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T-CONTRACTIVE MAPPING AND FIXED POINT THEOREMS IN CONE METRIC SPACES WITH C-DISTANCE

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Abstract. A new concept of the c-distance in cone metric space has been introduced recently in 2011. In this paper, we prove some fixed point theorems for different type of T-contractive conditions under c-distance in cone metric spaces. Our result improves and generalizes several well-known results in literature.

1. Introduction

The famous Banach contraction principle is the first important result on fixed points for contractive type mappings. There are many generalizations of Banach's contraction mapping principle in the literature. The concept of a cone metric space was introduced in the work of Huang and Zhang [12] which is more general than the concept of a metric space. They introduced cone metric space without c-distance.

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Recently, Cho et al. [2] introduced the concept of c-distance in a cone metric spaces and proved some fixed point theorems in ordered cone metric spaces. Then several authors have proved fixed point theorems for c-distance in cone metric spaces (see [6, 7, 8, 9, 10, 11, 14, 15] and [16]). In 2009, Beiranvand et al.[1] introduced new classes of contractive functions and established the Banach contraction principle. Since then, fixed point theorems for different classes of mappings on cone metric spaces have been appeared, see for instance [3], [4] and [5].

In this paper, we prove the existence of a unique fixed point for T-contractive mapping under the concept of c-distance in the setting of complete cone metric spaces. Throughout this paper, we do not impose the normality condition for the cones, but the only assumption is that the cone P is solid, that is $intP \neq \phi$.

2. Preliminaries

Let E be a real Banach space and θ denote to the zero element in E. A cone P is a subset of E such that:

(1) P is a nonempty, closed and $P \neq \{\theta\}$;
(2) If a, b are non-negative real numbers and $x, y \in P$ then $ax + by \in P$;
(3) $x \in P$ and $-x \in P \Rightarrow x = \theta$.

Given a cone $P \subseteq E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$, we write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, $intP$ denotes the interior of P.

A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E, \theta \preceq x \preceq y$ implies $||x|| \leq K||y||$. The least positive number K satisfying above is called the normal constant of P.

In the following we always suppose that E is a Banach space, P is a cone in E with $intP \neq \phi$ and $\preceq$ is partial ordering with respect to P.

Definition 2.1. ([12]) Let X be a nonempty set and E be a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone P. Suppose that the mapping $d : X \times X \to E$ satisfies the following conditions:

(i) If $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \preceq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Example 2.2. Let $E = \mathbb{R}^2$, where $\mathbb{R}$ is a set of real numbers and $P = \{(x, y) \in E : x, y \geq 0\} \subset \mathbb{R}^2$, $X = \mathbb{R}^2$ and suppose that $d : X \times X \to E$ is defined by $d(x, y) = d((x_1, x_2), (y_1, y_2)) = (|x_1 - y_1| + |x_2 - y_2|, \alpha \max\{|x_1 - y_1|, |x_2 - y_2|\})$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space. It is easy to see that d is a cone metric, and hence (X, d) becomes a cone metric space over (E, P). Also, we have P is a solid and normal cone where the normal constant $K = 1$.

Definition 2.3. ([12]) Let (X, d) be a cone metric space, let $\{x_n\}$ be a sequence in X and $x \in X$:

(1) The sequence $\{x_n\}$ is said to be convergent to x, if for all $c \in E$ with $\theta \ll c$, there exists a positive integer N such that $d(x_n, x) \ll c$ for all $n > N$. In this case, we denote $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
(2) The sequence $\{x_n\}$ is called Cauchy in X, if for all $c \in E$ with $\theta \ll c$, there exists a positive integer N such that for all $n, m > N$, $d(x_n, x_m) \ll c$.
(3) (X, d) is called a complete cone metric space, if every Cauchy sequence in X is convergent in X.

The following lemma is useful to prove our results.

