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SOME PROPERTIES OF COMMON FIXED POINT FOR TWO SELF-MAPPINGS ON SOME CONTRACTION MAPPINGS IN QUASI αb-METRIC SPACE

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Abstract. This paper proposes the common fixed point theorems and its proof for two self-mappings on some contraction mappings in quasi αb-metric space, through the coincidence point and the weakly compatible mapping.

1. Introduction

We know that b-metric space was introduced by Bakhtin in 1989 [1], and then in 1993 Czerwik [2] used this space to show the properties of fixed point of functions on some types of contraction mapping. However, many authors used b-metric space to show the existence and uniqueness of common fixed point on contraction mapping or expansive mapping [3, 4, 5, 6, 14], even with using the notion of incidence point and weakly compatible functions [7, 8]. This space developing continuously to become quasi b-metric space, this space is obtained with omitting the symmetric property in b-metric space conditions, and many authors used as dislocated quasi b-metric space to show the existence of fixed point on some contraction mappings [9, 10, 11]. While, quasi b-metric space is extension of quasi b-metric space [12, 13].

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The aim of this paper is to show the existence and uniqueness of common fixed point in quasi αb-metric space for some contraction mappings, by using coincidence and weakly compatible functions property.

2. Preliminaries

Definition 2.1. ([1, 2]) Let X be a nonempty set and let $b \geq 1$ be a given real number. Let $d : X \times X \to [0, \infty)$ be a mapping on X and for all $x, y, z \in X$ the following conditions are satisfied:

(1) $d(x, y) = d(y, x) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, y) \leq b(d(x, z) + d(z, y))$.

Then d is called a *b-metric* on X and (X, d) is called a *b-metric space*. And (X, d) is a called *quasi b-metric space*, if the conditions (1) and (3) hold. From the definition of b-metric shows that every b-metric is a quasi b-metric, but the converse is not true.

Now we introduce a generalization of quasi b-metric space by modifying the triangle inequality condition in quasi b-metric space.

Definition 2.2. ([12]) Let X be a nonempty set, $0 \leq \alpha < 1$ and $b \geq 1$ be given real number. Let $d : X \times X \to [0, \infty)$ be a mapping on X and for all $x, y, z \in X$ the following conditions are satisfied:

(1) $d(x, y) = d(y, x) = 0$ if and only if $x = y$;
(2) $d(x, y) \leq \alpha d(y, x) + \frac{1}{2} b(d(x, z) + d(z, y))$. (2.1)

Then d is called a *b-metric* on X and (X, d) is called a *b-metric space*. And (X, d) is called a *quasi b-metric space*, if the conditions (1) and (3) hold. From the definition of b-metric shows that every b-metric is a quasi b-metric, but the converse is not true.

Example 2.3. ([12, 13]) Let $X = R$ and defined $d : X \times X \to R^+$ as

$$d(x, y) = \begin{cases} 2x^2 + y^2, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

For the first condition of quasi αb-metric, it is clear from definition of function d. While for the second condition, we have to show as follows: For all $x, y, z \in X$ and $x \neq y$, we have

$$\begin{aligned} d(x,y) &= 2x^2 + y^2 \\ &\le \frac{5}{2}x^2 + 2y^2 + 3z^2 \\ &= \frac{1}{2}(2y^2 + x^2) + ((2x^2 + z^2) + (2z^2 + y^2)) \\ &= \frac{1}{2}d(y,x) + \frac{2}{2}(d(x,z) + d(z,y)). \end{aligned}$$

So we get

$$d(x,y) \le \frac{1}{2}d(y,x) + \frac{2}{2}(d(x,z) + d(z,y)).$$

Hence d is a quasi αb-metric with $\alpha = \frac{1}{2}$ and $b = 2$.

Definition 2.4. ([12, 13]) Let (X,d) be a quasi αb-metric space. Then a sequence $\{x_n\}$ in (X,d) is said to be *convergent* to $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0$, we write $\lim_{n\to\infty} x_n = x$.

Definition 2.5. ([12, 13]) Let (X,d) be a quasi αb-metric space. Then a sequence $\{x_n\}$ in (X,d) is said to be *Cauchy* in X if $\lim_{n,m\to\infty} d(x_n, x_m) = \lim_{n,m\to\infty} d(x_m, x_n) = 0$.

Definition 2.6. ([12, 13]) Let (X,d) be a quasi αb-metric space. Then (X,d) is said to be *complete* if every Cauchy sequence in X is convergent in X.

Definition 2.7. ([2]) Let X be a nonempty set and let T be a self-mapping on X. Then $x \in X$ is called a *fixed point* of T, if $Tx = x$. We define for all $x \in X$, $TT^{(n-1)}x = T^n x$ with $T^0 x = x$.

