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INVESTIGATION OF THE STABILITY AND THE ANTI-SYNCHRONIZATION OF THE BRUSSELATOR CHEMICAL REACTION MODEL

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Abstract. We consider the stability and bifurcation of the famous Brusselator chemical reaction-diffusion model. To reveal the mechanism with which the transition between the fast and slow process happens, we employ the slow-fast analysis method. For the stability, it is done through coordinate transformation in order to separate the system into slow and fast subsystems. We will use the integral sliding mode control law to show the anti-synchronization of the model for all initial conditions.

1. Introduction and Preliminaries

We consider a famous chemical reaction-diffusion model with oscillations commonly called the Brusselator model. This reaction usually reaches a state of homogeneity and equilibrium quickly which makes it different from most of chemical reactions, see [6, 8, 11, 13].

The partial differential equations governing this reaction have the form:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \alpha - (\beta+1)\,u + u^2 v + k_1 \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial v}{\partial t} &= \beta u - u^2 v + k_2 \frac{\partial^2 v}{\partial x^2} \end{aligned} \tag{1.1}$$

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where $\alpha,\ \beta,\ k_1,\ k_2$ are constants and $u(t),\ v(t)$ are activator and inhibitor variables respectively. In the absence of diffusion, that is, when $k_1 = k_2 = 0$, the system becomes

$$\frac{\partial u}{\partial t} = \alpha - (\beta + 1)\, u + u^2 v, \qquad \frac{\partial v}{\partial t} = \beta u - u^2 v. \tag{1.2}$$

Notice that $\alpha,\ \beta$ are external system parameters that determines the system dynamics. Hence the variation of these parameters have effect on the time scale of the system and the shape of the limit cycles.

Recently this model was investigated, both theoretically and numerically, to study its dynamics, stability of some equilibrium points, bifurcation, synchronization and forced brusselator reactions. For example, Hopf-bifurcation, double Hopf-bifurcation and steady state bifurcation were investigated by [6, 14, 15]. The dynamics and control of the system was investigated by [11], chaos was considered by [2, 5, 9] while synchronization was investigated by [5, 12]. For computation and numerical considerations one can refer to [1, 7].

As mentioned earlier, the parameters α and β determine the dynamics of the system. For example, if we choose $\alpha = 1$ and $\beta = 2$, the system will almost be based on a single time scale and the limit cycle will be similar to a simple harmonic vibration, see Figures 1 and 2.

While if α is held fixed and β is allowed to increase such that $\beta \ggg \alpha$, then the system will exhibit dynamical behavior with two time scales. In this case

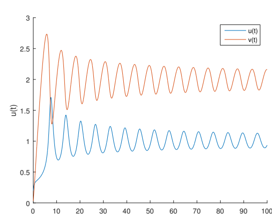

Figure 1: $\alpha = 1$ and $\beta = 2$

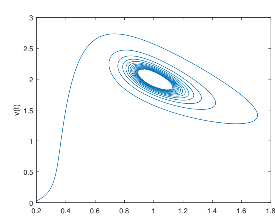

Figure 2: $\alpha = 1$ and $\beta = 2$

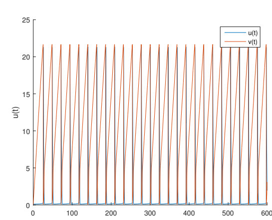

Figure 3: $\alpha = 1$ and $\beta = 8$

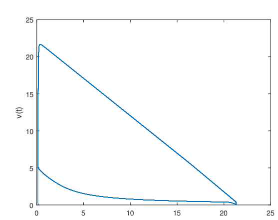

Figure 4: $\alpha = 1$ and $\beta = 8$

the system will interact with a slow and fast process, see for example Figures 3 and 4 ($\alpha = 1$ and $\beta = 8$).

Lemma 1.1. (i) *The system* (1.2) *has one steady state solution (referred to as fixed point in some cases) which is* $u = \alpha$ *and* $v = \dfrac{\beta}{\alpha}$.

(ii) *The system is stable if and only if* $\beta < \alpha^2 + 1$ *and unstable if* $\beta > \alpha^2 + 1$.

