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CONJUGATE DUALITY FOR CONCAVE MAXIMIZATION PROBLEMS AND APPLICATIONS

T. V. Thang[1] and N. D. Truong[2]

[1]Department of Mathematics, Electric Power University
Hanoi, Vietnam
e-mail: `thangtv@epu.edu.vn`

[2]Department of Mathematics, Haiphong University
Haiphong, Vietnam
e-mail: `truongnd@dhhp.edu.vn`

Abstract. In this article, we present a conjugate duality for scalarmaximization and vector-maximization problems involving concave increasing continuous homogeneous functions. We will show that the obtained conjugate duality has zero-gap and a duality inequality helps to characterize the weakly Pareto efficient for the vector-maximization problem. As a result, an optimization problem over the weakly efficient set reduces to a bilevel optimization problem solvable by monotonic optimization methods.

1. Introduction

The conjugate duality was constructed based on the conjugate of function $f:\ R^n_+ \to R_+$ in the form:

$$f^*(p) = \frac{1}{\sup\{f(x):\ p^T x \leq 1, x \geq 0\}}, \quad \forall\, p \in R^n_+,$$

by Thach, which has recently been applied to some optimization problems in economics (see for instance [12], [13], [14]). In [12], Thach applied the conjugate duality to study a maximization of a Leontief production function over a

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[0]Corresponding author: T. V. Thang(`thangtv@epu.edu.com`).

convex region of activities, he obtained a dual problem that is a maximization of an increasing linear function over a convex region. In [13], we developed the conjugate duality to a problem of maximizing a polyhedral concave increasing homogeneous function over a convex set, the obtained dual problem is also a problem of maximizing a polyhedral concave increasing homogeneous function over a convex feasible set. Moreover, when extended to a vector-maximization problem we obtained symmetric duality scheme and a duality equation that helps to characterize the weakly Pareto efficient. This duality scheme was applied to a nonlinear problem under resource allocation constraints. We shown that a nonlinear feasibility problem of planning under a single resource allocation constraint can be reduced to the maximization of a Cobb-Douglas function over a polytope and the feasibility problem of planning under multiple resource allocation constraints is reduced to a vector optimization problem involving k Cobb-Douglas functions (k being the number of resources) ([14]).

So far, this duality has been restricted to the classes of scalarmaximization and vector-maximization problems involving polyhedral concave criteria functions or Cobb-Douglas functions. However, many functions encountered in mathematical economics and other applications are not polyhedral concave but only concave. The first part of the paper is to extend the conjugate duality to the more general classes of concave scalar-maximization and concave vector-maximization problems involving concave functions. The extension will preserve the main properties earlier established in paper [13], specifically, the conjugate duality is symmetric and the gap duality is zezo. We also present duality relationships that can help to characterize the (weakly) Pareto efficient solutions in vector-maximization problems. Note that the duality theory for vector-optimization problems is often based on Lagrange duality applied for the scalarization depending on the weight parameters (cf. [5], [8], [9], [11]). In some extension the theory of set-valued map is used for vector-optimization problems (cf. [6], [7]), but the conjugate duality obtained by these way often requires convexity of the primal problem. The second part of this article is to apply the conjugate duality scheme to reduce an optimization problem over the weakly efficient set (this problem amounts to finding a feasible production program with maximal profit) to a bilevel optimization problem solvable by monotonic optimization methods.

The paper is organized as follows. After the introduction, in section 2, we present duality conjugate for scalar-maximization problems. Next in section 3, we will develop duality for vector-maximization problems. Finally, section 4 is devoted to an application of the duality to optimization over weakly efficient sets.