Definition 2.8. ([7, 8]) Let X be a nonempty set and $T_1, T_2 : X \to X$ be self-mapping. If $T_1 x = T_2 x = y$ for some $x \in X$, then y is called a *point of coincidence* of T_1 and T_2, and x is called a *coincidence point* of T_1 and T_2.

Definition 2.9. ([7, 8]) Let X be a nonempty set and $T_1, T_2 : X \to X$ be self-mapping. The pair $\{T_1, T_2\}$ is called *weakly compatible*, if for all $x \in X$, $T_1 x = T_2 x$ then $T_2 T_1 x = T_1 T_2 x$.

For the proof of main theorems, we need a lemma regarding sufficient conditions for the Cauchy sequence in quasi αb-metric space, this lemma has been proved in [12] as follows:

Lemma 2.10. ([12]) Let (X,d) be a quasi αb-metric space with $0 \leq \alpha < 1$ and $b \geq 1$, and let $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

Then $\{x_n\}$ is a Cauchy sequence in X.

3. Main Results

Now, we give the main results in this paper.

Theorem 3.1. *Let (X,d) be a quasi αb-metric space with $0 \leq \alpha < 1$ and $b \geq 1$. Let T_1 and T_2 be self-mappings on X such that*

$$d(T_1x, T_1y) \leq \frac{(pd(T_2x, T_1x)d(T_2x, T_1y) + qd(T_1x, T_2y)d(T_1y, T_2y))}{(1 + d(T_2x, T_1y) + d(T_1x, T_2y))} \tag{3.1}$$

for all $x, y \in X$, where $0 < p, q < 1$. If $\{T_1, T_2\}$ is weakly compatible, $T_1(X) \subseteq T_2(X)$ and T_2 (X) is complete, then T_1 and T_2 have a unique common fixed point in X.

Proof. Let $x_0 \in X$. Then $T_1x_0 \in T_1(X)$, it follows from $T_1(X) \subseteq T_2(X)$ that there exists $x_1 \in X$ such that $T_1x_0 = T_2x_1$. Since $x_1 \in X$, $T_1x_1 \in T_1(X)$. Similarly we have $T_1x_1 = T_2x_2$ for some $x_2 \in X$. With repeating this process, we can define a sequence $\{x_n\}$ such that $T_2x_n = T_1x_{n-1}$ for $n = 1, 2, 3, \cdots$. By using (3.1) we have

$$\begin{aligned}
&d(T_2x_n, T_2x_{n+1}) \\
&= d(T_1x_{n-1}, T_1x_n) \\
&\leq \frac{pd(T_2x_{n-1}, T_1x_{n-1})d(T_2x_{n-1}, T_1x_n) + qd(T_1x_{n-1}, T_2x_n)d(T_1x_n, T_2x_n)}{1 + d(T_2x_{n-1}, T_1x_n) + d(T_1x_{n-1}, T_2x_n)} \\
&= \frac{pd(T_2x_{n-1}, T_2x_n)d(T_2x_{n-1}, T_2x_{n+1}) + qd(T_2x_n, T_2x_n)d(T_1x_n, T_2x_n)}{1 + d(T_2x_{n-1}, T_2x_{n+1}) + qd(T_2x_n, T_2x_n)} \\
&= \frac{pd(T_2x_{n-1}, T_2x_n)d(T_2x_{n-1}, T_2x_{n+1})}{1 + d(T_2x_{n-1}, T_2x_{n+1})} \\
&\leq pd(T_2x_{n-1}, T_2x_n).
\end{aligned}$$

Thus we get

$$d(T_2x_n, T_2x_{n+1}) \leq pd(T_2x_{n-1}, T_2x_n).$$

Repeating this process for $n = 1, 2, 3, \cdots$, then we get

$$d(T_2x_n, T_2x_{n+1}) \leq p^n d(T_2x_0, T_2x_1).$$

Now we calculate for $d(T_2x_{n+1}, T_2x_n)$

$$\begin{aligned}
&d(T_2x_{n+1}, T_2x_n)\\
&= d(T_1x_n, T_1x_{n-1})\\
&\leq \frac{pd(T_2x_n, T_1x_n)d(T_2x_n, T_1x_{n-1}) + qd(T_1x_n, T_2x_{n-1})d(T_1x_{n-1}, T_2x_{n-1})}{1 + d(T_2x_n, T_1x_{n-1}) + d(T_1x_n, T_2x_{n-1})}\\
&= \frac{pd(T_2x_n, T_2x_{n+1})d(T_2x_n, T_2x_n) + qd(T_2x_{n+1}, T_2x_{n-1})d(T_2x_n, T_2x_{n-1})}{1 + d(T_2x_n, T_2x_n) + d(T_2x_{n+1}, T_2x_{n-1})}\\
&\leq \frac{qd(T_2x_{n+1}, T_2x_{n-1})d(T_2x_n, T_2x_{n-1})}{1 + d(T_2x_{n+1}, T_2x_{n-1})}.
\end{aligned}$$

