Nonlinear Functional Analysis and Applications
Vol. 25, No. 1 (2020), pp. 135-152
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2020.25.01.10
http://nfaa.kyungnam.ac.kr/journal-nfaa

ON THE RATE OF CONVERGENCE OF VISCOSITY IMPLICIT ITERATIVE ALGORITHMS

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Abstract. An essential numerical method for solving ordinary differential and differential algebraic equations is the implicit midpoint rule. Comparing the rate of convergence of the implicit midpoint rules by using numerical examples is common in the literatures. Under suitable conditions imposed on the control parameters, it is shown in this paper that certain two implicit iterative sequences converge to the same fixed point of a nonexpansive mapping in uniformly smooth Banach spaces. Moreover, analytical comparison for the rate of convergence of the implicit iterative sequences to a fixed point of a nonexpansive mapping in uniformly smooth Banach spaces is presented. The implicit iterative sequence which converges faster is determined by an analytical method which is more general than the numerical methods.

1. Introduction

In 2000, Moudafi [11] introduced a well-known iterative method known as the viscosity approximation method for approximating fixed points of a nonexpansive mapping. Later in 2004, Xu [20] applied a technique which uses (strict) contractions to regularize a nonexpansive mapping for the purpose of

[0]Received May 18, 2019. Revised August 5, 2019.
[0]2010 Mathematics Subject Classification: 47H06, 47J05, 47J25, 47H10, 47H17.
[0]Keywords: Convergence rate, nonexpansive, implicit iteration, viscosity method.
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selecting a particular fixed point of the nonexpansive mapping and studied the sequence

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n,\ n \in \mathbb{N}, \tag{1.1}$$

where f is a contraction on K and the nonexpansive mapping $T : K \to K$ is also defined on K, which is a nonempty closed convex subset of a real Hilbert space H. Xu [20] showed that under suitable conditions imposed on the parameters, the iterative sequence $\{x_n\}_{n=1}^{\infty}$ generated by (1.1), converges strongly to a fixed point p of a nonexpansive mapping T in Hilbert spaces that also solves the following variational inequality

$$\langle (I-f)p, x-p\rangle \geq 0,\ \forall\ x \in F(T), \tag{1.2}$$

where $F(T)$ is the set of fixed points of mapping T.

Recently, Xu et al. [22] introduced the implicit midpoint procedure

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right),\ n \in \mathbb{N}, \tag{1.3}$$

where f is a contraction and T is a nonexpansive mapping. They proved a strong convergence theorem for the sequence $\{x_n\}_{n=1}^{\infty}$ to a fixed point p of T which also solves the variational inequality (1.2) in Hilbert spaces. Yao et al. [24] extended the work of Xu et al. [22] and considered the implicit midpoint sequence

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T\left(\frac{x_n + x_{n+1}}{2}\right),\ n \in \mathbb{N}, \tag{1.4}$$

where T and f are as defined in (1.1) and $\alpha_n+\beta_n+\gamma_n = 1$ for all $n \in \mathbb{N}$. Under certain conditions on the parameters, they obtained that the sequence $\{x_n\}_{n=1}^{\infty}$ generated by (1.4) converges strongly to $p = P_{F(T)}f(p)$. In other words, the sequence $\{x_n\}_{n=1}^{\infty}$ generated by (1.4) converges in norm to a fixed point p of T, which is also the unique solution of the variational inequality (1.2). Luo et al. [9] studied the convergence of the sequence (1.3) in uniformly smooth Banach spaces. Furthermore, they used a numerical example to compare the rate of convergence of the sequences (1.1) and (1.3). Also, in uniformly smooth Banach spaces, numerical methods were used by Aibinu et al. [1] to compare the rate of convergence of the iteration procedures (1.3), (1.4) and

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T\left(\frac{x_n + x_{n+1}}{2}\right),\ n \in \mathbb{N}, \tag{1.5}$$

where T is a nonexpansive mapping and $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$, introduced by Alghamdi et al. [3] in 2014. Ke and Ma [8] chose $\{\delta_n\}_{n=1}^{\infty} \subset (0,1)$ and generalized the viscosity implicit midpoint rules of Xu et al. [22] and Yao et al. [24] to the two viscosity implicit rules

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\left(\delta_n x_n + (1-\delta_n)x_{n+1}\right),\ n \in \mathbb{N}, \tag{1.6}$$

and

$$y_{n+1} = \alpha_n f(y_n) + \beta_n y_n + \gamma_n T\left(\delta_n y_n + (1-\delta_n) y_{n+1}\right), \ n \in \mathbb{N}, \qquad (1.7)$$

where $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}, \{\gamma_n\}_{n=1}^{\infty} \subset [0,1]$ with $\alpha_n+\beta_n+\gamma_n = 1$. It was shown that the sequences generated by (1.6) and (1.7) converge strongly to a fixed point p of the nonexpansive mapping T, which solves the variational inequality (1.2). Extension of the main results of Ke and Ma [8] from Hilbert spaces to uniformly smooth Banach spaces was considered by Yan et al. [23]. Aibinu and Kim [2] recently studied the viscosity implicit iterative algorithms for nonexpansive mappings in Banach spaces (see also [17], [18]). Suitable conditions were imposed on the control parameters to prove a strong convergence theorem for the considered iterative sequence. Then, the following questions arise are of interest for the consideration in this paper:

