{\alpha-1}\mathcal{S}_\alpha(t-s)f(x)(s)ds,\ t\in J_i,\ 1\le i\le m,\end{cases}\tag{3.3}$$

where $f\in S^1_{F(\cdot,x(\cdot))}$. Obviously, any fixed point for G is a mild solution for the problem (Q). So, we will prove that G satisfies all the hypothesis of Theorem 2.16. The proof will be given in several steps. Let $W_0=\{x\in PC(J,E):\|x\|\le r\}$. Clearly, W_0 is bounded, convex and closed subset of $PC(J,E)$.

Step 1. We prove that $G(W_0)\subseteq W_0$. Let $x\in W_0$ and $t\in J$. Then, by using Lemma 2.6, (HF)(3), (Hg), (1) and Holder's inequality for $t\in J_0$,

$$\begin{aligned}\|G(x)(t)\|&\le\|\mathcal{T}_\alpha(t)g(x)+\int_0^t(t-s)^{\alpha-1}\mathcal{S}_\alpha(t-s)f(x)(s)ds\|\\&\le\|\mathcal{T}_\alpha(t)g(x)\|+\|\int_0^t(t-s)^{\alpha-1}\mathcal{S}_\alpha(t-s)f(x)(s)ds\|\\&\le MN+\frac{M}{\Gamma(1+\alpha)}\int_0^t(t-s)^{\alpha-1}\varsigma(s)ds\\&\le MN+\frac{M}{\Gamma(1+\alpha)}\frac{b^{(1+\omega)(1-q)}}{(1+\omega)^{(1-q)}}\|\varsigma\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)}\\&\le r,\end{aligned}$$

where $\omega=\dfrac{\alpha-1}{1-q}$. In addition, by using (HI) and for $t\in J_i$, $i=1,\cdots,m$, one can get by similar argument,

$$\|G(x)(t)\|\le MN+M\sum_{k=1}^{k=i}h_kr+\frac{M}{\Gamma(1+\alpha)}\frac{t^{(1+\omega)(1-q)}}{(1+\omega)^{(1-q)}}\|\varsigma\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)}\le r.$$

Then, $G(W_0)\subseteq W_0$. We define a sequence $W_n=\overline{conv}G(W_{n-1})$, $n\ge1$. By induction, one can easily show that $(W_n)_{n=1}^\infty$ is decreasing sequence. Now we prove that the set $W=\cap_{n=1}^\infty W_n$ is nonempty and compact. From Lemma 2.7, it is enough to prove that

$$\lim_{n\to\infty}\chi_{PC}(W_n)=0,\tag{3.4}$$

where χ_{PC} is defined in the previous section. Next, we prove (3.4) by step 2 and step 3.

Step 2. For each $n \in \mathbb{N}$, $J_i = (t_i, t_{i+1}]$ and $i = 0, 1, \cdots, m$, $W_{n|_{\overline{J_i}}}$ is equicontinuous, where

$$W_{n|_{\overline{J_i}}} = \{x^* \in C(\overline{J_i}, E) : x^*(t) = x(t),\ t \in J_i,\ x^*(t_i) = x(t_i^+), x \in W_n\}.$$

Without loss of generality we show that $W_{1|\overline{J_i}}$ is equicontinuous for all $0 \leq i \leq m$. Since $W_1 = \overline{conv}G(W_0)$, so it is enough to show that $G(W_0)_{|\overline{J_i}}$ is equicontinuous on $\overline{J_i}$. Let $x \in W_0$ and $y = G(x)$. Consider the following cases:

Case 1. If $i = 0$, we have the following subcases:

1. Let $t = 0$ and $\tau > 0$ with $t + \tau \in (0, t_1]$. By using (HA), Lemma 2.6 (vi) and Holder's inequality,

$$\begin{aligned}\|y^*(t+\tau) - y^*(t)\| &= \|y(\tau) - y(0)\| \\ &\leq \|\mathcal{T}_\alpha(\tau)g(x) - \mathcal{T}_\alpha(0)g(x)\| \\ &\quad + \|\int_0^\tau (\tau - s)^{\alpha-1}\mathcal{S}_\alpha(\tau - s)f(x)(s)ds\| \\ &\leq \|\mathcal{T}_\alpha(\tau) - \mathcal{T}_\alpha(0)\|\|g(x)\| + \frac{M}{\Gamma(\alpha)}\|\varsigma\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)}\frac{\tau^{\alpha-q}}{(\varpi+1)^{1-q}} \\ &\to 0\end{aligned}$$