Proof. The first part is clear and is obtained by substituting $u^2 v$ from the second equation of (1.2) in first equation. For the second part, notice that the Jacobian of the system at this solution is given by

$$J = \begin{bmatrix} \beta - 1 & \alpha^2 \\ -\beta & -\alpha^2 \end{bmatrix}.$$

The characteristic polynomial of J is

$$\lambda^2 + \left(\alpha^2 - \beta + 1\right)\lambda + \alpha^2 = 0. \tag{1.3}$$

The result is clear again since the system is stable if and only if $\beta < \alpha^2 + 1$ (see Figures 5 and 6) and unstable if $\beta > \alpha^2 + 1$ which means in the unstable case we will have a limit cycle (see Figures 7 and 8), Hairer et al [3]. ☐

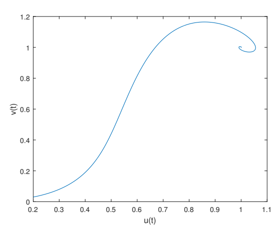

Figure 5: $\alpha = 1$ and $\beta = 1$

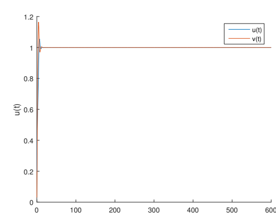

Figure 6: $\alpha = 1$ and $\beta = 1$

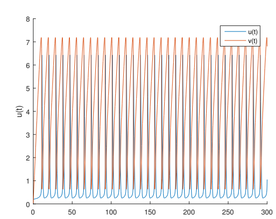

Figure 7: $\alpha = 1$ and $\beta = 4$

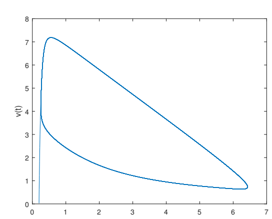

Figure 8: $\alpha = 1$ and $\beta = 4$

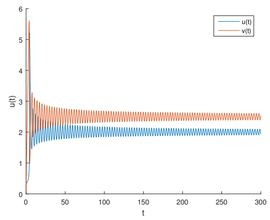

Figure 9: $\alpha = 2$ and $\beta = 5$

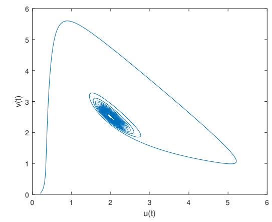

Figure 10: $\alpha = 2$ and $\beta = 5$

Notice also that the case $\beta = \alpha^2 + 1$ leads a characteristic polynomial with purely imaginary eigenvalues and leads to Hopf bifurcation, see Figures 9 and 10 .

To reveal the mechanism with which the transition between the fast and slow process happens, one needs to employ the slow-fast analysis method. It should be mentioned that this method can not be used on the system given in (1.2) directly since the parameters are present in both equations of (1.2). This suggests a coordinate transformation in order to facilitate the separation of the system into fast slow subsystems. Hence for $\beta \ggg \alpha$, introduce the transformation

$$x = v, \quad y = u + x \tag{1.4}$$

which when substituted into (1.2) we obtain

$$(a) \ \ \frac{dx}{dt} = \beta\,(y - x) - (y - x)^2\, x, \qquad (b) \ \ \frac{dy}{dt} = \alpha - (y - x)\,. \tag{1.5}$$

This leads to the separation of the parameters α and β. Note also that the new system (1.5) is topologically equivalent to (1.2) since the transformation is invertible. Here x and y denote the fast and slow subsystems respectively provided $\beta \ggg \alpha$ and $\beta \ggg 1$.

2. Main Results

In this section, we will discuss two results, one related to stability and bifurcation of the fast system and the other related to the anti-syncronization of the Brusselator system.

2.1. **Stability and Bifurcation of the Fast System:** We will discuss the stability and the bifurcation of the fast system given by (1.5)-(a). In this discussion, the slow variable y is considered as the bifurcation parameter. Now the equilibrium of the fast subsystem have to satisfy

$$\beta\,(y - x) - (y - x)^2\, x = 0 \ \text{ or } \ (y - x)\,[\beta - (y - x)\ \ x] = 0.$$

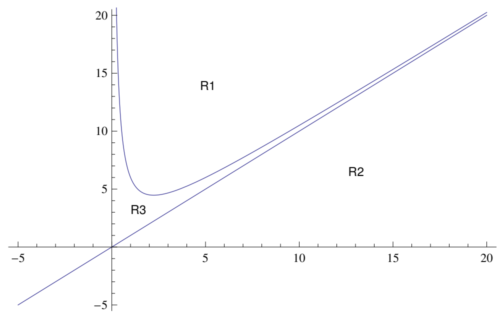

Figure 11: The three regions

Leading to

$$y = x, \quad y = \frac{x^2 + \beta}{x}, \tag{2.1}$$

see Figure 11.