2. Conjugate Duality

2.1. **Conjugate Duality for Scalar-Maximization Problems.** Let us consider the nonnegative $n-$dimensional orthant R_+^n of activities. A vector $x(x_1, x_2, ..., x_n)$ in R_+^n is called an activity vector where x_i indicaties the $i-$th type of activities. The production function for an activity vector x is $f(x)$, where $f(x)$ is a increasing continuous concave homogeneous finite-valued function defined on R_+^n and f is not identically zero on R_+^n. We recall that f is increasing if

$$f(x) \geq f(x^{'}), \quad \forall x \in R_+^n, \quad \forall x^{'} \in R_+^n : x \geq x^{'}.$$

X is said to be normal if $x \in X : 0 \leq x^{'} \leq x$, then $x^{'} \in X$. Let X be a compact convex normal set in R_+^n. Obviously, $f(x) \geq 0$ for all $x \in R_+^n$ (because $f(x)$ is increasing and homogeneous).

Consider the following maximization problem:

$$\max f(x), \quad s.t. \;\; x \in X. \tag{2.1}$$

Since f is continuous and X is compact, the problem (2.1) is solvable. Moreover, the optimal value of the problem is positive (because f is not identically zero on R_+^n).

Define

$$F = \left\{x \in R_+^n : f(x) \geq 1\right\}.$$

F is said to be conormal if for $x \in F$, $x \leq x^{'}$, then $x^{'} \in F$. Since $f(x)$ is increasing continuous concave, F is a closed convex conormal set in R_+^n. Since f is homogeneous, f is gauge function of F, that is,

$$f(x) = \max\left\{\gamma \geq 0 : x \in \gamma F\right\}, \;\; \forall x \in R_+^n.$$

We define F^* the upper conjugate of F.

$$F^* = \left\{p \in R_+^n : p^T x \geq 1, \;\; \forall x \in F\right\}.$$

Obviously, F^* is a subset in the dual space. Moreover, F^* is a closed convex conormal set in R_+^n. The conjugate of f is defined by

$$f^*(p) = \max\left\{\gamma \geq 0 : x \in \gamma F^*\right\}, \;\; \forall p \in R_+^n.$$

We can check that

$$F^* = \left\{p \in R_+^n : f^*(p) \geq 1\right\},$$

and f^* is a increasing continuous homogeneous finite-valued function defined on R_+^n.

Theorem 2.1. *We have*

$$f^*(p) = \frac{1}{\sup\{f(x) : \;\; p^T x \leq 1, x \geq 0\}}, \quad \forall p \in R_+^n,$$

(hear, we agree that $\frac{1}{+\infty} = 0$*).*

Proof. Define

$$\overline{f}(p) = \frac{1}{\sup\{f(x):\ p^T x \leq 1, x \geq 0\}}, \quad \forall p \in R^n_+.$$

If $q = 0$, we have $\overline{f}(0) = \frac{1}{+\infty} = 0$.
For $p \neq 0$ and $\theta > 0$ we have

$$\begin{aligned} \sup\{f(x):\ \theta p^T x \leq 1, x \geq 0\} &= \sup\{f(\frac{1}{\theta}x'):\ p^T x' \leq 1, x' \geq 0\} \\ &= \frac{1}{\theta}\sup\{f(x'):\ p^T x' \leq 1, x' \geq 0\}, \end{aligned}$$

hence, $\overline{f}(\theta p) = \theta\overline{f}(p)$. So, $\overline{f}$ is homogeneous. In order to prove $f^* = \overline{f}$ we need to prove that

$$F^* = \left\{p \in R^n_+ : \overline{f}(p) \geq 1\right\}.$$

For $p \geq 0$ we have

$$\begin{aligned} &\overline{f}(p) \geq 1 \\ &\Leftrightarrow \sup\{f(x):\ p^T x \leq 1, x \geq 0\} \leq 1 \\ &\Leftrightarrow f(x) \leq 1\ \forall x \geq 0:\ p^T x \leq 1 \\ &\Leftrightarrow p^T x > 1\ \forall x \geq 0:\ f(x) > 1 \\ &\Leftrightarrow p^T x \geq 1\ \forall x \geq 0:\ f(x) \geq 1 \\ &\Leftrightarrow p^T x \geq 1\ \forall x \in F \\ &\Leftrightarrow p \in F^*. \end{aligned}$$

This completes the proof. □

We obtain the following proposition.