as $\tau \to 0$. Therefore, independently on x we have,

$$\lim_{\tau\to 0}\|y^*(t+\tau) - y^*(t)\| = 0. \tag{3.5}$$

2. Let $t \in (0, t_1)$ and $\tau > 0$ provided that $t + \tau \in (0, t_1)$, then

$$\begin{aligned}\|y^*(t+\tau) - y^*(t)\| &= \|y(t+\tau) - y(t)\| \\ &\leq \|\mathcal{T}_\alpha(t+\tau)g(x) - \mathcal{T}_\alpha(t)g(x)\| \\ &\quad + \|\int_0^{t+\tau}(t+\tau-s)^{\alpha-1}\mathcal{S}_\alpha(t+\tau-s)f(x)(s)ds \\ &\quad - \int_0^t (t-s)^{\alpha-1}\mathcal{S}_\alpha(t-s)f(s)ds\| \\ &\leq G_1 + G_2 + G_3 + G_4,\end{aligned}$$

where

$G_1 = \|\mathcal{T}_\alpha(t+\tau)g(x) - \mathcal{T}_\alpha(t)g(x)\|$,

$G_2 = \|\int_0^t[(t+\tau-s)^{\alpha-1} - (t-s)^{\alpha-1}]\mathcal{S}_\alpha(t+\tau-s)f(x)(s)ds\|$,

$G_3 = \|\int_0^t(t-s)^{\alpha-1}[\mathcal{S}_\alpha(t+\tau-s) - \mathcal{S}_\alpha(t-s)]f(x)(s)ds\|$,

$G_4 = \|\int_t^{t+\tau}(t+\tau-s)^{\alpha-1}\mathcal{S}_\alpha(t+\tau-s)f(x)(s)ds\|$.

We will show that $G_i \to 0$ as $\tau \to 0$ for $i = 1, 2, 3, 4$. (HA) and Lemma 2.6 give

$$\begin{aligned}\lim_{\tau\to 0} G_1 &= \lim_{\tau\to 0} \|\mathcal{T}_\alpha(t+\tau)g(x) - \mathcal{T}_\alpha(t)g(x)\| \\ &\le \|g(x)\| \lim_{\tau\to 0} \|\mathcal{T}_\alpha(t+\tau) - \mathcal{T}_\alpha(t)\| \\ &= 0,\end{aligned}$$

independent on x. For G_2 and G_4, one can see the proof in details in ([22, Theorem 4]). For G_3, from the equicontinuity of $\{\mathcal{S}_\alpha(t) : t \in J\}$,

$$\begin{aligned}G_3 &\le \int_0^t \|(t-s)^{\alpha-1}[\mathcal{S}_\alpha(t+\tau-s) - \mathcal{S}_\alpha(t-s)]f(x)(s)\|ds \\ &\to 0, \quad \text{as } \tau \to 0.\end{aligned}$$

Therefore, we have

$$\lim_{\tau\to 0} \|y^*(t+\tau) - y^*(t)\| = 0. \tag{3.6}$$

3. When $t = t_1$. Let $\tau > 0$ and $\delta > 0$ provided that $t_1 + \tau \in J_1$ and $t_1 < \delta < t_1 + \tau \le t_2$. Then

$$\|y^*(t_1+\tau) - y^*(t_1)\| = \lim_{\delta\to t_1^+} \|y(t_1+\tau) - y(\delta)\|.$$

The definition of G implies that

$$\begin{aligned}\|y(t_1+\tau) - y(\delta)\| \le{}& \|\mathcal{T}_\alpha(t_1+\tau)g(x) - \mathcal{T}_\alpha(\delta)g(x)\| \\ &+ \sum_{k=1}^{k=i} \|\mathcal{T}_\alpha(t_1+\tau-t_k)I_k(x(t_k^-)) - \mathcal{T}_\alpha(\delta-t_k)I_k(x(t_k^-))\| \\ &+ \| \int_0^{t_1+\tau} (t_1+\tau-s)^{\alpha-1}\mathcal{S}_\alpha(t_1+\tau-s)f(x)(s)ds \\ &- \int_0^{\delta} (\delta-s)^{\alpha-1}\mathcal{S}_\alpha(\delta-s)f(x)(s)ds\|.\end{aligned}$$

With similar argument as in the previous way, we have

$$\lim_{\substack{\tau\to 0 \\ \delta\to t_1^+}} \|y(t_1+\tau) - y(\delta)\| = 0. \tag{3.7}$$

Case 2. In case of $1 \le i \le m$, use the same way as Case 1.

$$\lim_{\tau\to 0} \|y^*(t+\tau) - y^*(t)\| = 0. \tag{3.8}$$

From (3.5)-(3.8), $W_{1|_{\overline{J_i}}}$ is equicontinuous for every $0 \le i \le m$.

