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AN ITERATIVE SOLUTION OF A UNIFORMLY CONTINUOUS ACCRETIVE OPERATOR EQUATION

Niyati Gurudwan[1] and B. K. Sharma[2]

[1]School of Studies in Mathematics, Pt. Ravishankar Shukla University
Raipur,(C.G.) 492010, India,
email: niyati.kuhu@gmail.com

[2]School of Studies in Mathematics, Pt. Ravishankar Shukla University
Raipur,(C.G.) 492010, India,
email: sharmabk07@gmail.com

Abstract. A strong convergence theorem for the zero of a uniformly continuous accretive operator in a real normed space is proved using the iteration formula

$$x_{n+1} = x_n - \lambda_n \alpha_n A x_n - \lambda_n \theta_n (x_n - x_1), \quad \forall\, n \geq 1$$

where $\{\alpha_n\}_{n=1}^{\infty}, \{\lambda_n\}_{n=1}^{\infty}$ and $\{\theta_n\}_{n=1}^{\infty}$ are real sequences in $(0,1)$ satisfying certain conditions given by Chidume and Zegeye [9]. Similar result for uniformly continuous pseudocontractive map is also proved. Our result modifies the convergence results of Chidume and Ofoedu [7] and many others.

1. Introduction and Preliminaries

Let E be a real Banach space with dual E^*. The *normalized duality mapping* from E to 2^{E^*} is defined by

$$J(x) := \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2, \|x\| = \|x^*\|\},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between the elements of E and E^*.

Definition 1.1. *A mapping $A\colon D(A) \subseteq E \to E$ is said to be* accretive [2] *if for all $x, y \in E$, there exists $j(x-y) \in J(x-y)$ such that*

$$\langle Ax - Ay, j(x-y) \rangle \geq 0.$$

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When E is a Hilbert space, accretive operators are also called *monotone*. Let K be a nonempty subset of E.

Definition 1.2. *The mapping $T : K \to K$ is called pseudocontractive* [3] *if for all $x, y \in K$, there exists $j(x-y) \in J(x-y)$ such that*

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2.$$

In the sequel, we will need the following results.

Lemma 1.3. [15] *Let E be a real Banach space. Then for all $x, y \in E$, there exists $j(x+y) \in J(x+y)$ such that*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle.$$

2. **Main Results**

Theorem 2.1. *Let E be a real Banach Space and $A\colon E \to E$ be a uniformly continuous accretive operator such that $N(A) := \{x \in E : Ax = 0\} \neq \phi$. Suppose $\{\alpha_n\}_{n=1}^{\infty}, \{\lambda_n\}_{n=1}^{\infty}$ and $\{\theta_n\}_{n=1}^{\infty}$ be real sequences in $(0,1)$ satisfying the conditions*

$$\lim_{n\to\infty} \lambda_n = 0, \quad \alpha_n = o(\theta_n). \tag{2.1}$$

Let the sequence $\{x_n\}_{n=1}^{\infty}$ be generated iteratively from arbitrary $x_1 \in E$, by

$$x_{n+1} = x_n - \lambda_n \alpha_n A x_n - \lambda_n \theta_n (x_n - x_1), \quad \forall\, n \geq 1. \tag{2.2}$$

Then $\{x_n\}_{n=1}^{\infty}$ is bounded.

Proof. Let $x^* \in N(A)$. If $x_n = x^*$, for all $n \geq 1$, then the theorem is proved. So, let N_* be the first smallest integer such that $x_{N_*} \neq x^*$. So that there exists $N_0 \geq N_*$ and $r > 0$ be sufficiently large such that

$$x_{N_0} \in \overline{B}_r(x^*) = B := \{x \in E : \|x - x^*\| \leq r\},$$

and $x_1 \in B_{\frac{r}{2}}(x^*)$. In order to prove that $\{x_n\}_{n=1}^{\infty}$ is bounded, it is sufficient to show that $x_n \in B = \overline{B}_r(x^*)$, for all integers $n \geq N_0$, by induction. Now by our construction, $x_{N_0} \in B$. So we next assume that $x_n \in B$ for some $n > N_0$ and we shall prove that $x_{n+1} \in B$. Let $x_{n+1} \notin B$, that is,