Proposition 2.2. *If F^* is a upper conjugate of F, then F is the upper conjugate of F^*:*

$$F = \left\{x \in R^n_+ : p^T x \geq 1,\ \forall p \in F^*\right\}.$$

Proof. Setting $A = \left\{x \in R^n_+ : p^T x \geq 1\, \forall p \in F^*\right\}$, since F^* is upper conjugate of F, it is obvious that $F \subseteq A$. Suppose that $\overline{x} \geq 0$ and $\overline{x} \notin F$. Since F is a convex set in R^n_+, F does not intersect with the line segment $[0; \overline{x}]$. By the separation theorem, there is $q \in R^n \setminus \{0\}$ and number $\alpha \in R$ such that

$$q^T x \geq \alpha, \quad \forall x \in F; \tag{2.2}$$

$$q^T \overline{x} < \alpha, \quad \forall x \in [0; \overline{x}]. \tag{2.3}$$

From (2.2) it follows that $q \geq 0$, and from (2.3) it follows that $\alpha > 0$. Setting $p = \frac{1}{\alpha} q$, we have

$$p^T x \geq 1, \quad \forall x \in F; \tag{2.4}$$

$$p^T \overline{x} < 1. \tag{2.5}$$

From (2.4) it follows that $p \in F^*$. This together with (2.5) implies that $\overline{x}$ does not belong to A. $\square$

Since the upper conjugate of F^* is F, the conjugate of f^* is f, and so

$$f(x) = \frac{1}{\sup\{f^*(p) : \ p^T x \leq 1, p \geq 0\}}, \quad \forall x \in R^n_+.$$

Example 2.3. Let f be Leontief function on R^n_+:

$$f(x) = \min\{\frac{x_i}{c_i} : i = 1, 2, ..., n\},$$

where $c \in R^n, c > 0$. It can be check that (cf. [12])

$$f^*(x) = \sum_{i=1}^{n} p_i c_i.$$

Example 2.4. Let f be polyhedral concave increasing and homogeneous function defined on R^n, i.e., there are $q^1 \in R^n_+, q^2 \in R^n_+, ..., q^s \in R^n_+$ such that

$$f(x) = \min\{{q^i}^T x : \ i = 1, 2, ..., s\}.$$

Setting $Y = \{x \geq 0 : \ q^i x \geq 1 \ \forall i = 1, 2, ..., s\}$. The level set Y of f on R^n_+ is a polyhedral convex subset. Let $\{y^1, y^2, ..., y^r\}$ be the set of vertices of Y. It is clear that $y^i \in R^n_+$ and $y^i \neq 0$ for all $i = 1, 2, ..., r$. Define

$$f^*(p) = \min\{{y^i}^T p : \ i = 1, 2, ..., r\}.$$

Then, f^* is is conjugate of f on R^n_+ and f^* is also polyhedral concave increasing and homogeneous function defined on R^n (cf. [13]).

Define P the lower conjugate of X: i.e,

$$P = \{p \geq 0 : p^T x \leq 1, \quad \forall x \in X\}.$$

We can check that P is also a compact convex set in R^n_+. Moreover, we have the following lemma.

Lemma 2.5. ([13]) *If P is a lower conjugate of X, then X is the lower conjugate of P:*

$$X = \{x \geq 0 : p^T x \leq 1, \quad \forall p \in P\}.$$

We consider the following problem as the dual problem of the primal problem (2.1)

$$\max f^*(p),\ s.t.\ p \in P. \tag{2.6}$$

Since X and P are the conjugates of each other and the conjugate of f^* is f, the duality between (2.1) and (2.6) is involutory. Now we present strong duality for problems (2.1) and (2.6) in the following theorem.