Step 3. We show that (3.4) is satisfied. Set $W = \cap_{n=1}^{\infty} W_n$. We want to prove that W is nonempty and compact in $PC(J, E)$. By light of Lemma 2.7, it is enough to show that $\lim_{n\to\infty}\chi_{PC}(W_n)=0$. By Lemma 2.10, for arbitrary $\varepsilon>0$ there exists a sequence $\{y_k\}_{k=1}^{\infty}$ in $G(W_{n-1})$ such that

$$\chi_{PC}(W_n)=\chi_{PC}G(W_{n-1})\leq 2\chi_{PC}\{y_k:k\geq 1\}+\varepsilon.$$

It follows from definition of χ_{PC} that

$$\chi_{PC}(W_n)\leq 2\max_{0\leq i\leq m}\chi_i(v_{|\overline{J_i}})+\varepsilon,$$

where $v=\{y_k:k\geq 1\}$. By using the equicontinuity of $W_{n|\overline{J_i}}$, $i=0,1,\cdots,m$, we can apply Lemma 2.8 and we get

$$\chi_i(v_{|\overline{J_i}})=\sup_{t\in\overline{J_i}}\chi(v(t)).$$

Hence, using the nonsinglarity of χ we get

$$\chi_{PC}(W_n)\leq 2\max_{i=0,1,\cdots,m}[\sup_{t\in\overline{J_i}}\chi(v(t))]+\varepsilon=2\sup_{t\in J}\chi(v(t))+\varepsilon.$$

Then, we have

$$\chi_{PC}(W_n)\leq 2\sup_{t\in J}\chi\{y_k(t):k\geq 1\}+\varepsilon. \tag{3.9}$$

Since $y_k\in G(W_{n-1}), k\geq 1$ there is $x_k\in W_{n-1}$ such that $y_k\in G(x_k), k\geq 1$. From the definition of G, (3.9) can be written as:

$$\chi_{PC}(W_n)\leq 2\sup_{t\in J}\chi\{y_k(t):k\geq 1\}$$

$$\leq\begin{cases}\chi(\mathcal{T}_\alpha(t)g(x_k))+\chi(\int_0^t(t-s)^{\alpha-1}\mathcal{S}_\alpha(t-s)f(x_k)(s)ds),\ t\in J_0,\\[2mm] \chi(\mathcal{T}_\alpha(t)g(x_k))+\sum_{j=1}^{j=i}\chi(\mathcal{T}_\alpha(t-t_j)I_j(x_k(t_j^-)))\\[2mm] \quad+\chi(\int_0^t(t-s)^{\alpha-1}\mathcal{S}_\alpha(t-s)f(x_k)(s)ds),\ t\in J_i.\end{cases}$$

Since, g and I_i for all $1\leq i\leq m$ are compact, Lemma 2.2 implies

$$\chi\{\mathcal{T}_\alpha(t)g(x_k):k\geq 1\}=0, \tag{3.10}$$

$$\chi\{\mathcal{T}_\alpha(t-t_j)I_j(x_k(t_j^-)):k\geq 1\}=0. \tag{3.11}$$

Hence, by (3.10) and (3.11) for every $t\in J$, we have

$$\chi_{PC}(W_n)\leq\varepsilon+2\sup_{t\in J}\chi\{\int_0^t(t-s)^{\alpha-1}\mathcal{S}_\alpha(t-s)f(x_k)(s)ds:k\geq 1\}.$$

Now, to estimate $\chi\{\int_0^t (t-s)^{\alpha-1}\mathcal{S}_\alpha(t-s)f_k(s)ds : k \geq 1\}$, take the linear continuous map $\Omega : L^{\frac{1}{q}}(J,E) \to C(J,E)$, where

$$\Omega(f)(t) = \int_0^t (t-s)^{\alpha-1}\mathcal{S}_\alpha(t-s)f(s)ds.$$

For any $f_1, f_2 \in L^{\frac{1}{q}}(J,E)$, and $t \in J$, Holder's inequality implies