Question 1.1. *Do the sequences* (1.6) *and* (1.7) *always converge to the same fixed point of a nonexpansive mapping?*

Question 1.2. *Can one give an analytical proof, which is more general than numerical examples to show which sequence converges faster between* (1.6) *and* (1.7)*?*

In this paper, an affirmative answers are given to those questions raised above. Under suitable conditions imposed on the control parameters, the analytical proof is given to show that the two sequences converge to the same fixed point of a nonexpansive mapping. Moreover, it is shown analytically that the sequence (1.7) converges faster than (1.6) in approximating a fixed point of a nonexpansive mapping.

2. Preliminaries

A normed linear space E is said to be uniformly smooth whenever given $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in E$ with $\|x\| = 1$ and $\|y\| \leq \delta$, then

$$\|x+y\| + \|x-y\| < 2 + \epsilon\|y\|.$$

Let K be a nonempty closed convex subset of a real Banach space E and T be a self-mapping of K. The set of fixed points of T will be denoted by $F(T) := \{p \in K : Tp = p\}$. Recall that $T : K \to K$ is said to be L-Lipschitzian if there exists a constant $L > 0$ such that for all $x, y \in K$,

$$\|Tx - Ty\| \leq L\|x - y\|.$$

If $0 < L < 1$, then T is a contraction and it called nonexpansive mapping if $L = 1$.

Definition 2.1. ([4]) Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be two sequences of real numbers that converge to u and v respectively, and assume that

$$l = \lim_{n\to\infty} \frac{|u_n - u|}{|v_n - v|} \quad \text{exist.} \tag{2.1}$$

(i) The sequence $\{u_n\}_{n=1}^{\infty}$ is said to be convergent faster to u than $\{v_n\}_{n=1}^{\infty}$ to v, if $l = 0$.
(ii) The sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are said to have the same convergence rate, if $0 < l < \infty$.
(iii) The sequence $\{v_n\}_{n=1}^{\infty}$ is said to be convergent faster to v than $\{u_n\}_{n=1}^{\infty}$ to u.

Definition 2.2. ([4]) Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two fixed point iteration procedures that converge to the same fixed point p on a normed space X such that the error estimates

$$\|x_n - p\| \le u_n, \ n \in \mathbb{N} \tag{2.2}$$

and

$$\|y_n - p\| \le v_n, \ n \in \mathbb{N} \tag{2.3}$$

are available, where $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are two null sequences of positive numbers (that is, sequences of positive numbers that have zero as their limit). The sequence $\{x_n\}_{n=1}^{\infty}$ is said to be convergent faster to p than $\{y_n\}_{n=1}^{\infty}$, if $\{u_n\}_{n=1}^{\infty}$ converges faster than $\{v_n\}_{n=1}^{\infty}$.

The following lemma will also be needed in the sequel.

Lemma 2.3. ([21]) *Assume* $\{a_n\}_{n=1}^{\infty}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} = (1 - \sigma_n)a_n + \sigma_n\delta_n, \ n > 0,$$

where $\{\sigma_n\}_{n=1}^{\infty}$ *is a sequence in* $(0, 1)$ *and* $\{\delta_n\}_{n=1}^{\infty}$ *is a real sequence such that*

(i) $\displaystyle\sum_{n=1}^{\infty} \sigma_n = \infty$,

(ii) $\displaystyle\limsup_{n\to\infty} \delta_n \le 0 \quad$ *or* $\quad \displaystyle\sum_{n=1}^{\infty} \sigma_n|\delta_n| < \infty$.

Then $\displaystyle\lim_{n\to\infty} a_n = 0$.

3. Main Results

3.1. Analytical analysis for the convergence of sequences (1.6) and (1.7) to the same fixed point of a nonexpansive mapping.

Here, the analytical proof is given to ascertain that the implicit iterative sequences (1.6) and (1.7) converge to the same fixed point of a nonexpansive mapping in uniformly smooth Banach spaces.

Theorem 3.1. *Let K be a nonempty closed convex subset of a uniformly smooth Banach space E, T be a nonexpansive self-mapping defined on K with $F(T) \neq \emptyset$ and $f : K \to K$, be a c-contraction mapping. Suppose that $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $[0,1]$ with*

(a) $\alpha_n + \beta_n + \gamma_n = 1$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c) $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = 0$.