Thus we get

$$d(T_2x_{n+1}, T_2x_n) \leq qd(T_2x_n, T_2x_{n-1}).$$

Continuing in this process for $n = 1, 2, 3, \cdots$, then we get

$$d(T_2x_{n+1}, T_2x_n) \leq q^n d(T_2x_1, T_2x_0)$$

Thus we obtain

$$d(T_2x_n, T_2x_{n+1}) \leq p^n d(T_2x_0, T_2x_1)$$

and

$$d(T_2x_{n+1}, T_2x_n) \leq q^n d(T_2x_1, T_2x_0). \tag{3.2}$$

Since $0 < p, q < 1$, it implies from (3.2) that for $n \to \infty$

$$d(T_2x_n, T_2x_{n+1}) \to 0$$

and

$$d(T_2x_{n+1}, T_2x_n) \to 0. \tag{3.3}$$

Thus by using (3.3) and Lemma 2.10, we obtain that $\{T_2x_n\}$ is a Cauchy sequence in $T_2(X)$. Since $T_2(X)$ is complete, there exists $y^* \in T_2(X)$ such that

$$d(T_2x_n, y^*) \to 0$$

and

$$d(y^*, T_2x_n) \to 0, \tag{3.4}$$

for $n \to \infty$. Since $y^* \in T_2(X)$, there is $x^* \in X$ such that $y^* = T_2x^*$.

We claim that $d(T_1x_n, T_1x^*) \to 0$ and $d(T_1x^*, T_1x_n) \to 0$.
From (3.1) we have

$$
\begin{aligned}
&d(T_1x_n, T_1x^*) \\
&\leq \frac{pd(T_2x_n, T_1x_n)d(T_2x_n, T_1x^*) + qd(T_1x_n, T_2x^*)d(T_2x^*, T_1x^*)}{1 + d(T_2x_n, T_1x^*) + d(T_1x_n, T_2x^*)} \\
&= \frac{(pd(T_2x_n, T_2x_{n+1})d(T_2x_n, T_1x^*) + qd(T_2x_{n+1}, y^*)d(y^*, T_1x^*))}{(1 + d(T_2x_n, T_1x^*) + d(T_2x_{n+1}, y^*))} \\
&= \frac{pd(T_2x_n, T_2x_{n+1})d(T_2x_n, T_1x^*)}{1 + d(T_2x_n, T_1x^*) + d(T_2x_{n+1}, y^*)} + \frac{qd(T_2x_{n+1}, y^*)d(y^*, T_1x^*)}{1 + d(T_2x_n, T_1x^*) + d(T_2x_{n+1}, y^*)} \\
&\leq \frac{pd(T_2x_n, T_2x_{n+1})d(T_2x_n, T_1x^*)}{1 + d(T_2x_n, T_1x^*) + qd(T_2x_{n+1}, y^*)d(y^*, T_1x^*)} \\
&\leq pd(T_2x_n, T_2x_{n+1}) + qd(T_2x_{n+1}, y^*)d(y^*, T_1x^*).
\end{aligned}
$$

Thus we have

$$d(T_1x_n, T_1x^*) \leq pd(T_2x_n, T_2x_{n+1}) + qd(T_2x_{n+1}, y^*)d(y^*, T_1x^*).$$

By using (3.2), we get

$$d(T_1x_n, T_1x^*) \leq p^{n+1}d(T_2x_0, T_2x_1) + qd(T_2x_{n+1}, y^*)d(y^*, T_1x^*)$$

Since $0 < p < 1$, $d(T_2x_n, y^*) \to 0$, $d(y^*, T_2x_n) \to 0$ for $n \to \infty$, then we obtain