Theorem 2.6. *Let $\overline{x} \in X$ and $\overline{p} \in P$. Then, $\overline{x}$ is optimal to* (2.1) *and $\overline{p}$ is optimal to* (2.6) *if and only if*

$$f(\overline{x})f^*(\overline{p}) = 1. \tag{2.7}$$

Proof. For every $x \in X$ and $p \in P$, we have

$$f(x)f^*(p) \leq 1. \tag{2.8}$$

Indeed, if $f(x) = 0$ then $f(x)f^*(p) = 0 \leq 1$. If $f(x) > 0$ then, from $p^T x \leq 1$ (because $x \in X$ and $p \in P$) it follows that

$$f(x)f^*(p) = f(x)\frac{1}{\sup\{f(x) :\ p^T x \leq 1, x \geq 0\}} \leq f(x)\frac{1}{f(x)} = 1.$$

So, if $\overline{x} \in X$ and $\overline{p} \in P$ satisfies the dual equation (2.7) then, we have

$$f(\overline{x})f^*(\overline{p}) = \max\{f(x)f^*(p) :\ x \in X, p \in P\} = \max_{x\in X} f(x) \max_{p\in P} f^*(p),$$

consequently,

$$f(\overline{x}) = \max_{x\in X} f(x),\ \ f^*(\overline{p}) = \max_{p\in P} f^*(p).$$

Hence, $\overline{x}$ solves (2.1) and $\overline{p}$ solves (2.6).

Conversely, suppose that $\overline{x}$ solves (2.1) and $\overline{p}$ solves (2.6). We have $f(\overline{x}) > 0$. So, $intX \cap int(f(\overline{x})F) = \emptyset$. By the separation theorem there is $q \in R^n_+$ such that

$$q^T x \leq 1,\ \ \forall x \in X; \tag{2.9}$$

$$q^T x \geq 1,\ \ \forall x \in f(\overline{x})F. \tag{2.10}$$

From (2.8) it follows $q \in P$. From (2.10) we have $f(\overline{x})q^T x \geq 1$ for all $x \in Y$, it implies $f(\overline{x})q \in F^*$. Hence, $f^*(f(\overline{x})q) \geq 1$ or $f^*(q)f(\overline{x}) \geq 1$. By virtue of (2.8), it implies $f^*(q)f(\overline{x}) = 1$, leading to q is optimal solution of the problem (2.1). Since $\overline{p}$ also solves (2.1), we have $f^*(q) = f^*(\overline{p})$. Therefore, we have $f^*(\overline{p})f(\overline{x}) = 1$. □

By Theorem (2.6), we can says that the equation (2.7) is dual equation for the scalar-maximization problems.

2.2. **Duality for Vector-Maximization Problems.** Suppose that the space R^n is the Decarte product of the subspaces R^{n_i} $i = 1, 2, ..., k (k \geq 1)$

$$R^n = \prod_{i=1}^{k} R^{n_i},$$

where $n = \sum_{i=1}^{k} n_i$ and $n_i \geq 1, i = 1, 2, ..., k$. For any $i = 1, 2, ..., k$ let $x^i \in R^{n_i}_+$ be an activity subvector, and denote $x = (x^1, x^2, ..., x^k)$. Similarly, for any $i = 1, 2, ..., k$ let $p^i \in R^{n_i}_+$ be a dual activity subvector, and denote $p = (p^1, p^2, ..., p^k)$.

For any $i = 1, 2, ..., k$ we are given a production function f_i defined on $R^{n_i}_+$ that is a increasing continuous concave homogeneous finite-valued function and not identically zero on ${R^{n}}^{i}_{+}$. Denote by f_i^* the conjugate of f_i on $R^{n_i}_+$ for any $i = 1, 2, ..., k$. Let X be a compact convex normal set in R^n_+ and denote by P the lower conjugate of X. We consider the primal vector-maximization problem

$$\begin{aligned} & f_i(x^i) \to \max \;\; i = 1, 2, ..., k \\ & s.t \; x = (x^1, x^2, ..., x^k) \in X \end{aligned} \tag{2.11}$$

and the its dual problem

$$\begin{aligned} & f_i^*(p^i) \to \max \;\; i = 1, 2, ..., k \\ & s.t \; p = (p^1, p^2, ..., p^k) \in P. \end{aligned} \tag{2.12}$$

