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FIXED POINT THEOREM FOR WEAKLY CONTRACTIVE MAPS IN METRICALLY CONVEX SPACES UNDER C-CLASS FUNCTION

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Abstract. In this paper, we shall prove a fixed point theorem for weakly contractive mappings which satisfies a generalized contraction condition on a complete metrically convex metric spaces by using a function say ($C-$ class function). The result in this paper generalizes the relevant results due to Khan and Imdad [14], Rhoades [16], Alber and Guerre-Delabriere [2] and others.

1. Introduction

Metric fixed point theory is one of the most rapidly growing area of research in nonlinear functional analysis and emerged as a powerful tool in solving existence and uniqueness problems in many branches of mathematical analysis say variational analysis, integral and functional equations as applications of fixed points of contraction mappings defined for different spaces.

In 1997, Alber and Guerre-Delabriere [2] coined the concept of weakly contractive maps and obtained fixed point results in Hilbert spaces for self mappings. Rhoades [16] extended some of their works to Banach spaces for the same setting.

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Recently, Khan and Imdad [14] proved the result due to Rhoades [16] for nonself single valued mappings by using the concept of weakly contractive maps and obtained fixed point results in complete metrically convex spaces.

In this paper, we prove a fixed point theorem for single valued nonself mappings by using the concept of C-class functions, introduced by A. H. Ansari [3], which either partially or completely generalize the results due to Khan and Imdad [14], Rhoades [16], Alber and Guerre-Delabriere [2] and others.

Before proving the results, we collect the relevant definitions and example for future use.

Definition 1.1. ([5]) A metric space (X, d) is said to be metrically convex if for any $x, y \in X$ with $x \neq y$ there exists a point $z \in X, x \neq z \neq y$ such that

$$d(x, z) + d(z, y) = d(x, y).$$

Definition 1.2. ([3]) A mapping $F : [0, \infty) \times [0, \infty) \to \mathrm{R}$ is called a C-class function if it is continuous and satisfies following axioms:

(i) $F(s, t) \leq s$,

(ii) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$, for all $s, t \in [0, \infty)$.

An extra condition on F is that $F(0, 0) = 0$ could be imposed in some cases if required. The letter $\mathcal{C}$ denotes the set of all C-class functions. The following example shows that $\mathcal{C}$ is nonempty.

Example 1.3. ([3]) Define a function $F : [0, \infty) \times [0, \infty) \to \mathrm{R}$ by

(iii) $F(s, t) = s - t$, $F(s, t) = s \Rightarrow t = 0$,

(iv) $F(s, t) = ms$, $0 < m < 1$, $F(s, t) = s \Rightarrow s = 0$,

(v) $F(s, t) = \frac{s}{(1+t)^r}$, $r \in (0, \infty)$, $F(s, t) = s \Rightarrow s = 0$ or $t = 0$,

(vi) $F(s, t) = \log(t + a^s)/(1 + t)$, $a > 1$, $F(s, t) = s \Rightarrow s = 0$ or $t = 0$,

(vii) $F(s, t) = \ln(1 + a^s)/2$, $a > e$, $F(s, 1) = s \Rightarrow s = 0$,

(viii) $F(s, t) = (s + l)^{(1/(1+t)^r)} - l$, $l > 1, r \in (0, \infty)$, $F(s, t) = s \Rightarrow t = 0$,

(ix) $F(s, t) = s \log_{t+a} a$, $a > 1$, $F(s, t) = s \Rightarrow s = 0$ or $t = 0$,

(x) $F(s, t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$, $F(s, t) = s \Rightarrow t = 0$,

(xi) $F(s, t) = s\beta(s)$, $\beta : [0, \infty) \to [0, 1)$, and is continuous, $F(s, t) = s \Rightarrow s = 0$,

(xii) $F(s, t) = s - \frac{t}{k+t}, F(s, t) = s \Rightarrow t = 0$,

(xiii) $F(s, t) = s - \varphi(s), F(s, t) = s \Rightarrow s = 0$, here $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$,

(xiv) $F(s, t) = sh(s, t), F(s, t) = s \Rightarrow s = 0$, here $h : [0, \infty) \times [0, \infty) \to [0, \infty)$ is a continuous function such that $h(t, s) < 1$ for all $t, s > 0$,