Lemma 2.4. ([13])

(1) *If E be a real Banach space with a cone P and $a \preceq \lambda a$ where $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.*
(2) *If $c \in intP$, $\theta \preceq a_n$ and $a_n \to \theta$, then there exists a positive integer N such that $a_n \ll c$ for all $n \geq N$.*

Next, we give the notion of c-distance on a cone metric space (X, d) of Cho et al. in [2].

Definition 2.5. Let (X, d) be a cone metric space. A function $q : X \times X \to E$ is called a c-distance on X if the following conditions hold:

(q_1) $\theta \preceq q(x, y)$ for all $x, y \in X$,
(q_2) $q(x, z) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$,
(q_3) for each $x \in X$ and $n \geq 1$ if $q(x, y_n) \preceq u$ for some $u = u_x \in P$, then $q(x, y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$,
(q_4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Example 2.6. ([2]) Let $E = \mathbb{R}$ and $P = \{x \in E : x \geq 0\}$, $X = [0, \infty)$ and a mapping $d : X \times X \to E$ be defined by $d(x, y) = |x - y|$, for all $x, y \in X$. Then (X, d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = y$ for all $x, y \in X$. Then q is a c-distance on X.

Example 2.7. ([9, 10]) Let $E = \mathbb{R}^2$ and $P = \{(x, y) \in E : x, y \geq 0\}$. Let $X = [0, 1]$ and a mapping $d : X \times X \to E$ be defined by $d(x, y) = (|x-y|, |x-y|)$, for all $x, y \in X$. Then (X, d) is a complete cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = (y, y)$ for all $x, y \in X$. Then q is a c-distance on X.

The following lemma is very important to prove our results.

Lemma 2.8. ([2]) *Let (X, d) be a cone metric space and q be a c-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y, z \in X$. Suppose that $\{u_n\}$ is a sequence in P converging to θ. Then the following hold:*

(1) *If $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $y = z$.*
(2) *If $q(x_n, y_n) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $\{y_n\}$ converges to z.*
(3) *If $q(x_n, x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.*
(4) *If $q(y, x_n) \preceq u_n$, then $\{x_n\}$ is a Cauchy sequence in X.*

Remark 2.9. ([2])

(1) $q(x, y) = q(y, x)$ does not necessarily for all $x, y \in X$.
(2) $q(x, y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

Next definition taken from [1]:

Definition 2.10. Let (X, d) be a cone metric space, P a solid cone and $T : X \to X$. Then

(a) T is said to be continuous if $\lim_{n\to\infty} x_n = x^*$ implies that $\lim_{n\to\infty} Tx_n = Tx^*$ for all $\{x_n\}$ in X,
(b) T is said to be sequentially convergent if for every sequence $\{x_n\}$, $\{Tx_n\}$ is convergent, then $\{x_n\}$ is also convergent,
(c) T is said to be subsequentially convergent if for every sequence $\{x_n\}$, $\{Tx_n\}$ is convergent, then $\{x_n\}$ has a convergent subsequence.

3. Main Results

Now we are ready to state and prove our main results.

Theorem 3.1. *Let (X, d) be a complete cone metric space, P a solid cone and q be a c-distance on X. In addition let $T : X \to X$ be an one to one, continuous and subsequentially convergent function and $f : X \to X$ be a mapping satisfies the contractive condition:*

$$\begin{aligned} q(Tfx, Tfy) \;\preceq\;& a_1 q(Tx, Ty) + a_2 q(Tx, Tfx) + a_3 q(Ty, Tfy) \\ &+ a_4[q(Tfx, Ty) + q(Tfy, Tx)] \end{aligned}$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4 are non-negative real numbers such that $a_1 + a_2 + a_3 + 2a_4 < 1$. Then, f has a unique fixed point $x^* \in X$ and for any

$x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $u = fu$ then $q(Tu, Tu) = \theta$.