Example 2.7. Let f_i be Leontief functions on $R^{n_i}_+$, i.e.,

$$f_i(x^i) = \min\{\frac{x^i_j}{c^i_j} : j = 1, 2, ..., n_i\}, \quad i = 1, 2, ..., k;$$

where $c^i \in R^{n_i}, c^i > 0$. Then, we have primal vector-maximization problem

$$\begin{aligned} & \min\{\frac{x^i_j}{c^i_j} : j = 1, 2, ..., n_i\} \to \max \;\; i = 1, 2, ..., k \\ & s.t \; x = (x^1, x^2, ..., x^k) \in X. \end{aligned} \tag{2.13}$$

By example (2.3) the dual problem of (2.13) is

$$\begin{aligned} & {p^i}^T c^i \to \max \;\; i = 1, 2, ..., k \\ & s.t \; p = (p^1, p^2, ..., p^k) \in P. \end{aligned} \tag{2.14}$$

The first result is the so-called weak duality theorem.

Theorem 2.8. ([13]) *For any $x \in X$ and $p \in P$ we have*

$$\sum_{i=1}^{k} f_i(x^i) f_i^*(p^i) \leq 1. \tag{2.15}$$

Theorem 2.9. ([13]) *Let $\overline{x} \in X$ and $\overline{p} \in P$. If $(\overline{x}, \overline{p})$ satisfies the equation*

$$\sum_{i=1}^{k} f_i(x^i) f_i^*(p^i) = 1, \tag{2.16}$$

then $\overline{x}$ is primal weakly Pareto efficient and $\overline{p}$ is dual weakly Pareto efficient.

Theorem 2.10. ([13]) *Let $\overline{x} \in X$ and $\overline{p} \in P$ such that $(\overline{x}, \overline{p})$ satisfies the equation* (2.16). *If $g_i^*(\overline{p}^i) > 0$ for any $i = 1, 2, ..., k$, then $\overline{x}$ is primal Pareto efficient. If $f_i^*(\overline{x}^i) > 0$ for any $i = 1, 2, ..., k$, then $\overline{p}$ is dual Pareto efficient.*

The above theorems says that the equation (2.16) is called the dual equation that referee to the strong duality. For the strong duality we have the following theorem.

Theorem 2.11. *If $\overline{x} > 0$ is primal weakly Pareto efficient then there is $\overline{p} \in P$ such that $(\overline{x}, \overline{p})$ satisfies the dual equation* (2.16). *Similarly, If $\overline{p} > 0$ is dual weakly Pareto efficient then there is $\overline{x} \in X$ such that $(\overline{x}, \overline{p})$ satisfies the dual equation* (2.16).

Proof. Suppose $\overline{x} > 0$ is primal weakly Pareto efficient. Setting

$$\Omega_{\overline{x}} = \{z \in R^n_+ : \; f_i(z^i) > f_i(\overline{x}^i) \; i = 1, 2, ..., k\},$$

then $\Omega_{\overline{x}}$ is an open convex conormal set in R^n_+ and $\Omega_{\overline{x}}$ has no intersection with the X. By the separation theorem, there are $u \in R^n \setminus \{0\}$ and $\alpha \in R$ such that

$$u^T z \leq \alpha, \quad \forall z \in X \tag{2.17}$$

and

$$u^T z > \alpha, \quad \forall z \in \Omega \overline{z}. \tag{2.18}$$

Since $\Omega_{\overline{x}}$ is an open convex conormal set in R^n_+, from (2.18) it follows that $u \geq 0$. From (2.17) it follows that $\alpha > 0$ (because X is normal). Set $\overline{p} = \frac{1}{\alpha} u$. Then, (2.17) and (2.18) are equivalent to