(xv) $F(s, t) = s - (\frac{2+t}{1+t})t$, $F(s, t) = s \Rightarrow t = 0$,

(xvi) $F(s, t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$, $F(s, t) = s \Rightarrow t = 0$,

(xvii) $F(s,t) = \sqrt[n]{\ln(1+s^n)}, F(s,t) = s \Rightarrow s = 0$,
(xviii) $F(s,t) = \phi(s), F(s,t) = s \Rightarrow s = 0$, here $\phi : [0,\infty) \to [0,\infty)$ is a upper semicontinuous function such that $\phi(0) = 0$, and $\phi(t) < t$ for $t > 0$,
(xix) $F(s,t) = \frac{s}{(1+s)^r}$, $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$,
(xx) $F(s,t) = \frac{s}{\Gamma(1/2)} \int_0^\infty \frac{e^{-x}}{\sqrt{x}+t}\, dx$, where Γ is the Euler Gamma function.

Then F are elements of $\mathcal{C}$.

Definition 1.4. ([3]) A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:
(xxi) ψ is non-decreasing and continuous function,
(xxii) $\psi(t) = 0$ if and only if $t = 0$.

Remark 1.5. ([3]) We denote Ψ the class of altering distance functions.

Definition 1.6. ([3]) A function $\varphi : [0,\infty) \to [0,\infty)$ is said to be an ultra altering distance function, if it is continuous, non-decreasing such that $\varphi(t) > 0$ $t > 0$ and $\varphi(0) \geq 0$.

Remark 1.7. ([3]) We denote Φ_u the class of ultra altering distance functions.

Definition 1.8. Let (X,d) be a metric space and K be a nonempty subset of X. Let $T : K \to X$ be a mapping, T is said to be generalized weakly contractive on K, if $Tx \in K$ and

$$\psi(d(Tx,Ty)) \leq F(\psi(d(x,y)), \varphi(d(x,y))) \tag{1.1}$$

for all $x, y \in K$, where F is a C-class function, $\psi : [0,\infty) \to [0,\infty)$ is non-decreasing and continuous function with $\psi(t) = 0$ if and only if $t = 0$ and φ is an ultra altering distance function.

2. Main Result

The following theorem is the main result of this paper, the proof which proceeds by steps, is based on an argument similar to the one used by Khan [13].

Theorem 2.1. *Let (X,d) be a complete metrically convex metric space and K be a nonempty closed subset of X. Let $T : K \to X$ be a mapping satisfying for each $x \in \delta K, Tx \in K$, and*

$$\psi(d(Tx,Ty)) \leq F(\psi(d(x,y)), \varphi(d(x,y))) \tag{2.1}$$

where $F \in C, \psi \in \Psi$ and $\varphi \in \Phi_u$. Then T has a unique fixed point in K.

Proof. First, we proceed to construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way. Let $x_0 \in K$, define $y_1 = Tx_0$. If $y_1 \in K$ set $y_1 = x_1$. If $y_1 \notin K$, then choose $x_1 \in \delta K$ so that

$$d(x_0, x_1) + d(x_1, y_1) = d(x_0, y_1).$$

If $y_2 \in K$ then set $y_2 = x_2$. If $y_2 \notin K$, then choose $x_2 \in \delta K$ so that

$$d(x_1, x_2) + d(x_2, y_2) = d(x_1, y_2).$$

Thus, repeating the foregoing arguments, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ such that

(xxiii) $y_{n+1} = Tx_n$,
(xxiv) $y_n = x_n$ if $y_n \in K$,
(xxv) if $x_n \in \delta K$ then

$$d(x_{n-1}, x_n) + d(x_n, y_n) = d(x_{n-1}, y_n), \text{ where } y_n \notin K.$$

Here, we can obtain two types of sets which are denoted by

$$P = \{x_i \in \{x_n\} : x_i = y_i\}$$

and

$$Q = \{x_i \in \{x_n\} : x_i \neq y_i\}.$$

One can note that if $x_n \in Q$ then $x_{n-1} \in P$ and $x_{n+1} \in P$. We, wish to estimate $d(x_n, x_{n+1})$. Now, we distinguish the following three cases.