Then (1.7) *converges in norm to p if and only if* (1.6) *converges in norm to p.*

Proof. It is shown that the implicit iterative sequences (1.6) and (1.7) converge to the same fixed point of a nonexpansive mapping T.

$$\begin{aligned}
\|y_{n+1} - x_{n+1}\| &= ||\alpha_n f(y_n) + \beta_n y_n + \gamma_n T(\delta_n y_n + (1-\delta_n) y_{n+1}) \\
&\quad - (\alpha_n f(x_n) + (1-\alpha_n) T(\delta_n x_n + (1-\delta_n) x_{n+1}))\,|| \\
&= ||\alpha_n (f(y_n) - f(x_n)) + \beta_n (y_n - T(\delta_n x_n + (1-\delta_n) x_{n+1})) \\
&\quad + \gamma_n \left(T(\delta_n y_n + (1-\delta_n) y_{n+1}) - T(\delta_n x_n + (1-\delta_n) x_{n+1})\right) || \\
&\leq \alpha_n ||f(y_n) - f(x_n)|| + \beta_n \|y_n - T(\delta_n x_n + (1-\delta_n) x_{n+1})\| \\
&\quad + \gamma_n \|T(\delta_n y_n + (1-\delta_n) y_{n+1}) - T(\delta_n x_n + (1-\delta_n) x_{n+1})\| \\
&\leq c\alpha_n ||y_n - x_n|| + \beta_n \|y_n - T(\delta_n x_n + (1-\delta_n) x_{n+1})\| \\
&\quad + \gamma_n \|\delta_n (y_n - x_n) + (1-\delta_n)(y_{n+1} - x_{n+1})\| \\
&\leq c\alpha_n ||y_n - x_n|| + \beta_n \|y_n - T(\delta_n x_n + (1-\delta_n) x_{n+1})\| \\
&\quad + \gamma_n \delta_n \|y_n - x_n\| + \gamma_n (1-\delta_n) \|y_{n+1} - x_{n+1}\| \\
&\leq (c\alpha_n + \gamma_n \delta_n) ||y_n - x_n|| + \beta_n \|y_n - T(\delta_n x_n + (1-\delta_n) x_{n+1})\| \\
&\quad + \gamma_n (1-\delta_n) \|y_{n+1} - x_{n+1}\|.
\end{aligned}$$

Note that $\{y_n\}_{n=1}^{\infty}$ and $\{T(\delta_n x_n + (1-\delta_n) x_{n+1})\}_{n=1}^{\infty}$ are bounded [23], let

$$M := \sup_n \|y_n - T(\delta_n x_n + (1-\delta_n) x_{n+1})\|.$$

Then we have

$$
\begin{aligned}
&\|y_{n+1}-x_{n+1}\| \\
&\le \frac{c\alpha_n+\gamma_n\delta_n}{1-\gamma_n(1-\delta_n)}||y_n-x_n||+\frac{\beta_n}{1-\gamma_n(1-\delta_n)}M \\
&= 1+\frac{c\alpha_n-(1-\gamma_n)}{1-\gamma_n(1-\delta_n)}||y_n-x_n||+\frac{\beta_n}{1-\gamma_n(1-\delta_n)}M \\
&= 1+\frac{-\beta_n-(1-c)\alpha_n}{1-\gamma_n(1-\delta_n)}||y_n-x_n||+\frac{\beta_n}{1-\gamma_n(1-\delta_n)}M \\
&= \left(1-\frac{(1-c)\alpha_n+\beta_n}{1-\gamma_n(1-\delta_n)}\right)||y_n-x_n||+\frac{\beta_n}{1-\gamma_n(1-\delta_n)}M \\
&\le \left(1-\frac{(1-c)\alpha_n}{1-\gamma_n(1-\delta_n)}\right)||y_n-x_n||+\frac{\beta_n}{1-\gamma_n(1-\delta_n)}M \\
&= \left(1-\frac{(1-c)\alpha_n}{1-\gamma_n(1-\delta_n)}\right)||y_n-x_n||+\frac{(1-c)\alpha_n}{1-\gamma_n(1-\delta_n)}\frac{\beta_n}{(1-c)\alpha_n}M \\
&= (1-\sigma_n)||y_n-x_n||+\frac{\beta_n}{(1-c)\alpha_n}\sigma_nM, \qquad (3.1)
\end{aligned}
$$

where $\sigma_n = \frac{(1-c)\alpha_n}{1-\gamma_n(1-\delta_n)}$. Notice that $\limsup_{n\to\infty}\frac{\beta_n}{\alpha_n} \le 0$. Then, we can apply Lemma 2.3 to (3.1) in order to deduce that $||y_n-x_n|| \to 0$ as $n\to\infty$.

Furthermore, suppose $||x_n-p|| \to 0$ as $n\to\infty$, we have that

$$
\begin{aligned}
||y_n-p|| &= ||y_n-x_n+x_n-p|| \\
&\le ||y_n-x_n||+||x_n-p|| \\
&\to 0 \quad \text{as } n\to\infty.
\end{aligned}
$$

Similarly, suppose $||y_n-p|| \to 0$ as $n\to\infty$, it is obtained that

$$
\begin{aligned}
||x_n-p|| &= ||x_n-y_n+y_n-p|| \\
&\le ||x_n-y_n||+||y_n-p|| \\
&\to 0 \quad \text{as } n\to\infty.
\end{aligned}
$$

This completes the proof. □

3.2. **Comparison of the rate of convergence**

The terminologies and definitions of Berinde [4] will be adopted and the methodology used in Ding et al. [7] and Olaleru ([12], [13], [14]), Yildirim and Abbas [25] will be applied to compare the rate of convergence of the iterative sequences (1.6) and (1.7).

The next result deals with the rate of convergence of implicit iterative sequences.