$$\overline{p}^T z \leq 1, \quad \forall z \in X \tag{2.19}$$

and

$$\overline{p}^T z > 1, \quad \forall z \in \Omega \overline{z}. \tag{2.20}$$

Then, from (2.19) it follows $\overline{p} \in P$. From (2.20), there are $\mu \in R^k_+ \setminus \{0\}$ such that

$$\overline{p} = \bigoplus_{i=1}^{k} \mu_i q^i, \tag{2.21}$$

$$1 \leq \sum_{i=1}^{k} \mu_i f_i(\overline{x}^i), \tag{2.22}$$

$$q^i \in \partial f_i(\overline{x}^i) \quad i = 1, 2, ..., k, \tag{2.23}$$

where $\partial f_i(\overline{x}^i)$ is the subdifferential set of f_i at $\overline{x}^i$ for any $i = 1, 2, ..., k$ (see [16]). From (2.23) we have

$$f_i(0) \leq f_i(\overline{x}^i) - {q^i}^T \overline{x}^i,$$
$$f_i(2\overline{x}^i) \leq f_i(\overline{x}^i) + {q^i}^T \overline{x}^i.$$

Hence

$${q^i}^T \overline{x}^i \leq f_i(\overline{x}^i),$$
$$f_i(\overline{x}^i) \leq {q^i}^T \overline{x}^i,$$

consequently $f_i(\overline{x}^i) = {q^i}^T \overline{x}^i$. Moreover,

$$\begin{aligned} f_i(\overline{x}^i) &= \{\max\{f_i(x^i) : \ {p^i}^T x^i \leq {p^i}^T \overline{x}^i, x^i \geq 0\} \\ &= \{\max\{f_i(x^i) : \ {p^i}^T x^i \leq f_i(\overline{x}^i), x^i \geq 0\}. \end{aligned}$$

Let

$$I = \{i \in \{1, 2, ..., k\} : \ f_i(\overline{x}^i) > 0\}.$$

Then for $i \in I$ we have

$$\begin{aligned} &\sup\{f_i(x^i) : \ {p^i}^T x^i \leq 1, x^i \geq 0\} \\ &= \sup\{f_i(x^i) : \ {p^i}^T (f_i(\overline{x}^i) x^i) \leq f_i(\overline{x}^i), x^i \geq 0\} \\ &= \sup\{f_i(\frac{1}{f_i(\overline{x}^i)} x^i) : \ {p^i}^T x^i \leq f_i(\overline{x}^i), x^i \geq 0\} \\ &= \frac{1}{f_i(\overline{x}^i)} \sup\{f_i(x^i) : \ {p^i}^T x^i \leq f_i(\overline{x}^i), x^i \geq 0\} \\ &= 1. \end{aligned}$$

So,

$$f_i^*(q^i) = 1, \quad \forall x \in I.$$

Therefore,

$$f_i^*(\overline{p}^i) = f_i^*(\mu_i q^i) = \mu_i f_i^*(q^i) = \mu_i, \quad \forall i \in I. \tag{2.24}$$

From (2.22) and (2.24) we have

$$1 \leq \sum_{i \in I} f_i(\overline{x}^i) f_i^*(\overline{p}^i) = \sum_{i=1}^{k} f_i(\overline{x}^i) f_i^*(\overline{p}^i).$$

By Theorem (2.8), we have

$$1 = \sum_{i=1}^{k} f_i(\overline{x}^i) f_i^*(\overline{p}^i).$$

Now, suppose $\overline{x} > 0$ is primal weakly Pareto efficient. Since the duality scheme is symmetric, by the arguments similar to the above we can show that there is $\overline{x} \in X$ such that the duality equation (2.16) holds at $(\overline{x}, \overline{p})$. □

3. Optimization over weakly Pareto efficient set

In this last section, as an application of the above results, we present an approach to optimization over the weakly Pareto efficient set based on bilevel programming and monotonic optimization.