Proof. Choose $x_0 \in X$. Set $x_1 = fx_0, x_2 = fx_1 = f^2x_0, ..., x_{n+1} = fx_n = f^{n+1}x_0$. Then we have

$$\begin{aligned}
q(Tx_n, Tx_{n+1}) &= q(Tfx_{n-1}, Tfx_n)\\
&\preceq a_1 q(Tx_{n-1}, Tx_n) + a_2 q(Tx_{n-1}, Tfx_{n-1})\\
&\quad + a_3 q(Tx_n, Tfx_n) + a_4[q(Tfx_{n-1}, Tx_n) + q(Tfx_n, Tx_{n-1})]\\
&= a_1 q(Tx_{n-1}, Tx_n) + a_2 q(Tx_{n-1}, Tx_n) + a_3 q(Tx_n, Tx_{n+1})\\
&\quad + a_4 q(Tx_{n+1}, Tx_{n-1})\\
&\preceq a_1 q(Tx_{n-1}, Tx_n) + a_2 q(Tx_{n-1}, Tx_n) + a_3 q(Tx_n, Tx_{n+1})\\
&\quad + a_4[q(Tx_{n-1}, Tx_n) + q(Tx_n, Tx_{n+1})].
\end{aligned}$$

So,

$$\begin{aligned}
q(Tx_n, Tx_{n+1}) &\preceq \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} q(Tx_{n-1}, Tx_n)\\
&= hq(Tx_{n-1}, Tx_n),\\
&\preceq h^2 q(Tx_{n-2}, Tx_{n-1})\\
&\ \ \vdots\\
&\preceq h^n q(Tx_0, Tx_1),
\end{aligned}$$

where $h = \frac{a_1+a_2+a_4}{1-a_3-a_4} < 1$. Note that,

$$q(Tfx_{n-1}, Tfx_n) = q(Tx_n, Tx_{n+1}) \preceq hq(Tx_{n-1}, Tx_n). \tag{3.1}$$

Let $m > n \geq 1$. Then it follows that,

$$\begin{aligned}
q(Tx_n, Tx_m) &\preceq q(Tx_n, Tx_{n+1}) + q(Tx_{n+1}, Tx_{n+2}) + \cdots + q(Tx_{m-1}, Tx_m)\\
&\preceq (h^n + h^{n+1} + \cdots + h^{m-1}) q(Tx_0, Tx_1)\\
&\preceq \Big(\frac{h^n}{1-h}\Big) q(Tx_0, Tx_1) \to \theta \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Thus, Lemma 2.8(3) shows that $\{Tx_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $v \in X$ such that $Tx_n \to v$ as $n \to \infty$. Since T is subsequentially convergent, $\{x_n\}$ has a convergent subsequence. So, there are $x* \in X$ and $\{x_{n_i}\}$ such that $x_{n_i} \to x^*$ as $i \to \infty$. Since T is continuous, we obtain $\lim Tx_{n_i} = Tx^*$. The uniqueness of the limit implies that $Tx^* = v$. Then by (q_3), we have

$$q(Tx_n, Tx^*) \preceq \Big(\frac{k^n}{1-k}\Big) q(Tx_0, Tx_1). \tag{3.2}$$

Now by (3.1), we have

$$\begin{aligned} q(Tx_n, Tfx^*) &= q(Tfx_{n-1}, Tfx^*) \\ &\preceq hq(Tx_{n-1}, Tx^*) \\ &\preceq h\Big(\frac{k^{n-1}}{1-k}\Big)q(Tx_0, Tx_1) \\ &= \Big(\frac{h^n}{1-h}\Big)q(Tx_0, Tx_1). \end{aligned} \tag{3.3}$$

By Lemma 2.8(1), (3.2) and (3.3), we have $Tx^* = Tfx^*$. Since T is one to one, $x^* = fx^*$. Thus, x^* is a fixed point of f. Suppose that $u = fu$, then we have

$$\begin{aligned} q(Tu, Tu) &= q(Tfu, Tfu) \\ &\preceq a_1 q(Tu, Tu) + a_2 q(Tu, Tfu) + a_3 q(Tu, Tfu) \\ &\quad + a_4[q(Tfu, Tu) + q(Tfu, Tu)] \\ &= a_1 q(Tu, Tu) + a_2 q(Tu, Tu) + a_3 q(Tu, Tu) \\ &\quad + a_4[q(Tu, Tu) + q(Tu, Tu)] \\ &= (a_1 + a_2 + a_3 + 2a_4)q(Tu, Tu). \end{aligned}$$

Since $a_1 + a_2 + a_3 + 2a_4 < 1$, Lemma 2.4(1) shows that $q(Tu, Tu) = \theta$.