Theorem 3.2. *Let K be a nonempty closed convex subset of a uniformly smooth Banach space E, T be a nonexpansive self-mapping defined on K with $F(T) \neq \emptyset$ and $f : K \to K$ be a c-contraction mapping. Let $\{\delta_n\}_{n=1}^{\infty} \subset (0,1)$ and $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}, \{\gamma_n\}_{n=1}^{\infty} \subset [0,1]$ be real sequences which satisfy the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0,\ 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(iv) $0 < \epsilon \leq \delta_n \leq \delta_{n+1} < 1$ *for all* $n \in \mathbb{N}$.

Then for arbitrary $x_1, y_1 \in K$ with $x_1 = y_1$, the iterative sequence (1.7) *converges faster than* (1.6).

Proof. Suppose $p \in F(T)$, it can be obtained from the iterative scheme (1.6) that

$$\begin{aligned}
\|x_{n+1} - p\| &= ||\alpha_n (f(x_n) - p) + (1-\alpha_n)(T(\delta_n x_n + (1-\delta_n)x_{n+1}) - p)|| \\
&= ||\alpha_n (f(x_n) - f(p)) + \alpha_n (f(p) - p) \\
&\quad + (1-\alpha_n)(T(\delta_n x_n + (1-\delta_n)x_{n+1}) - p)|| \\
&\leq \alpha_n||f(x_n) - f(p)|| + \alpha_n||f(p) - p|| \\
&\quad + (1-\alpha_n)||T(\delta_n x_n + (1-\delta_n)x_{n+1}) - p|| \\
&\leq c\alpha_n||x_n - p|| + \alpha_n||f(p) - p|| \\
&\quad + (1-\alpha_n)||\delta_n x_n + (1-\delta_n)x_{n+1} - p|| \\
&\leq c\alpha_n||x_n - p|| + \alpha_n||f(p) - p|| \\
&\quad + (1-\alpha_n)[\delta_n||x_n - p|| + (1-\delta_n)||x_{n+1} - p||] \\
&= (c\alpha_n + \delta_n(1-\alpha_n))||x_n - p|| \\
&\quad + \alpha_n||f(p) - p|| + (1-\alpha_n)(1-\delta_n)||x_{n+1} - p||.
\end{aligned}$$

This leads to

$$\begin{aligned}
[1 - (1-\alpha_n)(1-\delta_n)]||x_{n+1} - p|| \leq (c\alpha_n + \delta_n(1-\alpha_n))\,||x_n - p|| \\
+ \alpha_n||f(p) - p||.
\end{aligned}$$

Equivalently,

$$\begin{aligned} ||x_{n+1}-p|| &\leq \frac{c\alpha_n+\delta_n(1-\alpha_n)}{1-(1-\alpha_n)(1-\delta_n)}||x_n-p|| \\ &\quad +\frac{\alpha_n}{1-(1-\alpha_n)(1-\delta_n)}||f(p)-p|| \\ &\leq \frac{c\alpha_n+(1-\alpha_n)}{1-(1-\alpha_n)(1-\delta_n)}||x_n-p|| \\ &\quad +\frac{\alpha_n}{1-(1-\alpha_n)(1-\delta_n)}||f(p)-p|| \\ &= \frac{1-(1-c)\alpha_n}{1-(1-\alpha_n)(1-\delta_n)}||x_n-p|| \\ &\quad +\frac{\alpha_n}{1-(1-\alpha_n)(1-\delta_n)}||f(p)-p||. \end{aligned} \tag{3.2}$$

Then from (3.2), we obtain that

$$\begin{aligned} ||x_{n+1}-p|| &\leq \frac{1}{1-(1-\alpha_n)(1-\delta_n)}||x_n-p|| \\ &\quad + \frac{\alpha_n}{1-(1-\alpha_n)(1-\delta_n)}||f(p)-p|| \\ &\leq \frac{1}{1-(1-\alpha_n)(1-\delta_n)}\ \Big[\frac{1-(1-c)\alpha_{n-1}}{1-(1-\alpha_{n-1})(1-\delta_{n-1})}||x_{n-1}-p|| \\ &\quad + \frac{\alpha_{n-1}}{1-(1-\alpha_{n-1})(1-\delta_{n-1})}||f(p)-p||\ \Big] \\ &\quad + \frac{\alpha_n}{1-(1-\alpha_n)(1-\delta_n)}||f(p)-p|| \\ &= \frac{1-(1-c)\alpha_{n-1}}{\prod_{j=n-1}^{n}(1-(1-\alpha_j)(1-\delta_j))}||x_{n-1}-p|| \\ &\quad + \frac{\alpha_{n-1}}{\prod_{j=n-1}^{n}(1-(1-\alpha_j)(1-\delta_j))}||f(p)-p|| \\ &\quad + \frac{\alpha_n}{1-(1-\alpha_n)(1-\delta_n)}||f(p)-p||. \end{aligned}$$

Since $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\delta_n\}_{n=1}^{\infty}$ are in $(0,1)$, we have

$$\prod_{j=1}^{n-1}(1-(1-\alpha_j)(1-\delta_j)) \geq \prod_{j=1}^{n}(1-(1-\alpha_j)(1-\delta_j)) > 0.$$