We consider the problem (2.11) with assumption that X is a compact convex set in R^n_+ such that $x > 0$ for all $x \in X$. Denote by X_{WE} the set of all weakly Pareto efficient of the vector-maximization problem (2.11). We consider the following optimization problem over the weakly Pareto efficient set

$$\max q(x), \quad s.t. \;\; x \in X_{WE}, \tag{3.1}$$

where $q(x)$ is a continuous concave function defined on R^n_+ such that $q(x) > 0$ for all $x > 0$.

Define X^- the normal hull of X in R^n_+, i.e.,

$$X^- = \{y : \exists\, x \in X, x \geq y \geq 0\}$$

and P the lower conjugate of X^-

$$P = \{p \geq 0 : p^T x \leq 1 \;\forall x \in X^-\}.$$

Obviously, P is also the lower conjugate of X.

We define the following problem:

$$\begin{aligned} & f_i(x^i) \to \max \;\; i = 1, 2, ..., k \\ & s.t\; x = (x^1, x^2, ..., x^k) \in X^- \end{aligned} \tag{3.2}$$

Theorem 3.1. *Let* $\overline{x} \in X$. $\overline{x}$ *is a solution weakly Pareto efficient of* (2.11) *if and only if there is* $\overline{p} \in P$ *such that* $(\overline{x}, \overline{p})$ *satisfies the dual equation* (2.16).

Proof. Let $\overline{x}$ be a solution weakly Pareto efficient of (2.11). Then, $\overline{x} > 0$ and since the function f_i^* is increasing on $R_+^{n_i}$ for any $i = 1, 2, ..., k$, it is easily see that $\overline{x}$ is also the solution weakly Pareto efficient of (3.2). By Theorem (2.11), there is $\overline{p} \in P$ such that $(\overline{x}, \overline{p})$ satisfies the dual equation (2.16).

Conversely, suppose for $\overline{x} \in X$, there is $\overline{p} \in P$ such that $(\overline{x}, \overline{p})$ satisfies the dual equation (2.16). Then, by Theorem (2.8), $\overline{x}$ is the solution weakly Pareto efficient of (3.2). Consequently, $\overline{x}$ is also the solution weakly Pareto efficient of (2.11). □

By Theorem 2.8 and Theorem 3.1, the problem (3.1) can be rewritten as Bilevel Programming problem:

$$\max q(x) \tag{3.3}$$

$$x \in X, p \in P, \ (x, p) \ solves \tag{3.4}$$

$$\max \left\{ \sum_{i=1}^{k} f_i(x^i) f_i^*(p^i) | \ x \in X, p \in P \right\}. \tag{3.5}$$

This problem belongs to the class which is studied in [15]. Using the method proposed in the paper for solving bilevel programs we define

$$\theta(p) = \max \left\{ q(x) | \ \sum_{i=1}^{k} f_i(x^i) f_i^*(p^i) \geq 1, \ x \in X \right\},$$

(hear we agree that $\max \emptyset = 0$).

By the definition of f_i^*, for each $p \in R_+^n$ we have $f_i^*(p^i) \geq 0$ for all $i = 1, 2, ..., k$. Hence, $\theta(p)$ is a increasing function (because f_i^* is the increasing function on $R_+^{n_i}$ for any $i = 1, 2, ..., k$). Moreover, for each $p \in R_+^k$ the function $\sum_{i=1}^k f_i(x^i) f_i^*(p^i)$ is concave, this lead to the feasible set of the subproblem that defines $\theta(p)$ is a convex subset of the X. Therefore, the value of $\theta(p)$ is obtained by solving a convex problem (maximizing a concave function over a convex compact set).

We now prove that problem (3.3)-(3.5) is equivalent to the following monotonic optimization problem:

$$\max\{\theta(p) | \ p \in P\}. \tag{3.6}$$

Theorem 3.2. *The optimal values in problem* (3.6) *and problem* (3.3)-(3.5) *are equal :* $\theta^* = q^*$. *Moreover, if* $\overline{p}$ *is an optimal solution of (3.6) then* x^* *is an optimal solution of* (3.3)-(3.5), *where* x^* *is a maximizer of the function* $q(x)$ *on* $\{x \in X : \sum_{i=1}^k f_i(x^i) f_i^*(\overline{p}^i) = 1\}$.