Case 1. If $x_n \in P$ and $x_{n+1} \in P$, then

$$\begin{aligned}\psi\big(d(x_n, x_{n+1})\big) &= \psi\big(d(Tx_{n-1}, Tx_n)\big)\\ &\leq F\Big(\psi\big(d(x_{n-1}, x_n)\big), \varphi\big(d(x_{n-1}, x_n)\big)\Big)\\ &\leq \psi\big(d(x_{n-1}, x_n)\big).\end{aligned}$$

Hence we have

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n).$$

Case 2. If $x_n \in P$ and $x_{n+1} \in Q$, then

$$d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}) = d(x_n, y_{n+1}).$$

Therefore

$$\begin{aligned}\psi\big(d(x_n,x_{n+1})\big) &\le \psi\big(d(x_n,y_{n+1})\big)\\ &= \psi\big(d(Tx_{n-1},Tx_n)\big)\\ &\le F\Big(\psi\big(d(x_{n-1},x_n)\big),\varphi\big(d(x_{n-1},x_n)\big)\Big)\\ &\le \psi\big(d(x_{n-1},x_n)\big).\end{aligned}$$

Hence we have

$$d(x_n,x_{n+1}) \le d(x_{n-1},x_n).$$

Case 3. Let $x_n \in Q$ and $x_{n+1} \in P$. Since $x_n \in Q$ and is a convex linear combination of x_{n-1} and y_n, it follows that

$$d(x_n,x_{n+1}) \le \max\{d(x_{n-1},x_{n+1}), d(y_n,x_{n+1})\}.$$

If $d(x_{n-1},x_{n+1}) \le d(x_{n+1},y_n)$, then

$$\begin{aligned}\psi\big(d(x_n,x_{n+1})\big) &\le \psi\big(d(x_{n+1},y_n)\big)\\ &= \psi\big(d(Tx_{n-1},Tx_n)\big)\\ &\le F\Big(\psi\big(d(x_{n-1},x_n)\big),\varphi\big(d(x_{n-1},x_n)\big)\Big)\\ &\le \psi\big(d(x_{n-1},x_n)\big).\end{aligned}$$

Hence we have

$$d(x_n,x_{n+1}) \le d(x_{n-1},x_n).$$

Otherwise if $d(x_{n+1},y_n) \le d(x_{n-1},x_{n+1})$, then

$$\begin{aligned}\psi\big(d(x_n,x_{n+1})\big) &\le \psi\big(d(x_{n-1},x_{n+1})\big)\\ &= \psi\big(d(Tx_{n-2},Tx_n)\big)\\ &\le F\Big(\psi\big(d(x_{n-2},x_n)\big),\varphi\big(d(x_{n-2},x_n)\big)\Big)\\ &\le \psi\big(d(x_{n-2},x_n)\big).\end{aligned}$$

Hence we have

$$d(x_n,x_{n+1}) \le d(x_{n-2},x_n).$$

Notice that

$$\begin{aligned}d(x_{n-2},x_n) &\le d(x_{n-2},x_{n-1}) \ + \ d(x_{n-1},x_n)\\ &\le \max\{d(x_{n-2},x_{n-1}), d(x_{n-1},x_n)\}.\end{aligned}$$

Here, if $d(x_{n-2},x_{n-1}) \le d(x_{n-1},x_n)$, then

$$d(x_n,x_{n+1}) \le d(x_{n-1},x_n).$$

Otherwise, if $d(x_{n-1}, x_n) \leq d(x_{n-2}, x_{n-1})$, then

$$d(x_n, x_{n+1}) \leq d(x_{n-2}, x_{n-1}).$$

Thus in all the cases, we have

$$d(x_n, x_{n+1}) \leq \max\{d(x_{n-1}, x_n), d(x_{n-2}, x_{n-1})\}.$$

It follows that the sequence $\{d(x_n, x_{n+1})\}$ is monotonically decreasing. Hence $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

