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COMMON FIXED POINT OF ABSORBING MAPPING SATISFYING IMPLICIT RELATION

Urmila Mishra[1], Abhay S. Ranadive[2], A. K. Dubey[3] and Won Hee Lim[4]

[1]Department of Mathematics, Vishwavidyalaya Engineering College
Sant Gahira Guru University Sarguja
Lakhanpur-Ambikapur-497116(Chhattisgarh), India
e-mail: mishra.urmila@gmail.com

[2]Department of Mathematics, Bhilai Institute of Technology
Bhilai House, Durg (Chhattisgarh), India
e-mail: anilkumardbye@rediffmail.com

[3]Department of Pure and Applied Mathematics
Guru Ghasidas Vishwavidyalaya (A Central University)
Koni, Bilaspur-495009 (Chhattisgarh), India
e-mail: asranadive04@yahoo.co.in

[4]Department of Mathematics Education, Kyungnam University
Changwon, Gyeongnam, 51767, Korea
e-mail: worry36@kyungnam.ac.kr

Abstract. The aim of this paper is to improve certain results proved in a recent paper of Ali et al. [2], Aliouche [3], Djoudi et al. [4]. These results are the outcome of utilizing the idea of absorbing pairs due to Ranadive et al. [5] as opposed to conditions namely, weak compatibility. Some illustrative examples are also furnished to highlight the realized improvements.

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[0]Corresponding author: Urmila Mishra(mishra.urmila@gmail.com).

1. Introduction

Popa [9], introduce the notion of implicit relation and obtained many interesting results. Proving fixed point theorems using an implicit relation is fascinating, as it satisfies several contractive conditions rather than one contractive condition.

In this paper we use the concept of absorbing maps introduced by Ranadive et al. [5] in metric spaces for replacing commutativity at coincidence point as used by authors Ali et al. [2], Aliouche [3], Djoudi et al. [4], etc.

In [5], Ranadive et al. defined a new notion called absorbing maps, the definition of such mapping is given as:

Definition 1.1. ([8]) Let A and S be two self maps of a metric space (X, d). The map A is said to be S-absorbing if there exists $\mathcal{R} > 0$ such that

$$d(Sx, SAx) \leq \mathcal{R}\ d(Sx, Ax), \quad \text{for} \quad \text{all} \quad x \in X.$$

Similarly the map S is said to be A-absorbing if there exists $\mathcal{R} > 0$ such that

$$d(Ax, ASx) \leq \mathcal{R}\ d(Ax, Sx), \quad \text{for} \quad \text{all} \quad x \in X.$$

Similarly, we can defined point wise absorbing map. A mapping A is said to be point-wise S-absorbing if for given $x \in X$ there exists $\mathcal{R} > 0$ such that,

$$d(Sx, SAx) \leq \mathcal{R}\ d(Sx, Ax),$$

and S is said to be point-wise A-absorbing if for given $x \in X$ there exists $\mathcal{R} > 0$ such that

$$d(Ax, ASx) \leq \mathcal{R}\ d(Ax, Sx).$$

Now we give some examples which are illustrate the properties of absorbing maps. Following example shows that the class of absorbing maps is neither a sub class of compatible maps nor a sub class of non-compatible maps.

Example 1.2. ([8]) Let $X = [0, 1]$ be a metric space and d be the usual metric on X. Define $A, S : X \to X$ by

$$Ax = \frac{x}{16} \quad \text{and} \quad Sx = 1 - \frac{x}{3}.$$

Then we can see that

(i) A and S are compatible pair of maps.
(ii) A is S - absorbing while S is A - absorbing.

(Hint: Range of $A = [0, 1/16]$ and range of $S = [2/3, 1]$).

Thus we conclude that A and S can be compatible as well as absorbing also.

Next examples shows that A and S are absorbing but not compatible.