$$d(T_1x_n, T_1x^*) \to 0. \tag{3.5}$$

And we have

$$
\begin{aligned}
&d(T_1x^*, T_1x_n) \\
&\leq \frac{pd(T_2x^*, T_1x^*)d(T_2x^*, T_1x_n) + qd(T_1x^*, T_2x_n)d(T_1x_n, T_2x_n)}{1 + d(T_2x^*, T_1x_n) + d(T_1x^*, T_2x_n)} \\
&= \frac{pd(y^*, T_1x^*)d(y^*, T_2x_{n+1}) + qd(T_1x^*, T_2x_n)d(T_2x_{n+1}, T_2x_n)}{1 + d(y^*, T_1x_n) + d(T_1x^*, T_2x_n)} \\
&= \frac{pd(y^*, T_1x^*)d(y^*, T_2x_{n+1})}{1 + d(y^*, T_1x_n) + d(T_1x^*, T_2x_n)} + \frac{qd(T_1x^*, T_2x_n)d(T_2x_n, T_2x_{n+1})}{1 + d(y^*, T_1x_n) + d(T_1x^*, T_2x_n)} \\
&\leq \frac{pd(y^*, T_1x^*)d(y^*, T_2x_{n+1})}{1 + d(y^*, T_1x_n)} + \frac{qd(T_1x^*, T_2x_n)d(T_2x_n, T_2x_{n+1})}{1 + d(T_1x^*, T_2x_n)} \\
&\leq pd(y^*, T_2x_{n+1}) + d(T_2x_n, T_2x_{n+1}).
\end{aligned}
$$

By using (3.2), we get

$$d(T_1x^*, T_1x_n) \leq d(y^*, T_2x_{n+1}) + qp^n d(T_2x_0, T_2x_1).$$

Since $0 < p < 1$, $(T_2x_n, y^*) \to 0, d(y^*, T_2x_n) \to 0$ for $n \to \infty$, then we obtain

$$d(T_1x^*, T_1x_n) \to 0. \tag{3.6}$$

We will show that y^* is a coincidence point of T_1 and T_2. From (2.1) we have

$$\begin{aligned}
&d(T_1x^*, T_2x^*)\\
&= d(T_1x^*, y^*)\\
&\leq \alpha d(y^*, T_1x^*) + \frac{b}{2}(d(T_1x^*, T_1x_n) + d(T_1x_n, y^*))\\
&= \alpha d(y^*, T_1x^*) + \frac{b}{2}(d(T_1x^*, T_1x_n) + d(T_2x_{n+1}, y^*))\\
&\leq \alpha[\alpha d(T_1x^*, y^*) + \frac{b}{2}(d(y^*, T_2x_{n+1}) + d(T_1x_n, T_1x^*))]\\
&\quad + \frac{b}{2}(d(T_1x^*, T_1x_n) + d(T_2x_{n+1}, y^*)).
\end{aligned}$$

Thus we get

$$\begin{aligned}
d(T_1x^*, y^*) \leq \alpha[\alpha d(T_1x^*, y^*) + \frac{b}{2}(d(y^*, T_2x_{n+1}) + d(T_1x_n, T_1x^*))]\\
+ \frac{b}{2}(d(T_1x^*, T_1x_n) + d(T_2x_{n+1}, y^*)).
\end{aligned}$$

By using (3.4), (3.5), (3.6) and $y^* = T_2x^*$ then we get for $n \to \infty$,

$$d(T_1x^*, T_2x^*) \to 0.$$

Then we obtain $d(T_1x^*, T_2x^*) = 0$. Thus we have $T_1x^* = T_2x^* = y^*$, this means that y^* is a coincidence point of T_1 and T_2.

We will show that T_1 and T_2 have only one coincidence point. Suppose there is another coincidence point w^* such that $T_1x = T_2x = w^*$ for some $x \in X$. Then we have

$$\begin{aligned}
d(y^*, w^*) &= d(T_1x^*, T_1x)\\
&\leq \frac{pd(T_2x^*, T_1x^*)d(T_2x^*, T_1x) + qd(T_1x^*, T_2x)d(T_1x, T_2x)}{1 + d(T_2x^*, T_1x) + d(T_1x^*, T_2x)}\\
&= \frac{pd(T_2x^*, T_2x^*)d(T_2x^*, T_1x) + qd(T_1x^*, T_2x)d(T_2x, T_2x)}{1 + d(T_2x^*, T_1x) + d(T_1x^*, T_2x)}\\
&= 0.
\end{aligned}$$

Thus we have $y^* = w^*$. Therefore T_1 and T_2 have only one coincidence point in X.

Next, we show that T_1 and T_2 have a unique common fixed point. Since T_1 and T_2 is weakly compatible, from $T_1x^* = T_2x^* = y^*$, we have $T_1T_2x^* = T_2T_1x^*$. Thus we get

$$T_1y^* = T_1T_2x^* = T_2T_1x^* = T_2y^*.$$

Therefore from (3.1) we have

$$\begin{aligned} d(y^*, T_1y^*) &= d(T_1x^*, T_1y^*) \\ &\leq \frac{pd(T_2x^*, T_1x^*)d(T_2x^*, T_1y^*) + qd(T_1x^*, T_2y^*)d(T_1y^*, T_2y^*)}{1 + d(T_2x^*, T_1y^*) + d(T_1x^*, T_2y^*)} \\ &= \frac{pd(y^*, y^*)d(y^*, T_1y^*) + qd(y^*, T_2y^*)d(T_2y^*, T_2y^*)}{1 + d(y^*, T_1y^*) + d(T_1x^*, T_2y^*)} \\ &= 0. \end{aligned}$$

Thus we get $d(y^*, T_1y^*) = 0$, it implies that $y^* = T_1y^*$ is a fixed point of T_1. Since $T_1y^* = T_2y^*$, we have $y^* = T_1y^* = T_2y^*$. Hence, y^* is a common fixed point of T_1 and T_2.