$$\|x_{n+1} - x^*\| > r.$$

Then by (2.2) and Lemma 1.3, we have

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|x_n - x^* - \lambda_n(\alpha_n A x_n + \theta_n(x_n - x_1))\|^2 \\
&\le \|x_n - x^*\|^2 - 2\lambda_n \langle \alpha_n A x_n + \theta_n(x_n - x_1), j(x_{n+1} - x^*)\rangle \\
&\le \|x_n - x^*\|^2 - 2\lambda_n\alpha_n \langle A x_{n+1}, j(x_{n+1} - x^*)\rangle - 2\lambda_n\theta_n\|x_{n+1} - x^*\|^2 \\
&\quad + 2\lambda_n\alpha_n\langle A x_{n+1} - A x_n, j(x_{n+1} - x^*)\rangle + 2\lambda_n\theta_n\langle x_{n+1} - x_n, j(x_{n+1} - x^*)\rangle \\
&\quad + 2\lambda_n\theta_n\langle x_1 - x^*, j(x_{n+1} - x^*)\rangle \\
&\le \|x_n - x^*\|^2 - 2\lambda_n\alpha_n \langle A x_{n+1}, j(x_{n+1} - x^*)\rangle - 2\lambda_n\theta_n\|x_{n+1} - x^*\|^2 \\
&\quad + 2\lambda_n\theta_n\|x_1 - x^*\|\|x_{n+1} - x^*\| + 2\lambda_n\theta_n\|x_{n+1} - x_n\|\|x_{n+1} - x^*\| \\
&\quad + 2\lambda_n\alpha_n\|A x_{n+1} - A x_n\|\|x_{n+1} - x^*\| \qquad (2.3)
\end{aligned}
$$

Since A is a bounded operator, so we can define

$$M_0 := \sup\{\|x - Ax\| : \|x - x^*\| \le 4r\}$$

and by uniform continuity of A, for given $\epsilon_0 > 0$, there exists $\delta > 0$ such that

$$\|Ax - Ay\| < \epsilon_0, \text{ whenever } \|x - y\| < \delta.$$

Define

$$\gamma_0 := \frac{1}{2}\min\left\{1, \frac{\delta}{M_0 + \frac{3}{2}r}\right\}$$

and let $\lambda_n \le \frac{r}{8(M_0 + \frac{3}{2}r)}$, $\frac{\alpha_n}{\theta_n} \le \frac{r}{4\epsilon_0}$, $\forall n \ge N_0$.

Also we have

$$
\begin{aligned}
\|x_{n+1} - x_n\| &\le \lambda_n\alpha_n\|x_n - Ax_n\| + \lambda_n\theta_n\|x_n - x_1\| \\
&\le \lambda_n\left[\alpha_n M_0 + \theta_n\|x_n - x^*\| + \theta_n\|x_1 - x^*\|\right] \\
&\le \lambda_n\left[\alpha_n M_0 + \frac{3}{2}\theta_n r\right] \\
&\le \gamma_0[M_0 + \frac{3}{2}r] \\
&\le \delta,
\end{aligned}
$$

which implies that

$$\|Ax_{n+1} - Ax_n\| < \epsilon_0.$$

Again since A is accretive, so

$$\langle Ax_{n+1}, j(x_{n+1} - x^*)\rangle \ge 0.$$

Hence using the above estimates, equation (2.3) becomes

$$\begin{aligned}\|x_{n+1}-x^*\|^2 \le \|x_n-x^*\|^2 &+ 2\lambda_n\alpha_n\epsilon_0\|x_{n+1}-x^*\| - 2\lambda_n\theta_n\|x_{n+1}-x^*\|^2\\ &+ 2\lambda_n^2\theta_n\left[\alpha_n M_0+\theta_n\|x_n-x^*\|+\theta_n\|x_1-x^*\|\right]\|x_{n+1}-x^*\|\\ &+ 2\lambda_n\theta_n\|x_{n+1}-x^*\|\|x_1-x^*\|.\end{aligned}$$

Since $\|x_{n+1}-x^*\| > \|x_n-x^*\|$ by our assumption,

$$2\lambda_n\theta_n\|x_{n+1}-x^*\| \le 2\lambda_n\alpha_n\epsilon_0 + 2\lambda_n\theta_n\frac{r}{2} + 2\lambda_n^2\theta_n\left(\alpha_n M_0+\frac{3}{2}\theta_n r\right)$$

where $x_n \in B$ and $x_1 \in B_{\frac{r}{2}}(x^*)$. This implies that

$$\begin{aligned}\|x_{n+1}-x^*\| &\le \frac{\alpha_n}{\theta_n}\epsilon_0 + \lambda_n\left(M_0+\frac{3}{2}r\right)+\frac{r}{2}\\ &\le \frac{r}{4}+\frac{r}{2}+\frac{r}{8}\\ &< r,\end{aligned}$$

which is a contradiction of our assumption that $x_{n+1} \notin B$. Hence, $x_n \in B$ for all $n \ge N_0$, which implies that $\{x_n\}_{n=1}^\infty$ is bounded. □