Example 1.3. ([8]) Let $X = [2, 20]$ be a metric space with usual metric d. We define mappings A and S by

(i) $Ax = 6\ if\ 2 \leq x \leq 5$; A6 = 6 ; Ax = 10 if $x > 6$; $Ax = \frac{(x-1)}{2}$ if $x \in (5, 6)$.

(ii) $Sx = 2\ if\ 2 \leq x \leq 5$; $Sx = \frac{(x+1)}{3}$ if $x > 5$.

Then it is easy to see that both pairs (A, S) and (S, A) are not compatible but A is S - absorbing and S is A - absorbing. $\Big[$Hint: Choose $x_n = \left(5 + \frac{1}{2n}\right)$ $: n \in N \Big]$.

Now we give an example to show that absorbing maps need not commute at their coincidence points. Thus, the notion of absorbing maps is different from other generalizations of commutativity which force the mappings to commute at coincidence points.

Example 1.4. ([8]) Let $X = [0, 1]$ and d be the usual metric on X, define $A, S : X \to X$ by $Ax = 1$ for $x \neq 1$, $A1 = 0$ and $Sx = 1$ for all $x \in X$. Then the map A is S- absorbing for any $\mathcal{R} > 1$ but the pair of maps (A, S) do not commute at their coincidence point $x = 0$.

Definition 1.5. Let A and S be two self maps of a fuzzy metric space (X, d) into itself. Pair of maps (A, S) are said to be weakly compatible if they commute at their coincidence point, that is $Ax = Sx$ implies that $ASx = SAx$.

Definition 1.6. ([6]) Let A and S be self maps of a metric space (X, d). They are compatible or asymptotically commuting if

$$\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$; for some $t \in X$.

It is known that a pair (A, S) of compatible maps is weakly compatible but converse is not true in general.

Definition 1.7. A pair of self maps (A, S) of a metric space (X, d) is said to be non-compatible if there exists at least one sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t,$$

for some $t \in X$, but $\lim_{n\to\infty} d(ASx_n, SAx_n)$ is either non zero or non existent.

Generalization of these notion is given by Amari-Moutawakil [1] as follows;

Definition 1.8. ([1]) A pair of self mappings (A, S) of a metric space (X, d) is said to satisfy the property (E. A.) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t, \quad \text{for} \quad \text{some} \quad t \in X.$$

Clearly a pair of non-compatible mappings satisfy the property (E. A.).

Definition 1.9. ([2]) Two pairs (A, S) and (B, T) of self-maps of a metric space (X, d) are said to satisfy the common property (E. A.) if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t, \quad \text{for} \quad \text{some} \quad t \in X.$$

Recently, in [2], Ali-Imdad gave implicit function as follows:

Let Ψ_6 be the family of lower semi-continuous functions $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(F_1) $F(u, 0, u, 0, 0, u) > 0$ for all $u > 0$,

(F_2) $F(u, 0, 0, u, u, 0) > 0$ for all $u > 0$,

(F_3) $F(u, u, 0, 0, u, u) > 0$ for all $u > 0$.

Now we give some examples for above implicit function;

Example 1.10. Define $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k \max\left[t_2t_3, t_3t_5, \frac{t_4 + t_6}{2}\right], \quad \text{where} \quad k \in (0, 1].$$

Example 1.11. Define $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - k_1 \max\left[t_3t_6, \frac{t_2^2}{2}, \frac{t_4^2}{2}\right] - k_2 \max\{t_3t_5, t_4t_6, t_2t_4\}$$

where $k_1, k_2 \in (0, 1)$.

Example 1.12. Define $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$\begin{aligned} F(t_1, t_2, t_3, t_4, t_5, t_6) &= (1 + \alpha t_2t_3)t_1 - \beta \max\{t_2t_3, t_3t_5, t_4t_6\} \\ &\quad -\gamma \max\{t_2, t_3, t_4, t_5, t_6\} \end{aligned}$$

where $\alpha, \beta, \gamma \geq 0$ and $\gamma < 1$.