Furthermore, we will show that T_1 and T_2 have a unique common fixed point in X. Suppose there is another common fixed point s^* such that $T_1s^* = T_2s^* = s^*$. Then, from (3.2) we have

$$\begin{aligned} d(y^*, s^*) &= d(T_1y^*, T_1s^*) \\ &\leq \frac{pd(y^*, y^*)d(T_2y^*, T_1s^*) + qd(T_1y^*, T_2s^*)d(s^*, s^*)}{1 + d(T_2y^*, T_1s^*) + d(T_1y^*, T_2s^*)} \\ &= 0. \end{aligned}$$

Thus we get $d(y^*, s^*) = d(T_1y^*, T_1s^*) = 0$, it implies $y^* = s^*$. Hence T_1 and T_2 have a unique common fixed point in X. □

Theorem 3.2. *Let (X, d) be a quasi αb-metric space with $0 \leq \alpha < 1$ and $b \geq 1$. Let T_1 and T_2 be self-mappings on X such that $T_1(X) \subseteq T_2(X)$, $T_2(X)$ be complete and*

$$d(T_1x, T_1y) \leq pd(T_2x, T_2y) + qd(T_2y, T_2x) \tag{3.7}$$

for all $x, y \in X$, where $p, q > 0$, $p + q < 1$. If $\{T_1, T_2\}$ is weakly compatible, then T_1 and T_2 have a unique common fixed point in X.

Proof. Let $x_0 \in X$. Then $T_1x_0 \in T_1(X)$, it implies there exists $x_1 \in X$ such that $T_1x_0 = T_2x_1$, from $T_1(X) \subseteq T_2(X)$. Since $x_1 \in X$, $T_1x_1 \in T_1(X)$. Similarly we have $T_1x_1 = T_2x_2$ for some $x_2 \in X$. With repeating this process, we can define a sequence $\{x_n\}$ such that $T_2x_n = T_1x_{n-1}$ for $n = 1, 2, 3, \cdots$. By using (3.7) we have that

$$\begin{aligned} d(T_2x_n, T_2x_{n+1}) &= d(T_1x_{n-1}, T_1x_n) \\ &\leq pd(T_2x_{n-1}, T_2x_n) + qd(T_2x_n, T_2x_{n-1}) \end{aligned}$$

and

$$\begin{aligned} d(T_2x_{n+1}, T_2x_n) &= d(T_1x_n, T_1x_{n-1}) \\ &\leq pd(T_2x_n, T_2x_{n-1}) + qd(T_2x_{n-1}, T_2x_n). \end{aligned}$$

So we have

$$
\begin{aligned}
& d(T_2x_n, T_2x_{n+1}) + d(T_2x_{n+1}, T_2x_n) \\
& \leq (p+q)(d(T_2x_{n-1}, T_2x_n) + qd(T_2x_n, T_2x_{n-1})).
\end{aligned}
$$

Continuning this process, we get

$$
d(T_2x_n, T_2x_{n+1}) + d(T_2x_{n+1}, T_2x_n) \leq (p+q)^n(d(T_2x_0, T_2x_1) + qd(T_2x_1, T_2x_0)).
$$

Thus we have

$$
d(T_2x_n, T_2x_{n+1}) \leq (p+q)^n(d(T_2x_0, T_2x_1) + qd(T_2x_1, T_2x_0))
$$

and

$$
d(T_2x_{n+1}, T_2x_n) \leq (p+q)^n(d(T_2x_0, T_2x_1) + qd(T_2x_1, T_2x_0)).
$$

From $0 < p+q < 1$, for $n \to \infty$, we get

$$
d(T_2x_n, T_2x_{(n+1)}) \to 0, \quad d(T_2x_{(n+1)}, T_2x_n) \to 0. \tag{3.8}
$$

By using Lemma 2.10, we get that $\{T_2x_n\}$ is a Cauchy sequence in $T_2(X)$. And since $T_2(X)$ is complete, there exists $y^* \in T_2(X)$ such that for $n \to \infty$

$$
d(T_2x_n, y^*) \to 0, \quad d(y^*, T_2x_n) \to 0. \tag{3.9}
$$

Since $y^* \in T_2(X)$, there is $x^*\ inX$ such that $y^* = T_2x^*$.