Finally, suppose there is another fixed point y^* of f, then we have

$$\begin{aligned} q(Tx^*, Ty^*) &= q(Tfx^*, Tfy^*) \\ &\preceq a_1 q(Tx^*, Ty^*) + a_2 q(Tx^*, Tfx^*) + a_3 q(Ty^*, Tfy^*) \\ &\quad + a_4[q(Tfx^*, Ty^*) + q(Tfy^*, Tx^*)] \\ &= a_1 q(x^*, y^*) + a_2 q(x^*, x^*) + a_3 q(y^*, y^*) \\ &\quad + a_4[q(x^*, y^*) + q(y^*, x^*)] \\ &= a_1 q(Tx^*, Ty^*) + 2a_4 q(Tx^*, Ty^*) \\ &\preceq a_1 q(Tx^*, Ty^*) + a_2 q(Tx^*, Ty^*) + a_3 q(Tx^*, Ty^*) \\ &\quad + 2a_4 q(Tx^*, Ty^*) \\ &= (a_1 + a_2 + a_3 + 2a_4)q(Tx^*, Ty^*). \end{aligned}$$

Since $a_1 + a_2 + a_3 + 2a_4 < 1$, Lemma 2.4(1) shows that $q(Tx^*, Ty^*) = \theta$. Also we have $q(Tx^*, Tx^*) = \theta$. Thus, Lemma 2.8(1), $Tx^* = Ty^*$. Since T is one to one, $x^* = fx^*$. Therefore the fixed point is unique. □

Remark 3.2. Put $a_4 = 0$ in Theorem 3.1, we get the result of Fadail et al. [11].

Now, from Theorem 3.1, we get the following corollaries.

Corollary 3.3. *Let (X, d) be a complete cone metric space, P be a solid cone and q be a c-distance on X. In addition let $T : X \to X$ be an one to one, continuous and subsequentially convergent function and $f : X \to X$ be a mapping satisfies the contractive condition:*

$$q(Tfx, Tfy) \preceq a_1 q(Tx, Ty)$$

for all $x, y \in X$, where $a_1 \in [0, 1)$. Then, f has a unique fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $u = fu$ then $q(Tu, Tu) = \theta$.

Proof. Put $a_2 = a_3 = a_4 = 0$ in Theorem 3.1, we get the required result. □

Corollary 3.4. *Let (X, d) be a complete cone metric space, P be a solid cone and q be a c-distance on X. In addition let $T : X \to X$ be an one to one, continuous' and subsequentially convergent function and $f : X \to X$ be a mapping satisfies the contractive condition*

$$q(Tfx, Tfy) \preceq a_2 q(Tx, Tfx) + a_3 q(Ty, Tfy),$$

for all $x, y \in X$, where $a_2, a_3 \in [0, 1)$ are constants such that $a_2 + a_3 < 1$. Then f has a unique fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $u = fu$ then $q(Tu, Tu) = \theta$.

Proof. Put $a_1 = a_4 = 0$ in Theorem 3.1, we get the result of Corollary 3.4. □

Remark 3.5. If we take $Tx = x$ in Theorem 3.1, we get the result of Dubey et al. [6].