Then, we obtain

$$
\begin{aligned}
&||x_{n+1}-p|| \\
&\leq \frac{1-(1-c)\alpha_{n-1}}{\prod\limits_{j=n-1}^{n}(1-(1-\alpha_j)(1-\delta_j))}||x_{n-1}-p|| \\
&\quad + \frac{\sum\limits_{j=n-1}^{n}\alpha_j}{\prod\limits_{j=n-1}^{n}(1-(1-\alpha_j)(1-\delta_j))}||f(p)-p|| \\
&\leq \frac{1}{\prod\limits_{j=n-1}^{n}(1-(1-\alpha_j)(1-\delta_j))}||x_{n-1}-p|| \\
&\quad + \frac{\sum\limits_{j=n-1}^{n}\alpha_j}{\prod\limits_{j=n-1}^{n}(1-(1-\alpha_j)(1-\delta_j))}||f(p)-p|| \\
&\leq \frac{1}{\prod\limits_{j=n-1}^{n}(1-(1-\alpha_j)(1-\delta_j))}\ [\frac{1-(1-c)\alpha_{n-2}}{1-(1-\alpha_{n-2})(1-\delta_{n-2})}||x_{n-2}-p|| \\
&\quad + \frac{\alpha_{n-2}}{1-(1-\alpha_{n-2})(1-\delta_{n-2})}||f(p)-p||\ ] \\
&\quad + \frac{\sum\limits_{j=n-1}^{n}\alpha_j}{\prod\limits_{j=n-1}^{n}(1-(1-\alpha_j)(1-\delta_j))}||f(p)-p|| \\
&= \frac{1-(1-c)\alpha_{n-2}}{\prod\limits_{j=n-2}^{n}(1-(1-\alpha_j)(1-\delta_j))}||x_{n-2}-p|| \\
&\quad + \frac{\alpha_{n-2}}{\prod\limits_{j=n-2}^{n}(1-(1-\alpha_j)(1-\delta_j))}||f(p)-p||
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\sum_{j=n-1}^{n} \alpha_j}{\prod_{j=n-1}^{n} (1-(1-\alpha_j)(1-\delta_j))} ||f(p)-p|| \\
&\leq \frac{1}{\prod_{j=n-1}^{n} (1-(1-\alpha_j)(1-\delta_j))} \ [\frac{1-(1-c)\alpha_{n-2}}{1-(1-\alpha_{n-2})(1-\delta_{n-2})} ||x_{n-2}-p|| \\
&+ \frac{\alpha_{n-2}}{1-(1-\alpha_{n-2})(1-\delta_{n-2})} ||f(p)-p|| \ ] \\
&+ \frac{\sum_{j=n-1}^{n} \alpha_j}{\prod_{j=n-1}^{n} (1-(1-\alpha_j)(1-\delta_j))} ||f(p)-p|| \\
&= \frac{1-(1-c)\alpha_{n-2}}{\prod_{j=n-2}^{n} (1-(1-\alpha_j)(1-\delta_j))} ||x_{n-2}-p|| \\
&+ \frac{\alpha_{n-2}}{\prod_{j=n-2}^{n} (1-(1-\alpha_j)(1-\delta_j))} ||f(p)-p|| \\
&+ \frac{\sum_{j=n-1}^{n} \alpha_j}{\prod_{j=n-1}^{n} (1-(1-\alpha_j)(1-\delta_j))} ||f(p)-p|| \\
&= \frac{1-(1-c)\alpha_{n-2}}{\prod_{j=n-2}^{n} (1-(1-\alpha_j)(1-\delta_j))} ||x_{n-2}-p|| \\
&+ \frac{\sum_{j=n-2}^{n} \alpha_j}{\prod_{j=n-2}^{n} (1-(1-\alpha_j)(1-\delta_j))} ||f(p)-p||.
\end{aligned}
$$

Hence, by induction, we have

$$||x_{n+1}-p|| \leq \frac{1-(1-c)\alpha_1}{\prod_{j=1}^{n}(1-(1-\alpha_j)(1-\delta_j))}||x_1-p|| + \frac{\sum_{j=1}^{n}\alpha_j}{\prod_{j=1}^{n}(1-(1-\alpha_j)(1-\delta_j))}||f(p)-p||. \tag{3.3}$$

From (1.7), we have that

$$\begin{aligned}
&\|y_{n+1}-p\| \\
&= ||\alpha_n\left(f(y_n)-p\right)+\beta_n(y_n-p)+\gamma_n\left(T(\delta_n y_n+(1-\delta_n)y_{n+1})-p\right)|| \\
&= ||\alpha_n\left(f(y_n)-f(p)\right)+\alpha_n\left(f(p)-p\right)+\beta_n(y_n-p) \\
&\quad +\gamma_n\left(T(\delta_n y_n+(1-\delta_n)y_{n+1})-p\right)|| \\
&\leq \alpha_n||f(y_n)-f(p)||+\alpha_n||f(p)-p||+\beta_n||y_n-p|| \\
&\quad +\gamma_n||T(\delta_n y_n+(1-\delta_n)y_{n+1})-p|| \\
&\leq c\alpha_n||y_n-p||+\alpha_n||f(p)-p||+\beta_n||y_n-p|| \\
&\quad +\gamma_n||\delta_n y_n+(1-\delta_n)y_{n+1}-p|| \\
&\leq (c\alpha_n+\beta_n)||y_n-p||+\alpha_n||f(p)-p||+\gamma_n[\delta_n||y_n-p|| \\
&\quad +(1-\delta_n)||y_{n+1}-p||] \\
&= (c\alpha_n+\beta_n+\gamma_n\delta_n)||y_n-p||+\alpha_n||f(p)-p|| \\
&\quad +\gamma_n(1-\delta_n)||y_{n+1}-p||.
\end{aligned}$$