$$\begin{aligned}
\|\Omega(f_1)(t) - \Omega(f_2\| &\leq \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f_1(s) - f_2(s)\|ds \\
&\leq \frac{M}{\Gamma(\alpha)}\|f_1 - f_2\|(\int_0^t (t-s)^{\frac{\alpha-1}{1-q}}ds)^{1-q} \\
&\leq L\|f_1 - f_2\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)},
\end{aligned} \tag{3.12}$$

where, $L = \dfrac{Mb^{\alpha-q}}{\Gamma(\alpha)(\varpi+1)^{1-q}}$. Let $z_k = f(x_k)(\cdot)$ for all $k \geq 1$. From (HF)(3), we have for almost $t \in J$, $\|f_k(t)\| \leq \varsigma(t) \in L^{\frac{1}{q}}(J,E)$. By using (HF)(4) and for $a.e.$ $t \in J$, we get

$$\begin{aligned}
\chi\{z_k(t) : k \geq 1\} &\leq \chi\{F(t,x_k) : k \geq 1\} \\
&\leq \mu(t)\chi\{x_k(t) : k \geq 1\} \\
&\leq \mu(t)\chi(W_{n-1}(t)) \\
&\leq \mu(t)\chi_{PC}(W_{n-1}) \\
&= \varrho(t) \in L^{\frac{1}{q}}(J,\mathbb{R}^+).
\end{aligned}$$

Lemma 2.15 guarantees the existence of a compact set $K_\varepsilon \subseteq E$, measurable set $J_\varepsilon \subset J$ with measure less than ε, and a sequence of functions $\{\zeta_k\} \subset L^{\frac{1}{q}}(J,E)$ such that for all $s \in J$, $\{\zeta_k(s) : k \geq 1\} \subseteq K_\varepsilon$, $\|z_k(s) - \zeta_k(s)\| < 2\varrho(s) + \varepsilon$, for every $s \in J - J_\varepsilon$. By using (3.12) we obtain for all $t \in J, k \geq 1$

$$\begin{aligned}
\|\Omega(z_k)(t) - \Omega(\zeta_k)(t)\| &\leq L\|z_k - \zeta_k\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} \\
&\leq L[\int_{J-J_\varepsilon} \|z_k(s) - \zeta_k(s)\|^{\frac{1}{q}}ds \\
&\quad + \int_{J_\varepsilon} \|z_k(s) - \zeta_k(s)\|^{\frac{1}{q}}ds]^q \\
&\leq L[\int_{J-J_\varepsilon} (2\varrho(s) + \varepsilon)^{\frac{1}{q}}ds + \int_{J_\varepsilon} \|z_k(s) - \zeta_k(s)\|^{\frac{1}{q}}ds]^q.
\end{aligned}$$

But ε is arbitrary, then for all $t \in J, k \geq 1$

$$\|\Omega(z_k)(t) - \Omega(\zeta_k)(t)\| \leq 2L\int_J \varrho(s)ds = 2L\chi_{PC}(W_{n-1})\|\mu\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)}. \tag{3.13}$$

Therefore, from (3.9), (3.10), (3.11) and (3.13) we get

$$\chi_{PC}(W_n) \le 4L\chi_{PC}(W_{n-1})\|\mu\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} + \epsilon.$$

Since ϵ is arbitrary, we find

$$\chi_{PC}(W_n) \le 4L\chi_{PC}(W_{n-1})\|\mu\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)}.$$

Clearly, after finite steps one can write

$$0 \le \chi_{PC}(W_n) \le (4L\|\mu\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)})^{n-1}\chi_{PC}(W_1).$$

By using (HF)(4), if we take the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \chi_{PC}(W_n) = 0.$$

Thus, $W = \cap_{n=1}^{\infty} W_n$ is nonempty and compact by Lemma 2.7.

Step 5. We will prove the continuity of G on W.

Let (x_n) be a sequence in W with $x_n \to x$ in $W \subset PC(J,E)$. Which follows that $\lim_{n\to\infty} x_n(t) = x(t)$, for $t \in J$. As consequence, for every $t \in J$, $\lim_{n\to\infty} f(x_n)(t) = f(x)(t)$. Now, for every $t, s \in J$ we have

$$\|(t-s)^{\alpha-1} f(x_n)(s)\| \le (t-s)^{\alpha-1}\varsigma(s) \in L^1(J,\mathbb{R}^+).$$

and

$$\|(t-s)^{\alpha-1} f(x)(s)\| \le (t-s)^{\alpha-1}\varsigma(s) \in L^1(J,\mathbb{R}^+).$$