Let us investigate the spectrum of the Jacobian of the system at the regions decided by the equilibrium conditions given in (2.1). Here the Jacobian is given by

$$J = \begin{bmatrix} -\beta - 2\,(y-x)\,(-x) - (y-x)^2 & \beta - 2\,(y-x)\ x \\ 1 & -1 \end{bmatrix}.$$

The determinant of the matrix J is

$$\det[J] = (y-x)^2$$

and its trace is $Tra[J] = -\beta - 1 + 2x\ \ (y-x) - (y-x)^2$ leading to the eigenvalues of J given by

$$\lambda = \frac{Tra[J] \pm \sqrt{Tra[J]^2 - 4\det[J]}}{2}.$$

Consider the following cases:

(a) When $y = x$, then $Tra[J] = -(\beta+1)$ and $\det[J] = 0$. Hence $\lambda^+ = 0, \quad \lambda^- = -\beta - 1$.

(b) When $0 < x < \sqrt{\beta}$ and since $y = \dfrac{x^2+\beta}{x}$, then $0 < y < 2\sqrt{\beta}$. Hence

$$\det[J] \leq \left(2\sqrt{\beta} - \sqrt{\beta}\right)^2 = \beta$$

and

$$\begin{aligned} Tra[J] &\leq -\beta - 1 + 2\sqrt{\beta}\ \left(2\sqrt{\beta} - \sqrt{\beta}\right) - \left(2\sqrt{\beta} - \sqrt{\beta}\right)^2 \\ &= -\beta - 1 + 2\beta - \beta = -1. \end{aligned}$$

This leads to $\lambda \leq \dfrac{-1 \pm \sqrt{1-4\beta}}{2}$.

(c) When the parameter $y > 2\sqrt{\beta}$, consider (1.5)-(a) given by $\frac{dx}{dt} = \beta\,(y-x) - (y-x)^2\,x$.To analyze the attraction domain of this equation, if we let $y - x = z$ and with y considered the bifurcation parameter, then $dz = -dx$ and the equation reduces to a variable separable first order differential equation of the form

$$\frac{dz}{\beta z + z^2\,(z-y)} = -dt \quad \text{or} \quad \frac{dz}{z\,[z^2 - 2z + \beta]} = -dt.$$

Integrating using partial fractions on the left leads to

$$\frac{1}{\beta}\ln|y-x| - \frac{1}{2\beta}\left|x^2 - xy + \beta\right| + \frac{y}{2\beta\sqrt{y^2-4\beta}}\ln\left|\frac{y-2x-\sqrt{y^2-4\beta}}{y-2x+\sqrt{y^2-4\beta}}\right| \tag{2.2}$$
$$= -t + C$$

with C the constant of integration.

Theorem 2.1. *If $y > 2\sqrt{\beta}$, then the attraction domain of the equilibrium $x = \dfrac{y-\sqrt{y^2-4\beta}}{2}$ is $x < \dfrac{y+\sqrt{y^2-4\beta}}{2}$ and the attraction domain for the other equilibrium $y = x$ is $x > \dfrac{y+\sqrt{y^2-4\beta}}{2}$. Furthermore, the system* (1.5)-(a) *possesses only one stable attractor $y = x$ when $y < 2\sqrt{\beta}$ and the attraction for the single attractor is R^+.*

Proof. From (2.2) and if $y > 2\sqrt{\beta}$ and as $t \to \infty$, there are two stable attractors when $y = x$ and $y - 2x - \sqrt{y^2-4\beta} = 0$ or $x = \dfrac{y-\sqrt{y^2-4\beta}}{2}$. In this case the region is divided into three subregions as follows, see Figure 11:

(1) $R_1 : x > \sqrt{\beta}$ and $2\sqrt{\beta} < y < x + \dfrac{\beta}{x}$. In this region notice that

$$\left|\frac{y-2x-\sqrt{y^2-4\beta}}{y-2x+\sqrt{y^2-4\beta}}\right| \neq 0$$

and hence when $t \to \infty$, we have $y = x$ or starting from any initial point in R_1 the solution converges into the stable attractor $y = x$.

(2) $R_2 : y > x + \dfrac{\beta}{x}$. Again, in this region $\left|\dfrac{y-2x-\sqrt{y^2-4\beta}}{y-2x+\sqrt{y^2-4\beta}}\right| \neq 0$. Then, as $t \to \infty$, the left side will approach $-\infty$. This means either $x = y$ or $x^2 - xy + \beta = 0$. Since $x = y$ is not in the region, then we have

$x^2 - xy + \beta = 0$ or $y = x + \dfrac{\beta}{x}$. As a result the stable contractor is $y = x + \dfrac{\beta}{x}$ or $x = \dfrac{y - \sqrt{y^2 - 4\beta}}{2}$.