Example 1.13. Define $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$\begin{aligned} F(t_1, t_2, t_3, t_4, t_5, t_6) &= t_1 - \alpha[\max(t_2^2, t_3t_4, t_3t_5, t_4t_6)]^{1/2} \\ &\quad -\beta[\max(t_2^3, t_1t_3t_6, t_1t_4t_5, t_2^2t_6, t_6^3)]^{1/3} \end{aligned}$$

where $\alpha + \beta < 1$, $\alpha \geq 0$ and $\beta \in (0, 1)$.

Example 1.14. Define $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - \alpha\frac{t_2t_3 + t_4t_5}{t_3t_5 + 1}, \qquad \alpha \in [0, 1).$$

Example 1.15. Define $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \alpha(t_2 + t_3) - \beta\frac{t_3^2 + t_4^2}{t_2 + t_4 + t_6}$$

where $\alpha, \beta \in (0, 1)$ and $\alpha + \beta < 1$.

Example 1.16. Define $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \max\left[t_2t_3, t_3t_5, \alpha(t_3t_4)^{1/2}, \frac{t_4 + t_6}{2}\right],$$

where $\alpha > 0$.

Example 1.17. Define $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \phi[\max(t_2^3, t_3^2t_4, t_3^2t_5, t_4^2t_6)].$$

where $\phi : \mathbb{R}_+^5 \to \mathbb{R}$ is an upper semi-continuous function such that, $\max\{\phi(0, 0, 0, t, 0), \phi(0, 0, 0, 0, t), \phi(t, 0, t, 0, 0)\} < t$ for each $t > 0$.

Example 1.18. Define $F(t_1, t_2, t_3, t_4, t_5, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = \begin{cases} t_1^2 - \alpha\max\{t_2t_3, t_4t_6\} - \beta\frac{t_4^3 + t_5^3}{t_4 + t_5} & \text{if} \quad t_4 + t_5 \neq 0 \\ t_1 \quad \text{if} \quad t_4 + t_5 = 0. \end{cases}$$

where $\alpha + \beta < 1$ and $\beta \in (0, 1)$.

2. Main Result

In this section, we obtain some unique common fixed point results which generalizes and extends the results of Ali et al. [2], Aliouche [3], Djoudi et al. [4].

Theorem 2.1. *Let A, B, S and T be four self maps on a complete metric space (X, d) satisfying:*

(1) *$S(X)$ and $T(X)$ are closed subsets of X,*
(2) *the pairs (A,S), (B,T) enjoy the common property (E.A.),*
(3) *for all $x,y \in X$ and $F \in \Psi_6$,*

$$F(d(Ax,By), d(Sx,Ty), d(Ax,Sx), d(By,Ty), d(Sx,By), d(Ty,Ax)) \leq 0.$$

Then the pairs (A,S) and (B,T) have coincidence point. Moreover if A is point wise S-absorbing and B is point wise T-absorbing then A,B,S and T have unique common fixed point.

Proof. Using (2) it can be seen that there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t,$$

for some $t \in X$. Since $S(X)$ is closed subset of X, $\lim_{n\to\infty} Sx_n = t \in S(X)$, therefore there exists a point $u \in X$ such that $Su = t$.

Now we claim that $Au = Su$, for if not then by condition (3) we have,

$$F(d(Au,By_n), d(Su,Ty_n), d(Au,Su), d(By_n,Ty_n), d(Su,By_n), d(Ty_n,Au)) \leq 0.$$

Letting $n \to \infty$, we get,

$$F(d(Au,t), d(Su,t), d(Au,t), d(t,t), d(t,t), d(t,Au)) \leq 0.$$

That is, $F(d(Au,t), 0, d(Au,t), 0, 0, d(t,Au)) \leq 0$.

So by (F_1) we get a contradiction. Hence $Au = Su$. Thus u is a coincidence point of A and S.