This leads to

$$[1-\gamma_n(1-\delta_n)]||y_{n+1}-p|| \leq \left(c\alpha_n+\beta_n+\gamma_n\delta_n\right)||y_n-p||+\alpha_n||f(p)-p||.$$

From $\alpha_n+\beta_n+\gamma_n=1$, we obtain that

$$
\begin{aligned}
||y_{n+1}-p|| &\leq \frac{c\alpha_n+\beta_n+\gamma_n\delta_n}{1-\gamma_n(1-\delta_n)}||y_n-p||+\frac{\alpha_n}{1-\gamma_n(1-\delta_n)}||f(p)-p|| \\
&= \frac{c\alpha_n+(1-\alpha_n-\gamma_n)+\gamma_n\delta_n}{1-(1-\alpha_n-\beta_n)(1-\delta_n)}||y_n-p|| \\
&\quad +\frac{\alpha_n}{1-(1-\alpha_n-\beta_n)(1-\delta_n)}||f(p)-p|| \\
&= \frac{1-(1-c)\alpha_n-\gamma_n(1-\delta_n)}{1-(1-\alpha_n)(1-\delta_n)+\beta_n(1-\delta_n)}||y_n-p|| \\
&\quad +\frac{\alpha_n}{1-(1-\alpha_n)(1-\delta_n)+\beta_n(1-\delta_n)}||f(p)-p|| \\
&= \frac{1-(1-c)\alpha_n-\gamma_n(1-\delta_n)}{1-(1-\alpha_n)(1-\delta_n)+\beta_n(1-\delta_n)}||y_n-p|| \\
&\quad +\frac{\alpha_n}{1-(1-\alpha_n)(1-\delta_n)+\beta_n(1-\delta_n)}||f(p)-p|| \\
&\leq \frac{1-(1-c)\alpha_n-\gamma_n(1-\delta_n)}{1-(1-\alpha_n)(1-\delta_n)}||y_n-p|| \\
&\quad +\frac{\alpha_n}{1-(1-\alpha_n)(1-\delta_n)}||f(p)-p||. \qquad (3.4)
\end{aligned}
$$

Clearly, it follows from (3.4) that

$$
\begin{aligned}
&||x_{n+1}-p|| \\
&\leq \frac{1}{1-(1-\alpha_n)(1-\delta_n)}||y_n-p||+\frac{\alpha_n}{1-(1-\alpha_n)(1-\delta_n)}||f(p)-p|| \\
&\leq \frac{1}{1-(1-\alpha_n)(1-\delta_n)}\ [\frac{1-(1-c)\alpha_{n-1}-\gamma_{n-1}(1-\delta_{n-1})}{1-(1-\alpha_{n-1})(1-\delta_{n-1})}||y_{n-1}-p|| \\
&\quad +\frac{\alpha_{n-1}}{1-(1-\alpha_{n-1})(1-\delta_{n-1})}||f(p)-p||\ ] \\
&\quad +\frac{\alpha_n}{1-(1-\alpha_n)(1-\delta_n)}||f(p)-p|| \\
&= \frac{1-(1-c)\alpha_{n-1}-\gamma_{n-1}(1-\delta_{n-1})}{\prod_{j=n-1}^{n}(1-(1-\alpha_j)(1-\delta_j))}||y_{n-1}-p|| \\
&\quad +\frac{\alpha_{n-1}}{\prod_{j=n-1}^{n}(1-(1-\alpha_j)(1-\delta_j))}||f(p)-p|| \\
&\quad +\frac{\alpha_n}{1-(1-\alpha_n)(1-\delta_n)}||f(p)-p||.
\end{aligned}
$$

Since $\{\alpha_n\}, \{\delta_n\} \subset (0,1)$. we know that

$$\prod_{j=1}^{n-1}(1-(1-\alpha_j)(1-\delta_j)) \geq \prod_{j=1}^{n}(1-(1-\alpha_j)(1-\delta_j)) > 0.$$

Therefore, we have

$$||y_{n+1}-p|| \leq \frac{1-(1-c)\alpha_{n-1}-\gamma_{n-1}(1-\delta_{n-1})}{\prod_{j=n-1}^{n}(1-(1-\alpha_j)(1-\delta_j))}||y_{n-1}-p|| + \frac{\sum_{j=n-1}^{n}\alpha_j}{\prod_{j=n-1}^{n}(1-(1-\alpha_j)(1-\delta_j))}||f(p)-p||.$$