(3) $R_3 : 0 < x < \sqrt{\beta}$ and $2\sqrt{\beta} < y < x + \dfrac{\beta}{x}$. This case is similar to the case of R_2 and since $x = y$ is not in the region and using the same argument, we conclude that the stable attractor is $x = \dfrac{-y - \sqrt{y^2 - 4\beta}}{2}$.

This completes the proof. □

2.2. **Anti-Synchronization of the Brusselator System.** Here we consider the anti-synchronization of two identical systems evolving from two different initial conditions. As a master or the drive system, consider the Brusselator chemical reaction system of the form:

$$\frac{dx_1}{dt} = \alpha - (\beta + 1)\, x_1 + x_1^2 y_1, \quad \frac{dy_1}{dt} = \beta x_1 - x_1^2 y_1. \tag{2.3}$$

and the slave or the response system as

$$\frac{dx_2}{dt} = \alpha - (\beta + 1)\, x_2 + x_2^2 y_2 + u_x, \quad \frac{dy_2}{dt} = \beta x_2 - x_2^2 y_2 + u_y. \tag{2.4}$$

where u_x and u_y are the control input to be determined.

Define the anti-synchronization error between the systems (2.3) and (2.4) by

$$e_x = x_2 + x_1, \qquad e_y = y_2 + y_1. \tag{2.5}$$

Here $e_x \to 0$ and $e_y \to 0$ if and only if $x_2 \to -x_1$ and $y_2 \to -y_1$, respectively. This means when the Brusselator systems (2.3) and (2.4) are anti-synchronized, their states will be equal in magnitude but opposite in sign. Here

$$\frac{de_x}{dt} = \frac{dx_1}{dt} + \frac{dx_2}{dt} = 2\alpha + x_2^2 y_2 + x_1^2 y_1 - (\beta + 1)\, e_x + u_x$$

and

$$\frac{de_y}{dt} = \frac{dy_1}{dt} + \frac{dy_2}{dt} = -x_2^2 y_2 - x_1^2 y_1 + \beta e_x + u_y.$$

Using the sliding mode control theory, see [10], the integral sliding surface of each error variable is defined by

$$S_x = \left[\frac{d}{dt} + \lambda_x\right] \left(\int_0^t e_x(r)\, dr\right) = e_x + \lambda_x \int_0^t e_x(r)\, dr$$

and

$$S_y = \left[\frac{d}{dt} + \lambda_y\right]\left(\int_0^t e_y(r)\,dr\right) = e_y + \lambda_y \int_0^t e_y(r)\,dr. \tag{2.6}$$

Differentiating with respect to t leads to

$$\frac{dS_x}{dt} = \frac{de_x}{dt} + \lambda_x e_x \quad \text{and} \quad \frac{dS_y}{dt} = \frac{de_y}{dt} + \lambda_y e_y. \tag{2.7}$$

The Hurwitz condition is satisfied if λ_x and λ_y are positive constants. Now using the exponential reaching law, see Slotine and Li[10], we set

$$\frac{dS_x}{dt} = -\eta_x sgn(S_x) - k_x S_x = \frac{de_x}{dt} + \lambda_x e_x$$

and

$$\frac{dS_y}{dt} = -\eta_y sgn(S_y) - k_y S_y = \frac{de_y}{dt} + \lambda_y e_y. \tag{2.8}$$

Substituting $\frac{de_x}{dt}$ and $\frac{de_y}{dt}$ from (2.6) into (2.7), we obtain

$$2\alpha + x_2^2 y_2 + x_1^2 y_1 - (\beta+1)e_x + u_x + \lambda_x e_x = -\eta_x sgn(S_x) - k_x S_x$$

and

$$-x_2^2 y_2 - x_1^2 y_1 + \beta e_x + u_y + \lambda_y e_y = -\eta_y sgn(S_y) - k_y S_y. \tag{2.9}$$

Hence the control laws are obtained as

$$u_x = -2\alpha - x_2^2 y_2 - x_1^2 y_1 + (\beta+1)e_x - \lambda_x e_x - \eta_x sgn(S_x) - k_x S_x$$

$$u_y = x_2^2 y_2 - x_1^2 y_1 + \beta e_x - \lambda_y e_y - \eta_y sgn(S_y) - k_y S_y. \tag{2.10}$$