Now if $T(X)$ is closed subset of X, then $\lim_{n\to\infty} Ty_n = t \in T(X)$. Therefore there exists a point w in X such that $Tw = t$. We claim that $Bw = Tw$, for if not then by (3), we have,

$$F(d(Ax_n,Bw), d(Sx_n,Tw), d(Ax_n,Sx_n), d(Bw,Tw), d(Sx_n,Bw), d(Tw,Ax_n)) \leq 0.$$

Letting $n \to \infty$, we get,

$$F(d(t,Bw), d(t,Tw), d(t,t), d(Bw,Tw), d(t,Bw), d(Tw,t)) \leq 0.$$

That is,

$$F(d(Tw,Bw), 0, 0, d(Bw,Tw), d(Bw,Tw), 0) \leq 0,$$

which is a contradiction to (F_2). Hence $Bw = Tw$. Thus w is a coincidence point of the pair (B,T).

Since A is point wise S-absorbing, there exists $\mathcal{R} > 0$ such that,

$$d(Su, SAu) \leq \mathcal{R}\ d(Su, Au) \quad \text{i.e.} \quad Su = SAu \Rightarrow t = St.$$

Now we assert $At = t$, for if not then by inequality (3) we get,

$$F(d(At,By_n), d(St,Ty_n), d(At,St), d(By_n,Ty_n), d(St,By_n), d(Ty_n,At)) \leq 0.$$

Letting $n \to \infty$,

$$F(d(At,t), d(t,t), d(At,t), d(t,t), d(t,t), d(t,At)) \leq 0.$$

That is,

$$F(d(At,t),0,d(At,t),0,0,d(t,At)) \leq 0,$$

which is a contradiction to (F_2), hence $At = St = t$. Also, B is point wise T-absorbing, there exists $\mathcal{R} > 0$ (not necessarily the same as above) such that

$$d(Tw,TBw) \leq \mathcal{R}\ d(Tw,Bw) \quad \text{i.e.} \quad Tw = BTw \Rightarrow t = Tt.$$

Now suppose $t \neq Bt$, then by condition (3) we get,

$$F(d(Au,Bt),d(Su,Tt),d(Au,Su),d(Bt,Tt),d(Su,Bu),d(Tt,Au)) \leq 0$$

or

$$F(d(t,Bt),d(t,t),d(t,t),d(Bt,t),d(t,Bt),d(t,t)) \leq 0,$$

which is a contradiction to (F_2), thus $Bt = Tt = t$. Therefore we have $t = At = Bt = St = Tt$ and hence t is a common fixed point of A, B, S and T. Uniqueness follows from condition (3) and (F_3). □

Theorem 2.2. *Theorem* 2.1 *remains true if we replace condition* (1) *by following condition*

(1*) $\overline{A(X)} \subset T(X)$ *and* $\overline{B(X)} \subset S(X)$.

Theorem 2.3. *Let A, B, S and T be four self-maps on a complete metric space (X,d) satisfying:*

(1) *$A(X) \subset T(X)$ (or $B(X) \subset S(X)$)*
(2) *$S(X)$ (or $T(X)$) is closed subsets of X*
(3) *the pairs (A,S) and (B,T) enjoy the property (E.A.),*
(4) *for all $x,y \in X$ and $F \in \Psi_6$,*

$$F(d(Ax,By),d(Sx,Ty),d(Ax,Sx),d(By,Ty),d(Sx,By),d(Ty,Ax)) \leq 0.$$

Then the pairs (A,S) and (B,T) have coincidence point. Moreover if A is point wise S-absorbing and B is point wise T-absorbing then A, B, S and T have unique common fixed point.

Proof. The proof of this theorem can be obtain in a similar way of above Theorem 2.1. □

Corollary 2.4. *The conclusion of Theorem* 2.1 *remains true if inequality* (3) *is replaced by one of the following contraction conditions: For all $x,\ y \in X$ and $k \in (0,1]$,*

$$(1) \quad d(Ax,By) \leq k\ \max\{d(Sx,Ty)d(Ax,Sx), d(Ax,Sx)d(Sx,By), [d(By,Ty)+d(Ty,Ax)]/2\}.$$

$$(2) \quad d^2(Ax,By) \leq k_1\ \max\{d(Ax,Sx)d(Ty,Ax), d^2(Sx,Ty)/2, d^2(By,Ty)/2\} - k_2\ \max\{d(Ax,Sx)d(Sx,By), d(By,Ty)d(Ty,Ax),$$

$d(Sx,Ty)d(By,Ty)\}$, *where* $k_1, k_2 \in (0,1)$.