Theorem 3.6. *Let (X, d) be a complete cone metric space, P be a solid cone and q be a c-distance on X. In addition let $T : X \to X$ be an one to one, continuous and subsequentially convergent function and $f : X \to X$ be a mapping satisfies the contractive condition*

$$\begin{aligned} q(Tfx, Tfy) \quad &\preceq \quad a_1 q(Tx, Ty) + a_2[q(Tx, Tfy) + q(Ty, Tfx)] \\ &\quad\quad + a_3[q(Tx, Tfx) + q(Ty, Tfy)] \end{aligned}$$

for all $x, y \in X$, where a_1, a_2, a_3 are non-negative real numbers such that $a_1 + 2a_2 + 2a_3 < 1$. Then, f has a unique fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $u = fu$ then $q(Tu, Tu) = \theta$.

Proof. Proof of this theorem is same as Theorem 3.1. □

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References

[1] A. Beiranvand, S. Moradi, M. Omid and H. Pazandeh, *Two fixed point theorem for special mapping,* arXiv:0903.1504v1[math. FA].

[2] Y.J. Cho, R. Saadati and S. Wang, *Common fixed point theorems on generalized distance in ordered cone metric spaces,* Comp. Math. with Appl., **61**(4) (2011), 1254-1260.

[3] A.K. Dubey, R. Shukla and R.P. Dubey, *Common fixed point theorems for T-Reich contraction mapping in cone metric spaces,* Adv. in Fixed Point Theory, **3**(2) (2013), 315–326.

[4] A.K. Dubey, R. Shukla and R.P. Dubey, *Cone metric spaces and fixed point theorems of generalized T-Zamfirescu mappings,* Int. J. Appl. Math. Research, **2**(1) (2013), 151–156.

[5] A.K. Dubey, R. Shukla and R.P. Dubey, *Common fixed point theorem for generalized T-Hardy-Rogers contractive mapping in a cone metric space,* Adv. Inequal. and Appl., **2014:18** (2014), 1–16.

[6] A.K. Dubey, Rohit Verma and R.P. Dubey, *Cone metric spaces and fixed point theorems of contractive mapping for c-distance,* Int. J. of Math. Appl., **3**(1) (2015), 83–88.

[7] A.K. Dubey and U. Mishra, *Some fixed point results of single-valued mapping for c-distance in TVS-cone metric spaces,* Filomat, **30**(11) (2016), 2925–2934.

[8] A.K. Dubey and U. Mishra, *Some fixed point results for c-distance in cone metric spaces,* Nonlinear Funct. Anal. Appl., **22**(2) (2017), 275–286.

[9] Z.M. Fadail, A.G.B. Ahmad and Z. Golubovic, *Fixed point theorems of single valued mapping for c-distance in cone metric spaces,* Abst. and Appl. Anal., **2012** (2012), Article ID 826815, 11 pages.

[10] Z.M. Fadail, A.G.B. Ahmad and L. Paunovic, *New fixed point results of single valued mapping for c-distance in cone metric spaces,* Abst. and Appl. Anal., **2012** (2012), Article ID 639713, 12 pages.

[11] Z.M. Fadail and S.M. Abusalim, *T-Reich contraction and fixed point results in cone metric spaces with c-distance,* Int. Jour. of Math. Anal., **11**(8) (2017), 397–405, http://doi.org/10.12988/ijma.2017, 7338.

[12] L.G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings,* J. Math. Anal. Appl., **332** (2007), 1468–1476.

[13] G. Jungck, S. Redenović, S. Radojević and V. Rakoćević, *Common fixed point theorems for weakly compatible pairs on cone metric spaces,* Fixed Point Theory and Appl., **2009** (2009), Article ID 643840.

[14] W. Shatanawi, E. Karapinar and H. Aydi, *Coupled coincidence points in partially ordered cone metric spaces with c-distance,* J. Appl. Math., **2012** (2012), Article ID 312078, 15 pages.

[15] W. Sintunavarat, Y.J. Cho and P. Kumam, *Common fixed point theorems for c-distance in ordered cone metric spaces,* Comp. Math. Appl., **62**(4) (2011), 1969–1978.

[16] S. Wang, and B. Guo, *Distance in cone metric spaces and common fixed point theorems,* Appl. Math. Letters, **24** (2011), 1735–1739.