Now we have the following result:

Theorem 2.2. *The Brusselator chemical reaction* (2.3) *and* (2.4) *are globally and asymptotically anti-synchronized for all initial conditions by the integral sliding mode control law* (2.10)*, where the constants* λ_x, λ_y, η_x, η_y, k_x *and* k_y *are all positive.*

Proof. The result is proved using Lyapunov stability theory, see Khalil[4]. To do so consider a Lyapunov function of the form $V(S_x,\ S_y) = \frac{1}{2}\left(S_x^2 + S_y^2\right)$ where $S_x,\ S_y$ are as given in (2.7). This means the variation of V is given by

$$\frac{dV}{dt} = S_x \frac{dS_x}{dt} + S_y \frac{dS_y}{dt}. \tag{2.11}$$

Substituting (2.8) into (2.11), we have

$$\frac{dV}{dt} = S_x \left[-\eta_x sgn\left(S_x\right) - k_x S_x\right] + S_y \left[-\eta_y sgn\left(S_y\right) - k_y S_y\right]$$

or

$$\frac{dV}{dt} = -\eta_x \left|S_x\right| - k_x S_x^2 - \eta_y \left|S_y\right| - k_y S_y^2. \tag{2.12}$$

Since the constants η_x, η_y, k_x and k_y are all positive, this means that $\frac{dV}{dt}$ is a negative definite function. Now using the Lyapunov stability theory[4], it follows that $(S_x,\ S_y) \to (0,\ 0)$ as $t \to \infty$ and as a result $(e_x,\ e_y) \to (0,\ 0)$ as $t \to \infty$. Hence the systems (2.3) and (2.4) are global asymptotically stable. □

References

[1] B. Attili, *A Second-Order Difference Scheme for a Singularly Perturbed Reaction-Diffusion Problem*, Differential and Difference Equ. Appl., (2013), 239-249.

[2] C. Grebogi and Y.-C. Lai, *Controlling Chaotic Dynamical Systems*, Syst. and Contr. Lett., **31** (1997), 307-312.

[3] E. Hairer, S. Norsett and G. Wanner, *Stability, Periodic Solutions, Limit Cycles, Strange attractors (Solving Ordinary Differential Equations I - Non-stiff Problems),* 2nd edition, Germany, Springer 2000.

[4] H. K. Khalil, *Nonlinear Systems,* 3rd ed., Prentice Hall, New Jersey, USA, 2002.

[5] J. Kurths, S. Boccalletti, C. Grebogi and Y.-C. Lai, *Introduction: Control and Synchronization in Chaotic Dynamical Systems*, Chaos, **13** (2003), 126-127.

[6] B. Li and M.X. Wang, *Diffusion-Driven Instability and Hopf -Bifurcation in Brusselator Systems,* Appl. Math. Mechanics, **29** (2008), 825-832.

[7] Z. Lin, R. Ruiz-Baier and C. Tian, *Finite Volume Element Approximation of an Inhomogeneous Brusselator Model With Cross-Diffusion,* J. Comput. Phys., **256** (2014), 806–823.

[8] T. Ma and Sh. Wang, *Phase Transition for the Brusselator Model,* J. Math. Phys., **52** (2011), 033501 [doi:10.1063/1.3559120].

[9] E. Ott, C. Grebogi and J. Yorke, *Controlling Chaos,* Phys. Rev. Lett., **64**1196 (1990).

[10] J. Slotine and W. Li, *Applied Nonlinear Control, Prentice Hall,* New Jersey, USA, 1991.

[11] S. Vaidyanathan, *Dynamics and Control of Brusselator Chemical Reaction,* Inter. J. of Chemtech Research, **8** (2015), 740-749.

[12] S. Vaidyanathan, *Anti-Synchronization of Brusselator Chemical Reaction Systems via Integral Sliding Mode Control,* Inter. J. Chemtech Research, **8** (2015), 700-713.

[13] L. Xianghong, H. Jingyu and S. Yongjun, *Slow-Fast Effect and Generation Mechanism of Brusselator Based on Coordinate Transformation,* Open Physics, **14** (2016), 261-268.

[14] P. Yu and A. B. Gumal, *Bifurcation and Stability Analysis for a Coupled Brusselator Model,* J. Sound Vibration, **244** (2001), 795-820.

[15] W. Zuo and J. Wei, *Multiple Bifurcation and Spatiotemporal Patterns for a Coupled Two-Cell Brusselator Model,* Dyn. of Part. Diff. Equat., **8** (2011), 363-384.