Hence, in a similar manner to (3.3) and by induction,

$$||y_{n+1}-p|| \leq \frac{1-(1-c)\alpha_1-\gamma_1(1-\delta_1)}{\prod_{j=1}^{n}(1-(1-\alpha_j)(1-\delta_j))}||y_1-p|| + \frac{\sum_{j=1}^{n}\alpha_j}{\prod_{j=1}^{n}(1-(1-\alpha_j)(1-\delta_j))}||f(p)-p||. \tag{3.5}$$

Since $\gamma_1(1-\delta_1) > 0$, $\quad 0 < 1-(1-c)\alpha_1-\gamma_1(1-\delta_1) < 1-(1-c)\alpha_1$. Hence

$$\frac{1-(1-c)\alpha_1-\gamma_1(1-\delta_1)}{1-(1-\alpha_j)(1-\delta_j)} \quad < \quad \frac{1-(1-c)\alpha_1}{1-(1-\alpha_j)(1-\delta_j)},$$

it implies that for all $n > 0$,

$$\frac{1-(1-c)\alpha_1-\gamma_1(1-\delta_1)}{(1-(1-\alpha_j)(1-\delta_j))^n} \quad < \quad \frac{1-(1-c)\alpha_1}{(1-(1-\alpha_j)(1-\delta_j))^n}.$$

Let for all $n > 0$,

$$u_n = \frac{1-(1-c)\alpha_1}{\prod_{j=1}^{n}(1-(1-\alpha_j)(1-\delta_j))} \geq 0$$

and

$$v_n = \frac{1-(1-c)\alpha_1-\gamma_1(1-\delta_1)}{\prod_{j=1}^{n}(1-(1-\alpha_j)(1-\delta_j))} \geq 0.$$

Then we have

$$\lim_{n\to\infty}\frac{v_n}{u_n} = 0.$$

It follows from (3.3), (3.5) and Definition 2.2 that the iterative scheme (1.7) converges faster than (1.6). □

In view of Theorems 3.1 and 3.2, the following results hold.

Corollary 3.3. *Let K be a nonempty closed convex subset of a uniformly smooth Banach space E, T be a nonexpansive self-mapping defined on K with $F(T) \neq \emptyset$ and $f : K \to K$ be a c-contraction mapping. Let $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be real sequences in $[0,1]$ which satisfy the following conditions:*

(i) $\alpha_n+\beta_n+\gamma_n=1$,

(ii) $\lim_{n\to\infty}\alpha_n=0,\ \sum_{n=1}^{\infty}\alpha_n=\infty$,

(iii) $\lim_{n\to\infty}|\beta_{n+1}-\beta_n|=0,\ 0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$,

Then for arbitrary $x_1, y_1 \in K$ with $x_1 = y_1$, the iterative sequence (1.4), *given by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T\left(\frac{x_n+x_{n+1}}{2}\right),\ n \in \mathbb{N},$$

converges faster than (1.3), *given by*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T\left(\frac{x_n+x_{n+1}}{2}\right),\ n \in \mathbb{N}.$$

Proof. The desired result follows from Theorem 3.2 by taking $\delta_n = \frac{1}{2}$ for all $n \in \mathbb{N}$. □

4. Applications

The results in this section show an improvement and generalization of the main results of Xu et al. [22], Yao et al. [24] and Ke and Ma [8]. It will be assumed that the real sequences $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty} \subset [0,1]$ and $\{\delta_n\}_{n=1}^{\infty} \subset (0,1)$ satisfy the following conditions:

(i) $\alpha_n+\beta_n+\gamma_n=1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty,$

(iii) $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0,\ 0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1,$

(iv) $0 < \epsilon \le \delta_n \le \delta_{n+1} < 1$ for all $n \in \mathbb{N}$.

4.1. **Finite combination of nonexpansive mappings**

The proof of the proposition below is given in Wong et al. [19].

Proposition 4.1. *Let K be a nonempty closed convex subset of a strictly convex and uniformly smooth Banach space E and let $\theta_i > 0$ $(i = 1, 2, \ldots, r)$ such that $\sum_{i=1}^{r} \theta_i = 1$. Let $T_1, T_2, \ldots, T_r : K \to K$ be nonexpansive mappings with $\cap_{i=1}^r F(T_i) \neq \emptyset$ and let $T = \sum_{i=1}^{r} \theta_i T_i$. Then T is nonexpansive from K into itself and $F(T) = \cap_{i=1}^r F(T_i)$.*

Therefore the following result holds.