Proof. Let $\overline{x}$ be an optimal solution of (3.3)-(3.5), i.e., $q(\overline{x}) = q^*$ and exits $\overline{p} \in P$ such that $(\overline{x}, \overline{p})$ is a maximizer of the function $\sum_{i=1}^k f_i(x^i) f_i^*(p^i)$ on $X \times P$, this is equivalent to $\sum_{i=1}^k f_i(\overline{x}^i) f_i^*(\overline{p}^i) = 1$. Then,

$$\begin{aligned} q^* &= q(\overline{x}) \\ &\leq \max\left\{ q(x) : \sum_{i=1}^k f_i(x^i) f_i^*(\overline{p}^i) \geq 1,\ x \in X \right\} \\ &= \theta(\overline{p}) \\ &\leq \theta^*. \end{aligned}$$

This particularly implies $\theta^* > 0$.

Conversely, let $\overline{p}$ be an optimal solution of (3.6), i.e., $\overline{p} \in P$ and $\theta(\overline{p}) = \theta^*$. We have $\{x \in X : \sum_{i=1}^k f_i(x^i) f_i^*(\overline{p}^i) = 1\}$ is nonempty set. Indeed, if $\{x \in X : \sum_{i=1}^k f_i(x^i) f_i^*(\overline{p}^i) = 1\} = \emptyset$. This, together with Theorem 2.8 we have

$$\begin{aligned} \theta^* &= \theta(\overline{p}) \\ &= \max\left\{ q(x) : \sum_{i=1}^k f_i(x^i) f_i^*(\overline{p}^i) \geq 1,\ x \in X \right\} \\ &= \max\left\{ q(x) : \sum_{i=1}^k f_i(x^i) f_i^*(\overline{p}^i) = 1,\ x \in X \right\} \text{(Because } \overline{p} \in P) \\ &= \max \emptyset \\ &= 0 \\ &< \theta^*, \end{aligned}$$

which is a contradiction. Let $\overline{x}$ be a maximizer of the function $q(x)$ on $\{x \in X : \sum_{i=1}^k f_i(x^i) f_i^*(\overline{p}^i) = 1\}$, we have $x \in X_{WE}$ and

$$\begin{aligned} \theta^* &= \theta(\overline{p}) \\ &= \max\left\{ q(x) : \sum_{i=1}^k f_i(x^i) f_i^*(\overline{p}^i) \geq 1,\ x \in X \right\} \\ &= \max\left\{ q(x) : \sum_{i=1}^k f_i(x^i) f_i^*(\overline{p}^i) = 1,\ x \in X \right\} \\ &= q(\overline{x}) \\ &\leq q^*. \end{aligned}$$

In summary we have $\theta^* = q^*$. This particularly implies that $\overline{x}$ is an optimal solution of the problem (3.3)-(3.5). $\square$

□

Note that if $\overline{p}$ be an optimal solution of (3.6), then $\{x \in X : \sum_{i=1}^k f_i(x^i) f_i^*(\overline{p}^i) = 1\}$ is the set of all solutions of the concave maximization problem:

$$\max\left\{ \sum_{i=1}^k f_i(x^i) f_i^*(\overline{p}^i) : \ x \in X \right\},$$

so $\{x \in X : \sum_{i=1}^k f_i(x^i) f_i^*(\overline{p}^i) = 1\}$ is a compact convex set in X. Therefore,

$$\max\left\{ q(x) : \sum_{i=1}^k f_i(x^i) f_i^*(\overline{p}^i) = 1, \ x \in X \right\}$$

is the problem of maximizing the concave function $q(x)$ over convex compact set (equivalent to a convex minimizing problem).

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