Theorem 2.2. *Let E be a real normed linear space and let $A : E \to E$ be a uniformly continuous accretive operator with $N(A) \ne \phi$. Let $x_1 \in E$ be fixed and $\{x_n\}_{n=1}^\infty$ be generated iteratively by*

$$x_{n+1} = x_n - \lambda_n(\alpha_n Ax_n + \theta_n(x_n - x_1)),\ n \ge 1, \tag{2.4}$$

where $\{\lambda_n\}_{n=1}^\infty$, $\{\theta_n\}_{n=1}^\infty \in (0,1)$ satisfy the conditions (2.1) of Theorem 2.1 with $\sum_{n=1}^\infty \lambda_n\theta_n = \infty$. Then $\{x_n\}_{n=1}^\infty$ converges strongly to the unique element $x^* \in N(A)$.

Proof. The existence of a solution for the equation $Ax = 0$, for a continuous accretive operator A follows from [13] and the uniqueness is obvious from the definition of accretivity. Let x^* be the unique solution and set

$$M_1 := 2\sup_n \|x_1-x^*\|\|x_{n+1}-x^*\|,$$

$$M_2 := 2\sup_n\{\|x_n-Ax_n\|+\|x_n-x_1\|\}\|x_{n+1}-x^*\|$$

and

$$M_3 := 2\sup\|x_{n+1}-x^*\|.$$

Using (2.4) and Lemma 1.3 again, we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\le \|x_n - x^*\|^2 - 2\lambda_n\alpha_n\langle Ax_{n+1}, j(x_{n+1} - x^*)\rangle - 2\lambda_n\theta_n\|x_{n+1} - x^*\|^2 \\
&\quad + 2\lambda_n\theta_n\langle x_{n+1} - x_n, j(x_{n+1} - x^*)\rangle + 2\lambda_n\theta_n\langle x_1 - x^*, j(x_{n+1} - x^*)\rangle \\
&\quad + 2\lambda_n\alpha_n\langle Ax_{n+1} - Ax_n, j(x_{n+1} - x^*)\rangle \\
&\le \|x_n - x^*\|^2 - 2\lambda_n\theta_n\|x_{n+1} - x^*\|^2 + 2\lambda_n\theta_n\|x_1 - x^*\|\|x_{n+1} - x^*\| \\
&\quad + 2\lambda_n^2\theta_n[\alpha_n\|x_n - Ax_n\| + \theta_n\|x_n - x_1\|]\|x_{n+1} - x^*\| \\
&\quad + 2\lambda_n\alpha_n\|Ax_{n+1} - Ax_n\|\|x_{n+1} - x^*\|.
\end{aligned}$$

Thus

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 \le &\|x_n - x^*\|^2 - 2\lambda_n\theta_n\|x_{n+1} - x^*\|^2 \\
&+ \lambda_n M_3\|Ax_{n+1} - Ax_n\| + \lambda_n^2\theta_n M_2 + \lambda_n M_1.
\end{aligned} \tag{2.5}$$

Next we claim that $\inf\{\|x_{n+1} - x^*\|; n \ge 0\} = 0$. Let

$$\inf\{\|x_{n+1} - x^*\|; n \ge 0\} = \delta > 0.$$

Then we have,

$$\|x_{n+1} - x^*\| > \delta,\ \forall\, n \ge 0.$$

Also, since

$$\|x_{n+1} - x^*\|\lambda_n[\alpha_n\|x_n - Ax_n\| + \theta_n\|x_n - x_1\|] \to 0 \text{ as } n \to \infty,$$

so that by uniform continuity of A, there exists $N_0 > 0$ such that

$$\|Ax_{n+1} - Ax_n\| < \frac{\delta^2}{M_3},\ \forall\, n \ge N_0$$

. Hence, for all $n \ge N_0$

$$\|x_{n+1}\| \le \|x_n - x^*\|^2 - 2\lambda_n\theta_n\delta^2 + \lambda_n\delta^2 + \lambda_n^2\theta_n M_2 + \lambda_n M_1$$

and

$$\lambda_n\theta_n\delta^2 \le (\|x_n - x^*\|^2 - \|x_{n+1}\|) + \lambda_n(\delta^2 + M_1) + \lambda_n^2\theta_n M_2,$$

that is,

$$\delta^2\sum_{i=1}^{n}\lambda_i\theta_i \le \sum_{i=1}^{n}(\|x_i - x^*\|^2 - \|x_{i+1}\|) + \sum_{i=1}^{n}\lambda_i(\delta^2 + M_1 + M_2),$$

which implies that $\sum_{i=1}^{n}\lambda_i\theta_i < \infty$, which is a contradiction for $\sum_{i=1}^{n}\lambda_i\theta_i = \infty$. Hence our claim is true. Thus, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{n\to\infty}\|x_{n_j} - x^*\| = 0$.