Corollary 4.2. *Suppose K is a nonempty closed convex subset of a strictly convex and uniformly smooth Banach space E, $f : K \to K$ is a c-contraction and let $\theta_i > 0$ $(i = 1, 2, \ldots, r)$ such that $\sum_{i=1}^{r} \theta_i = 1$. Let $T_1, T_2, \ldots, T_r : K \to K$ be nonexpansive mappings with $\cap_{i=1}^r F(T_i) \neq \emptyset$. Then the iterative sequence $\{x_n\}_{n=1}^{\infty}$ which is defined from an arbitrary $x_1 \in K$ by*

$$x_{n+1} \quad = \quad \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^{r} \theta_i T_i \left(\delta_n x_n + (1-\delta_n) x_{n+1}\right), \quad (4.1)$$

converges strongly to a fixed point $p \in \cap_{i=1}^r F(T_i)$, which solves the variational inequality

$$\langle (I-f)p, J(x-p)\rangle \ge 0,\ \text{for all } x \in \cap_{i=1}^r F(T_i). \quad (4.2)$$

Proof. Define $T := \sum_{i=1}^{r} \theta_i T_i$. It suffices to show that T is a nonexpansive mapping and $\cap_{i=1}^r F(T_i) \subseteq F(T)$. This is true by Proposition 4.1. □

4.2 **Composition of finite family of nonexpansive mappings**

Corollary 4.3. *Suppose K is a nonempty closed convex subset of a uniformly smooth Banach space E and $\{T_i\}_{i=1}^N$ a finite family of nonexpansive self-mappings of K such that $F := \cap_{i=1}^N F(T_i) \neq \emptyset$. Let $f : K \to K$ be a* *c-contraction. Then the iterative sequence* $\{x_n\}_{n=1}^{\infty}$ *which is defined from an arbitrary* $x_1 \in K$ *by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_N T_{N-1} T_{n-2} ... T_1 \left(\delta_n x_n + (1-\delta_n) x_{n+1}\right),$$

converges strongly to a fixed point $p \in F$, *which solves the variational inequality*

$$\langle (I-f)p, J(x-p)\rangle \geq 0, \text{ for all } x \in F. \tag{4.3}$$

Proof. It is known that a composition T of finite family of nonexpansive self-mappings $\{T_i\}_{i=1}^N$ on K is nonexpansive with $F(T) \supseteq \cap_{i=1}^N F(T_i) \neq \emptyset$. □

4.3 **Monotone mappings**

Let E be a real Banach space with the duality pairing $\langle .,. \rangle$ and norm $\|.\|$. The dual of E is denoted by E^*. Let A be a set-valued mapping and denote the domain and range of A by $D(A)$ and $R(A)$, respectively. The set $G(A)$ defined by

$$G(A) = \{(u,v) \in E \times E^* : u \in D(A), v \in R(A)\}$$

is called the graph of A.

A mapping A is said to be monotone if

$$\langle u - v, x - y\rangle = 0, \ (u,x), (v,y) \in G(A).$$

A is said to be maximal monotone if it is not properly contained in any other monotone mapping. Monotone mappings have been studied extensively (see, e.g., Bruck [5], Chidume [6], Martinet [10], Reich [15], Rockafellar [16]) due to their role in convex analysis, in nonlinear analysis, in certain partial differential equations and optimization theory. For a maximal monotone mapping $A : D(A) \to 2^{E^*}$, we define the resolvent of A by

$$J_t^A = (J + tA)^{-1} J, \ t > 0. \tag{4.4}$$

It is well known that $J_t^A : E \to D(A)$ is nonexpansive, and $F(J_t^A) = A^{-1}0$, where $F(J_t^A)$ denotes the set of fixed points of J_t^A.

We have the following.

Corollary 4.4. *Suppose* K *is a nonempty closed convex subset of a uniformly smooth Banach space* E, $f : K \to K$ *is a c-contraction and* $\theta_i > 0$ $(i = 1, 2, \dots, r)$ *such that* $\sum_{i=1}^{r} \theta_i = 1$. *Let* $\{A_i\} \subset E \times E^*$ *be a family of maximal monotone mappings with resolvent* $J_t^{A_i}$ *for* $t > 0$ *such that* $\cap_{i=1}^r A_i^{-1} 0 \neq \emptyset$. *Then the iterative sequence* $\{x_n\}_{n=1}^{\infty}$ *which is defined from an arbitrary* $x_1 \in K$ *by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^{r} \theta_i J_t^{A_i} \left(\delta_n x_n + (1-\delta_n) x_{n+1}\right),$$

converges strongly to a unique point $p \in \cap_{i=1}^{r} A_i^{-1}0$, *which solves the variational inequality problem: find* $p \in \cap_{i=1}^{r} A_i^{-1}0$ *such that*

$$\langle (I-f)p, J((x-p)\rangle \geq 0 \text{ for all } x \in \cap_{i=1}^{r} A_i^{-1}0.$$

Proof. Define $T := \sum_{i=1}^{r} \theta_i J_t^{A_i}$. Then T is nonexpansive self-mapping of K and $F(T) \supseteq \cap_{i=1}^{r} F(J_t^{A_i}) \neq \emptyset$. □

Acknowledgments: The first author acknowledges with thanks the bursary and financial support from Department of Science and Technology and National Research Foundation, Republic of South Africa Center of Excellence in Mathematical and Statistical Sciences (DST-NRF CoE-MaSS) Doctoral Bursary. Opinions expressed and conclusions arrived at are those of the author and are not necessarily to be attributed to the CoE-MaSS. The second author was supported by the Basic Science Research Program through the National Research Foundation(NRF) Grant funded by Ministry of Education of the republic of Korea (2018R1D1A1B07045427).

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