(3) $(1+\alpha\ d(Sx,Ty)d(Ax,Sx))d(Ax,By) \leq \beta\ \max\{d(Sx,Ty)d(Ax,Sx), d(Ax,Sx)d(Sx,By), d(By,Ty)d(Ty,Ax)\} - \gamma \max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), d(Sx,By), d(Ty,Ax)\}$,
where $\alpha, \beta, \gamma \geq 0$ *and* $\gamma < 1$.

(4) $d(Ax,By) \leq \alpha[\max\{d^2(Sx,Ty), d(Ax,Sx)d(By,Ty), d(Ax,Sx)d(Sx,By), d(By,Ty)d(Ty,Ax)\}]^{1/2} - \beta[\max\{d^3(Sx,Ty), d(Ax,By)d(Ax,Sx)d(Ty,Ax), d(Ax,By)d(By,Ty)d(Sx,By), d^2(Sx,Ty)d(Ty,Ax), d^3(Ty,Ax)\}]^{1/3}$,
where $\alpha+\beta < 1$, $\alpha \geq 0$ *and* $\beta \in (0,1)$.

(5) $d^2(Ax,By) \leq \alpha\ \frac{d(Sx,Ty)d(Ax,Sx)+d(By,Ty)d(Sx,By)}{d(Ax,Sx)d(Sx,By)+1}$, *where* $\alpha \in [0,1)$.

(6) $d(Ax,By) \leq \alpha[d(Sx,Ty)+d(Ax,Sx)] - \beta\ \frac{d^2(Ax,Sx)+d^2(By,Ty)}{d(Sx,Ty)+d(By,Ty)+d(Ty,Ax)}$,
where $\alpha, \beta \in (0,1)$ *and* $\alpha+\beta < 1$.

(7) $d(Ax,By) \leq \max\ \{d(Sx,Ty)d(Ax,Sx), d(Ax,Sx)d(Sx,By), \alpha(d(Ax,Sx)d(By,Ty))^{1/2}, \frac{d(By,Ty)+d(Ty,Ax)}{2}\}$,
where $\alpha > 0$.

(8) $d(Ax,By) \leq \phi\{\ \max[d(Sx,Ty), d^2(Ax,Sx)d(By,Ty), d^2(Ax,Sx)d(Sx,By), d^2(By,Ty)d(Ty,Ax)]\}$
where $\phi : \mathbb{R}^5_+ \to \mathbb{R}$ *is an upper semi continuous function such that,* $\max\{\phi(0,0,0,t,0), \phi(0,0,0,0,t), \phi(t,0,t,0,0)\} < t$ *for each* $t > 0$.

(9) $$d^2(Ax,By) \leq \begin{cases} \alpha\ \max\{d(Sx,Ty)d(Ax,Sx), d(By,Ty)d(Ty,Ax) \\ -\beta\ \frac{d^3(By,Ty)+d^3(Sx,By)}{d(By,Ty)+d(Sx,By)}\}, & \text{if } D \neq 0, \\ 0, & \text{if } D = 0. \end{cases}$$

Now we give an example which illustrates our Theorems 2.1, 2.2 and 2.3. We find a coincidence point of mapping pairs (A,S) and (B,T) and show that at that coincidence point they do not commute but they are point wise absorbing.