Claim that $d(T_1x_n, T_1x^*) \to 0$ and $d(T_1x^*, T_1x_n) \to 0$.
From (3.7) we have

$$
d(T_1x_n, T_1x^*) \leq pd(T_2x_n, T_2x^*) + qd(T_2x^*, T_2x_n)
$$

and

$$
d(T_1x^*, T_1x_n) \leq pd(T_2x^*, T_2x_n) + qd(T_2x_n, T_2x^*).
$$

Thus we have

$$
\begin{aligned}
d(T_1x_n, T_1x^*) + d(T_1x^*, T_1x_n) & \leq (p+q)(d(T_2x_n, T_2x^*) + d(T_2x^*, T_2x_n)) \\
& = (p+q)(d(T_2x_n, y^*) + d(y^*, T_2x_n)).
\end{aligned}
$$

Thus we get

$$
d(T_1x_n, T_1x^*) \leq (p+q)(d(T_2x_n, y^*) + d(y^*, T_2x_n))
$$

and

$$
d(T_1x^*, T_1x_n)) \leq (p+q)(d(T_2x_n, y^*) + d(y^*, T_2x_n)).
$$

By using (3.9), for $n \to \infty$ we get

$$
d(T_1x_n, T_1x^*) \to 0 \ \text{ and } \ d(T_1x^*, T_1x_n) \to 0. \tag{3.10}
$$

From (2.1) we have

$$d(T_1x^*, T_2x^*) \leq \frac{1}{(1-\alpha^2)}\frac{b}{2}(d(y^*, T_2x_{n+1}) + d(T_1x_n, T_1x^*)) + \frac{b}{2}(d(T_1x^*, T_1x_n) + d(T_2x_{n+1}, y^*)).$$

Thus by (3.9), (3.10) and for $n \to \infty$, we get $d(T_1x^*, T_2x^*) = 0$, this means that $T_1x^* = T_2x^* = y^*$. Thus T_1 and T_2 have coincidence point in X.

Next, we show the uniqueness of the coincidence point of T_1 and T_2. Suppose there is another coincidence point w^* such that $T_1x = T_2x = w^*$ for some $x \in X$. Then we have

$$\begin{aligned} d(y^*, w^*) &= d(T_1x^*, T_1x) \\ &\leq pd(T_2x^*, T_2x) + qd(T_2x, T_2x^*) \\ &= pd(y^*, w^*) + qd(w^*, y^*)d(w^*, y^*) \\ &\leq pd(w^*, y^*) + qd(y^*, w^*). \end{aligned}$$

Thus we have

$$d(y^*, w^*) + d(w^*, y^*) \leq (p+q)(d(y^*, w^*) + d(w^*, y^*)).$$

Thus we get

$$(1-p-q)(d(y^*, w^*) + d(w^*, y^*)) \leq 0.$$

Since $p + q < 1$, we get $d(y^*, w^*) = 0$, this means that $y^* = w^*$.

Claim that T_1 and T_2 have a unique common fixed point in X. From (3.7) we have

$$\begin{aligned} d(y^*, T_1y^*) &= d(T_1x^*, T_1y^*) \\ &\leq pd(T_2x^*, T_2y^*) + qd(T_2y^*, T_2x^*). \end{aligned} \tag{3.11}$$

Since $\{T_1, T_2\}$ is weakly compatible, from $T_1x^* = T_2x^* = y^*$, we can have $T_1T_2x^* = T_2T_1x^*$. Thus we get

$$T_1y^* = T_1T_2x^* = T_2T_1x^* = T_2y^*. \tag{3.12}$$

Hence, from (3.11) we get

$$\begin{aligned} d(y^*, T_1y^*) &\leq pd(T_2x^*, T_2y^*) + qd(T_2y^*, T_2x^*) \\ &= pd(y^*, T_1y^*) + qd(T_1y^*, y^*). \end{aligned}$$

Similarly, we have

$$d(T_1y^*, y^*) \leq pd(T_1y^*, y^*) + qd(y^*, T_1y^*).$$

Thus we have

$$(1-p-q)(d(y^*, T_1y^*) + d(T_1y^*, y^*)) \leq 0.$$

Since $p+q<1$, we get $d(y^*,T_1y^*)=0$ and $d(T_1y^*,y^*)=0$, this means that $y^*=T_1y^*$. From using (3.12) we get $T_2y^*=T_1y^*=y^*$. Thus y^* is a unique common fixed point of T_1 and T_2. □