Let $\epsilon > 0$ be given. Since $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$ and A is uniformly continuous, we can choose an integer $N_1 > 0$ such that $\forall n \geq N_1$,

$$\|Ax_{n+1} - Ax_n\| \leq \frac{\epsilon}{M_3}.$$

Now choose an integer $N_2 > N$, such that $\|x_{n_i} - x^*\| < \epsilon$, $\forall\, i \geq N_2$.
Fix $i_* \geq N_2$. Then

$$\|x_{n_{i_*}} - x^*\| < \epsilon.$$

We next claim that $\|x_{n_{i_*}+m} - x^*\| < \epsilon$, $\forall\ m = 1, 2, \cdots$. We prove it by induction on m. So we first show that $\|x_{n_{i_*}+1} - x^*\| < \epsilon$. Suppose this is not true. Then, $\|x_{n_{i_*}+1} - x^*\| \geq \epsilon$.

On the other hand, we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \leq &\|x_{n+1} - x^*\| - 2\lambda_n\theta_n\|x_{n+1} - x^*\|^2 \\ &+ \lambda_n M_3\|Ax_{n+1} - Ax_n\| + \lambda_n^2\theta_n M_2 + \lambda_n M_1.\end{aligned}$$

So that

$$\|x_{n_{i^*}+1} - x^*\|^2 \leq \|x_{n_{i^*}} - x^*\| - 2\lambda_n\theta_n\epsilon^2 + \lambda_n\epsilon^2 + \lambda_n^2\theta_n M_2 + \lambda_n M_1$$

or

$$2\lambda_n\theta_n\epsilon^2 \leq \|x_{n_i - x^*}\| - \|x_{n_j - x^*}\| + \frac{3}{2}\epsilon^2\lambda_n.$$

This implies that

$$\begin{aligned}2\epsilon^2\sum_{i=1}^{n}\lambda_n\theta_n &\leq (\|x_{n_i - x^*}\| - \|x_{n_j - x^*}\|) + \frac{3}{2}\epsilon^2\sum_{i=1}^{n}\lambda_n \\ &< \infty,\end{aligned}$$

which is a contradiction. Hence, the claim holds for $m = 1$. Assume now it holds for $m = k$. Following the similar arguments as above, we can show that the claim holds for $m = k + 1$ also. Hence it is true for all the values of m. This implies that $\{x_n\}$ converges strongly to x^* as $n \to \infty$. □

Remark 2.3. *Our result modifies the corresponding results of* [6, 7, 20] *and the references therein, to uniformly continuous accretive operators with weaker conditions on the parameters. Also the results of* [10, 9, 17] *and many others are extended to a more general reflexive Banach space.*

Next we prove the strong convergence for pseudocontractive mappings.

Theorem 2.4. *Let E be a real normed linear space and $T : E \to E$ be a uniformly continuous pseudocontractive mapping such that*

$$F(T) = \{x \in E : Tx = x\} \neq \phi.$$

Let $x_1 \in E$ be fixed and $\{x_n\}_{n=1}^{\infty}$ be generated iteratively by

$$x_{n+1} = (1 - \lambda_n\alpha_n)x_n + \lambda_n(\alpha_n Tx_n - \theta_n(x_n - x_1)),\ n \geq 1, \tag{2.6}$$

where $\{\lambda_n\}_{n=1}^{\infty}$, $\{\theta_n\}_{n=1}^{\infty} \in (0,1)$ *satisfy the conditions (2.1) of Theorem 2.1 with* $\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty$. *Then* $\{x_n\}_{n=1}^{\infty}$ *converges strongly to some* $x^* \in F(T)$.

Proof. We first observe that T is pseudocontractive if and only $A\colon = I - T$ is accretive [2]. Again, $x^* \in K_{\min} \bigcap F(T)$ implies $Tx = x$, which again implies $Ax = 0$. Thus $K_{\min} \bigcap N(A) \neq \phi$. Clearly, T is also continuous. Hence, replacing T by $I - A$ in (2.4), then boundedness of $\{x_n\}_{n=1}^{\infty}$ follows from Theorem 2.1 and rest of the result follows from Theorem 2.2. □

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