Then, Lebesgue dominated convergence theorem concludes that

$$\lim_{n\to\infty} \int_0^t (t-s)^{\alpha-1}\|f(x_n)(s) - f(x)(s)\|ds = 0.$$

Therefore, if $t \in J_0$, continuity of g gives

$$\begin{aligned}\lim_{n\to\infty} \|G(x_n)(t) - G(x)(t)\| &\le \lim_{n\to\infty} M\|g(x_n) - g(x)\| \\ &\quad + \lim_{n\to\infty} \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\|f(x_n)(s) - f(x)(s)\|ds \\ &= 0.\end{aligned}$$

Similarly, if $t \in J_i,\ 1 \le i \le m$, then by the continuity of I_i, we get

$$\begin{aligned}\lim_{n\to\infty} \|G(x_n)(t) - G(x)(t)\| &\le \lim_{n\to\infty} M\|g(x_n) - g(x)\| \\ &\quad + M\sum_{k=1}^{k=i} \lim_{n\to\infty} \|I_k(x_n(t_k)) - I_k(x_n(t_k))\| \\ &\quad + \lim_{n\to\infty} \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\|f(x_n)(s) - f(x)(s)\|ds \\ &= 0.\end{aligned}$$

Hence G is continuous. Thus, $G : W \to W$ satisfies the hypotheses in Theorem 2.16, so there is $x \in W \subset PC(J, E)$ which is a mild solution for (Q). □

4. Example

We consider the following system:

$$\begin{cases} \partial_t^\alpha y(t,z) \in \partial_z^2 y(t,z) + R(t, y(t,z)), \ t \neq t_i, \\ y(t,0) = y(t,1) = 0, \\ y((\dfrac{i}{m+1})^+, z) = y(\dfrac{i}{m+1}, z) + \dfrac{1}{2^i}, \\ y(0,z) = \displaystyle\sum_{j=0}^{j=q} \int_0^1 k_j(z,v) tan^{-1}(y(s_j, v)) dv, \end{cases} \tag{4.1}$$

where q is a positive integer, $0 < s_0 < s_1 < \cdots < s_q < 1$, $i = 1, \cdots, m$, $t, z \in [0,1]$, $k_j \in C([0,1] \times [0,1], \mathbb{R})$, $j = 0, 1, \cdots, q$, ∂_t^α is the Caputo fractional partial derivative of order α, where $0 < \alpha < 1$ and $R : [0,1] \times E \to P(E)$.

In order to rewrite (4.1) in the abstract form, we put $E = L^2([0,1], \mathbb{R})$, and A is the Laplace operator, i.e., $A = \dfrac{\partial^2}{\partial z^2}$ on the domain $D(A) = \{x \in E : x, x'$ are absolutely continuous, and $x'' \in E, x(0) = x(1) = 0\}$. From [30], A generates an analytic and compact semigroup $\{T(t)\}_{t \geq 0}$ in E. This leads to A satisfies the assumption (HA).

For every $i = 1, \cdots, m$ define $I_i : E \to E$ by

$$I_i(x)(z) = \frac{1}{2^i}, \ z \in [0,1].$$

Note that the assumption (HI) is valid. For every $j = 0, 1, \cdots, q$, define $H_j : E \to E$ as

$$(H_j(x))(z) = \int_0^1 k_j(z,v) tan^{-1}(x(v)) dv, z \in [0,1].$$

Now take $g : PC([0,1], E) \to E$ as

$$g(x) = \sum_{j=0}^{j=q} H_j(x(s_j)).$$

Finally, let $F(t,x)(z) = R(t, x(z))$ and $x(t)(z) = x(t,z)$, where $z \in [0,1]$. Then, the system (4.1) takes the form:

$$\begin{cases} {}^cD^\alpha x(t) \in Ax(t) + F(t, x(t)) \ t \in J = [0,1], \ t \neq t_i, \\ x(t_i^+) = x(t_i) + I_i(x(t_i^-)), \\ x(0) = g(x). \end{cases}$$

If we put some conditions on F as in Theorem 3.1, then (4.1) possesses a mild solution on $[0, 1]$.

Conclusion

The present article discussed the existence of mild solutions of nonlocal impulsive differential inclusions in Banach space in case when the operator semigroup is not necessarily compact and the multivalued function is lower semicontinuous and nonconvex. We used methods and results of NCHM, and theorems of fixed point in order to determine sufficient conditions that guarantee the existence of mild solutions for (Q). The results given in this study developed and extended some previous results. An example was presented to support our main results.

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