Theorem 3.3. *Let (X,d) be a quasi αb-metric space with $0\le\alpha<1$ and $b\ge 1$. Let T_1 and T_2 be self-mappings on X such that $T_1(X)\subseteq T_2(X)$, $T_2(X)$ be complete and*

$$d(T_1x,T_1y)\le \frac{pd(T_2x,T_2y)+qd(T_1x,T_2y)d(T_2x,T_1y)}{1+d(T_1x,T_2y)+d(T_2x,T_1y)} \tag{3.13}$$

for all $x,y\in X$, where $p,q>0$, $p+q<1$. If $\{T_1,\ T_2\}$ is weakly compatible, then T_1 and T_2 have a unique common fixed point in X.

Proof. Let $x_0\in X$, then $T_1x_0\in T_1(X)$, it implies there exists $x_1\in X$ such that $T_1x_0=T_2x_1$, since $T_1(X)\subseteq T_2(X)$. From $x_1\in X$, $T_1x_1\in T_1(X)$. Similarly we have $T_1x_1=T_2x_2$ for some $x_2\in X$. With repeating this process, we can define a sequence $\{x_n\}$ such that $T_2x_n=T_1x_{n-1}$ for $n=1,2,3,\cdots$. So by using (3.13) we have

$$\begin{aligned}
&d(T_2x_n,T_2x_{n+1})\\
&=d(T_1x_{n-1},T_1x_n)\\
&\le pd(T_2x_{n-1},T_2x_n)+\frac{qd(T_1x_{n-1},T_2x_n)d(T_2x_{n-1},T_1x_n)}{1+d(T_1x_{n-1},T_2x_n)+d(T_2x_{n-1},T_1x_n)}\\
&\le pd(T_2x_{n-1},T_2x_n)+\frac{qd(T_2x_n,T_2x_n)d(T_2x_{n-1},T_1x_n)}{1+d(T_2x_n,T_2x_n)+d(T_2x_{n-1},T_1x_n)}\\
&=pd(T_2x_{n-1},T_2x_n).
\end{aligned}$$

Continuning this process, we get

$$d(T_2x_n,T_2x_{(n+1)})\le p^nd(T_2x_0,T_2x_1).$$

And also we consider that

$$\begin{aligned}
&d(T_2x_{n+1},T_2x_n)\\
&=d(T_1x_n,T_1x_{n-1})\\
&\le pd(T_2x_n,T_2x_{n-1})+\frac{qd(T_1x_n,T_2x_{n-1})d(T_2x_n,T_1x_{n-1})}{1+d(T_1x_n,T_2x_{n-1})+d(T_2x_n,T_1x_{(n-1)})}\\
&\le pd(T_2x_n,T_2x_{n-1})+\frac{qd(T_1x_n,T_2x_{n-1})d(T_2x_n,T_2x_n)}{1+d(T_1x_n,T_2x_{n-1})+d(T_2x_n,T_2x_n))}\\
&=pd(T_2x_n,T_2x_{n-1}).
\end{aligned}$$

Continuing this process, we get

$$d(T_2x_{n+1},T_2x_n)\le p^nd(T_2x_1,T_2x_0).$$

Since $0 < p + q < 1$, for $n \to \infty$ we get

$$d(T_2x_n, T_2x_{(n+1)}) \to 0 \quad \text{and} \quad d(T_2x_{(n+1)}, T_2x_n) \to 0.$$

So from Lemma 2.10, we know that $\{T_2x_n\}$ is a Cauchy sequence in $T_2(X)$. Since $T_2(X)$ is complete, there exists $y^* \in T_2(X)$ such that for $n \to \infty$

$$d(T_2x_n, y^*) \to 0, \quad d(y^*, T_2x_n) \to 0. \tag{3.14}$$

Since $y^* \in T_2(X)$, there is $x^* \in X$ such that $y^* = T_2x^*$.

We will show that for $n \to \infty$,

$$d(T_1x_n, T_1x^*) \to 0 \quad \text{and} \quad d(T_1x^*, T_1x_n) \to 0.$$

From (3.13) we have

$$\begin{aligned} d(T_1x_n, T_1x^*) &\le pd(T_2x_n, T_2x^*) + \frac{qd(T_1x_n, T_2x^*)d(T_2x_n, T_1x^*)}{1 + d(T_1x_n, T_2x^*) + d(T_2x_n, T_1x^*)} \\ &\le pd(T_2x_n, T_2x^*) + \frac{qd(T_1x_n, T_2x^*)d(T_2x_n, T_1x^*)}{1 + d(T_2x_n, T_1x^*)} \\ &\le pd(T_2x_n, T_2x^*) + qd(T_1x_n, T_2x^*) \\ &= pd(T_2x_n, y^*) + qd(T_1x_n, y^*). \end{aligned}$$