Now, we prove that the sequence $\{x_n\}$ is Cauchy. Let on contrary that the sequence $\{x_n\}$ is not Cauchy. Then there exists $\epsilon > 0$ for which we can find subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$. Further, corresponding to each $m(k)$, we can find $n(k)$ in such a way that the smallest positive integer $n(k) > m(k)$ satisfying $d(x_{m_k}, x_{n_{k-1}}) < \epsilon$. Now, we have

$$\epsilon \leq d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{n_k}) < \epsilon + d(x_{n_{k-1}}, x_{n_k}).$$

On letting $k \to \infty$, we have $d(x_{m_k}, x_{n_k}) = \epsilon$. Again,

$$d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{m_{k-1}}) + d(x_{m_{k-1}}, x_{m_k})$$

whereas

$$d(x_{n_{k-1}}, x_{m_{k-1}}) \leq d(x_{n_{k-1}}, x_{n_k}) + d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_{k-1}}).$$

Now on letting $k \to \infty$ in the above inequalities, we obtain,

$$\lim_{k\to\infty} d(x_{n_{k-1}}, x_{m_{k-1}}) = \epsilon.$$

By setting $x = x_{m_{k-1}}$ and $y = x_{n_{k-1}}$ in (2.1). On letting $k \to \infty$, we obtain,

$$\psi(\epsilon) \leq F\big(\psi(\epsilon), \varphi(\epsilon)\big),$$

which implies $\psi(\varepsilon) = 0$ or $\varphi(\varepsilon) = 0$. That is $\varepsilon = 0$, which is a contradiction. Thus the sequence $\{x_0, x_1, x_2, \cdots, x_{n-1}, x_n, x_{n+1}, \cdots\}$ is Cauchy and hence convergent. Let $x_n \to z$ as $n \to \infty$. Substituting $x = x_{n-1}$ and $y = z$ in equation (2.1), we obtain

$$\psi\big(d(Tz, x_n)\big) \leq F\Bigg(\psi\big(d(z, x_{n-1})\big), \varphi\big(d(z, x_{n-1})\big)\Bigg).$$

Letting $n \to \infty$ and using continuity of ϕ, we have

$$\psi\big(d(Tz, z)\big) \leq F\big(\psi(0), \varphi(0)\big) \leq \psi(0) = 0,$$

implying thereby $Tz = z$. This shows that z is a fixed point T.

Next, to prove that the uniqueness of fixed points. Let us suppose that z_1 and z_2 are two fixed points of T. Then

$$\begin{aligned}\psi\big(d(z_1,z_2)\big) &= \psi\big(d(Tz_1,Tz_2)\big)\\ &\le F\Big(\psi\big(d(z_1,z_2)\big),\varphi\big(d(z_1,z_2)\big)\Big)\\ &\le \psi\big(d(z_1,z_2)\big).\end{aligned}$$

which implies that $\psi\big(d(z_1,z_2)\big)=0$. Hence $d(z_1,z_2)=0$, that is $z_1=z_2$. This completes the proof.

Remark 2.2. By setting $K=X, F(s,t)=s-t$ and $\psi(t)=t$ in the Theorem 2.1, then we deduce a theorem due to Rhoades [16].

Remark 2.3. By setting $K=X, F(s,t)=s-t$ and $\psi(t)=t$ in the Theorem 2.1, then we deduce a partial generalization of theorem due to Alber and Guerre-Delabriere [2].

By setting $F(s,t)=s-t$, $\psi(t)=t$ in the Theorem 2.1, then we deduce the following corollary in the form of the result due to Khan and Imdad [14].

Corollary 2.4. Let (X,d) be a complete metrically convex metric space and K be a nonempty closed subset of X. Let $T:K\to X$ be a mapping satisfying for each $x\in\delta K, Tx\in K$,

$$d(Tx,Ty)\le d(x,y)-\varphi\big(d(x,y)\big)$$

where $\varphi\in\Psi$. Then T has a unique fixed point in K.

By setting $F(s,t)=ks, 0<k<1$ in the Theorem 2.1, then we deduce the following corollary in the form of Banach Contraction Principle.

Corollary 2.5. Let (X,d) be a complete metrically convex metric space and K be a nonempty closed subset of X. Let $T:K\to X$ be a mapping satisfying for each $x\in\delta K, Tx\in K$,

$$d(Tx,Ty\le kd(x,y)$$

where $0<k<1$. Then T has a unique fixed point in K.

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