Example 2.5. Let $X = [2,20]$ be a metric space with usual metric d. Define A, B, S and T self mappings of (X,d) as;

$Ax = 2$ if $2 \leq x \leq 5$, $Ax = 3$ if $x > 5$;

$Bx = 2$ if $2 \leq x < 5$, $Bx = 3$ if $x \in [5,20]$;

$S2 = 2$, $Sx = 3$ if $2 < x \leq 5$, $Sx = \dfrac{x+1}{3}$ if $x > 5$ and

$T2 = 2$, $T3 = 3$, $Tx = 12 + x$ if $3 < x < 8$, $Tx = x - 5$ if $x \geq 8$.

Clearly, both the pairs (A, S) and (B, T) satisfy the common property $(E.A.)$ as there exists two sequences $\{x_n = 8 + \frac{1}{n}\}$, $\{y_n = 8 + \frac{1}{n}\} \in X$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 3.$$

Also $A(x) = \{2, 3\} \subset [2, 20] = T(X)$ and $B(X) = \{2, 3\} = [2, 7] \bigcup \{3\} = S(X)$. Define $F(t_1, t_2, \ldots, t_6) : \mathbb{R}^6_+ \to \mathbb{R}$ with $t_3 + t_4 \neq 0$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \alpha t_2 - \beta\Big[\frac{t_3^2 + t_4^2}{t_3 + t_4}\Big] - \gamma[t_5 + t_6]$$

where $\alpha, \beta, \gamma \geq 0$ with at least one is nonzero and $\beta + \gamma < 1$.

By routine calculation we can verify that contraction condition of theorem is satisfied for $\alpha = 1, \quad \beta = \frac{1}{4}, \quad \gamma = \frac{1}{2}$. If $x,\ y \in [2, 5)$ then $d(Ax, By) = 0$ and also if $x,\ y \in (5, 20]$ then $d(Ax, By) = 0$ and verification is of contractive condition is trivial. If $x = 5$ and $y \in [5, 8)$, then

$$\alpha d(Sx, Ty) + \beta\Big[\frac{d^2(Sx, Ax) + d^2(Ty, By)}{d(Sx, Ax) + d(Ty, By)}\Big] + \gamma[d(Sx, By) + d(Ty, Ax)]$$

$$= 1.|3 - 12 - y| + \frac{1}{4}\Big[\frac{|3 - 2|^2 + |12 + y - 3|^2}{|3 - 2| + |12 + y - 3|}\Big] + \frac{1}{2}[|3 - 3| + |12 + y - 2|\ ]$$

$$= | - 9 - y| + \frac{1}{4}\Big[\frac{1 + |9 + y|^2}{1 + |9 + y|}\Big] + \frac{1}{2}[|10 + y|\ ] \geq 1 = d(Ax, By).$$

Similarly, one can verify the other cases. Thus all the conditions of above Theorems are satisfied and 2 is the unique common fixed point of A, B, S and T. Here one may notice that all the mappings in this example are even discontinuous at their unique common fixed point 2.

Mapping pairs (A, S) and (B, T) are non-commuting: In this example, '2' is the common coincidence point of mappings A, B, S and T and since it is a fixed point of mappings then at this point mapping pairs (A, S) and (B, T) commute necessarily. However, 8 is the coincidence point of mapping pair (A, S) and (B, T), on which they do not commute, as $A8 = S8 = 3$ but $AS8 = A3 = 2$ and $SA8 = S3 = 3$, i.e., $SA8 \neq AS8$. Similarly $T8 = B8 = 3$ but $TB8 = T3 = 3$ and $BT8 = B3 = 2$, i.e., $BT8 \neq TB8$. But it is easy to verify that at 8, mapping pairs are point wise absorbing. Thus our theorems improve the result of Ali et al.[2] and similar results too.

3. Conclusion

In this attempt we prove common fixed point theorems by using the new notion called absorbing mappings which generalizes and improves the results of Ali et al. [2], Aliouche [3], Djoudi et al. [4]. Our results are the outcome of utilizing the idea of absorbing pair of mappings. Since implicit functions implies several contraction conditions, employing these contraction conditions we can get many results as corollary of our main theorem. In the last we have given one interesting example in support of our theorem.

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