By using (3.14), for $n \to \infty$, we get

$$d(T_1x_n, T_1x^*) \to 0. \tag{3.15}$$

From (3.13) we have

$$\begin{aligned} d(T_1x^*, T_1x_n) &\le pd(T_2x^*, T_2x_n) + \frac{qd(T_1x^*, T_2x_n)d(T_2x^*, T_1x_n)}{1 + d(T_1x^*, T_2x_n) + d(T_2x^*, T_1x_n)} \\ &\le pd(T_2x^*, T_2x_n) + \frac{qd(T_1x^*, T_2x_n)d(T_2x^*, T_1x_n)}{1 + d(T_1x^*, T_2x_n)} \\ &\le pd(y^*, T_2x_n) + qd(y^*, T_2x_{n+1}). \end{aligned}$$

By using (3.14), for $n \to \infty$, we get

$$d(T_1x^*, T_1x_n) \to 0. \tag{3.16}$$

Claim that T_1 and T_2 have only one coincidence point.
From (2.1), for $y^* = T_2x^*$, we can have

$$\begin{aligned} d(T_1x^*, T_2x^*) \le \frac{1}{(1-\alpha^2)} &\frac{\alpha b}{2}(d(y^*, T_2x_{(n+1)}) + d(T_1x_n, T_1x^*)) \\ &+ \frac{b}{2}(d(T_1x^*, T_1x_n) + d(T_2x_{(n+1)}, y^*)). \end{aligned}$$

Then by using (3.14), (3.15), (3.16) and for $n \to \infty$, we obtain

$$d(T_1x^*, T_2x^*) = 0.$$

Thus we have $T_1x^* = T_2x^* = y^*$. This means that y^* is a coincidence point of T_1 and T_2.

Suppose there is another coincidence point w^* such that $T_1x = T_2x = w^*$ for some $x \in X$. Then we have

$$\begin{aligned} d(y^*, w^*) &= d(T_1x^*, T_1x) \\ &\leq pd(T_2x^*, T_2x) + \frac{qd(T_1x^*, T_2x)d(T_2x^*, T_1x)}{1 + d(T_1x^*, T_2x) + d(T_2x^*, T_1x)} \\ &= pd(y^*, w^*) + \frac{qd(y^*, w^*)d(y^*, w^*)}{1 + d(y^*, w^*) + d(y^*, w^*)} \\ &\leq pd(y^*, w^*) + qd(y^*, w^*). \end{aligned}$$

Thus we get

$$(1 - p - q)d(y^*, w^*) \leq 0.$$

Since $0 < p+q < 1$, we get $d(y^*, w^*) = 0$. This means that $y^* = w^*$. Therefore T_1 and T_2 have only one coincidence point in X.

Claim that T_1 and T_2 have a unique common fixed point in X. From (3.14) we have

$$\begin{aligned} d(y^*, T_1y^*) &= d(T_1x^*, T_1y^*) \\ &\leq pd(T_2x^*, T_2y^*) + \frac{qd(T_1x^*, T_2y^*)d(T_2x^*, T_1y^*)}{1 + d(T_1x^*, T_2y^*) + d(T_2x^*, T_1y^*)}. \end{aligned} \tag{3.17}$$

Since $\{T_1, T_2\}$ is weakly compatible, from $T_1x^* = T_2x^* = y^*$, we have $T_1T_2x^* = T_2T_1x^*$. Thus we get

$$T_1y^* = T_1T_2x^* = T_2T_1x^* = T_2y^*. \tag{3.18}$$

Thus from (3.17) and (3.18) we get

$$\begin{aligned} d(y^*, T_1y^*) &\leq pd(T_2x^*, T_1y^*) + \frac{qd(T_1x^*, T_1y^*)d(T_2x^*, T_1y^*)}{1 + d(T_1x^*, T_1y^*) + d(T_2x^*, T_1y^*)} \\ &= pd(y^*, T_1y^*) + \frac{qd(y^*, T_1y^*)d(y^*, T_1y^*)}{1 + d(y^*, T_1y^*) + d(y^*, T_1y^*)} \\ &\leq pd(y^*, T_1y^*) + qd(y^*, T_1y^*). \end{aligned}$$

Thus we get

$$(1 - p - q)d(y^*, T_1y^*) \leq 0.$$

Since $0 < p + q < 1$, we get $d(y^*, T_1y^*) = 0$. This means that $y^* = T_1y^*$. So by using (3.18), we get $T_2y^* = T_1y^* = y^*$. Hence y^* is a unique common fixed point of T_1 and T_2. □

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