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CONVERGENCE RATES OF THE TIKHONOV REGULARIZATION FOR ILL-POSED MIXED VARIATIONAL INEQUALITIES WITH INVERSE-STRONGLY MONOTONE PERTURBATIONS

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Abstract. In this paper, we study the convergence rates for an operator method of regularization to solve ill-posed mixed variational inequalities involving monotone operators in Banach spaces, in case perturbative operators are inverse-strongly monotone. Our results are presented in the form of combination of finite-dimensional approximations of spaces. An illustrative numerical result is given.

1. Introduction

Variational inequality problems appear in many fields of applied mathematics such as convex programming, nonlinear equations, equilibrium models in economics, technics (see [2], [9]). These problems are studied in finite-dimensional spaces as well as infinite-dimensional spaces.

In this paper, they are considered in a real reflexive Banach space X having a property that weak and norm convergence of any sequence in X imply its strong convergence, and the dual space X^* of X is strictly convex. For the sake of simplicity, the norms of X and X^* are denoted by the symbol $\|\cdot\|$. We write $\langle x^*, x\rangle$ instead of $x^*(x)$ for $x^* \in X^*$ and $x \in X$. Then, the mixed variational inequality problem can be formulated as follows: for a given $f \in X^*$, find an

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element $x_0 \in X$ such that

$$\langle A(x_0) - f, x - x_0\rangle + \varphi(x) - \varphi(x_0) \geq 0, \quad \forall x \in X. \tag{1.1}$$

where A is a hemi-continuous and monotone operator from X into X^*, and $\varphi(x)$ is a weakly lower semicontinuous and proper convex functional on X. We will suppose that Problem (1.1) has at least one solution. For existence theorems, we refer the reader to [5]. Many problems can be seen as special cases of the problem (1.1). When φ is the indicator function of a closed convex set K in X, that is

$$\varphi(x) = I_K(x) = \begin{cases} 0, & \text{if} \quad x \in K, \\ +\infty, & \text{otherwise}, \end{cases}$$

then the problem (1.1) is equivalent to that of finding $x_0 \in K$ such that

$$\langle A(x_0) - f, x - x_0\rangle \geq 0, \quad \forall x \in K.$$

When K is the whole space X, this variational inequality is of the form of operator equation $A(x_0) = f$. When A is the Gâteaux derivative of a finite-valued convex function F defined on X, Problem (1.1) becomes the nondifferentiable convex optimization problem (see [5]):

$$\min_{x\in X}\{F(x) + \varphi(x)\}. \tag{1.2}$$

Some methods have been proposed for solving Problem (1.1), for example, the proximal point method (see [12]), and the auxiliary subproblem principle (see [8]). However, the problem (1.1) is in general ill-posed, as its solutions do not depend continuously on the data (A, f, φ), we have use stable methods for solving it. A widely used and efficient method is the regularization method introduced by Liskoves using the perturbative mixed variational inequality (see [10]):

$$\langle A_h(x_\alpha^\tau) + \alpha U^s(x_\alpha^\tau - x_*) - f_\delta, x - x_\alpha^\tau\rangle + \varphi_\varepsilon(x) - \varphi_\varepsilon(x_\alpha^\tau) \geq 0, \quad \forall x \in X, \tag{1.3}$$

where α is a regularization parameter, U^s is a generalized duality mapping of X, i.e., U^s is a mapping from X onto X^* satisfying

$$\langle U^s(x), x\rangle = \|x\|^s, \ \|U^s(x)\| = \|x\|^{s-1}, \ s \geq 2,$$

$(A_h, f_\delta, \varphi_\varepsilon)$ are approximations of (A, f, φ), $\tau = (h, \delta, \varepsilon)$ and x_* is in X which plays the role of a criterion of selection. By the choice of x_* we can obtain approximate solutions.

In this paper, we use the inequality (1.3) with the following conditions posed on the perturbations: $A_h : X \to X^*$ is the hemi-continuous monotone operator and (A_h, f_δ) are approximations for (A, f) in the sense that

$$\|A_h(x) - A(x)\| \leq hg(\|x\|), \ h \to 0, \quad \|f_\delta - f\| \leq \delta, \ \delta \to 0, \tag{1.4}$$

where $g(t)$ is a nonnegative function satisfying the condition $g(t) \le g_0 + g_1 t^\eta$, $\eta = s-1$, $g_0, g_1 \ge 0$, φ_ε are functionals defined on X having the same properties as φ, and

$$\begin{aligned} |\varphi(x) - \varphi_\varepsilon(x)| &\le \varepsilon d(\|x\|), \quad \varepsilon \to 0, \\ |\varphi_\varepsilon(x) - \varphi_\varepsilon(y)| &\le C_0\|x-y\|, \quad \forall x, y \in X, \end{aligned} \tag{1.5}$$

for some positive constant C_0 and $d(t)$ has the same properties as $g(t)$.

The existence and uniqueness of solutions x_α^τ for every $\alpha > 0$ are shown in [10]. The regularized solutions x_α^τ converges to $x_0 \in S_0$, where S_0 is the set of solutions of (1.1) which is assumed to be nonempty with x_*-minimum norm solution, i.e.

$$\|x_0 - x_*\| = \min_{x \in S_0} \|x - x_*\|,$$

if $(h+\delta+\varepsilon)/\alpha, \alpha \to 0$. The parameter choice and the convergence rate for the regularized solution x_α^τ are considered in [4] under conditions of inverse-strongly monotonicity for A. The question arises as to what happens if A_h are inverse-strongly monotones, too.

Our main purpose of this paper is to solve problem (1.2) in Banach space X for inverse-strongly monotonicity perturbations, i.e. A_h possesses the following property

$$\langle A_h(x) - A_h(y), x-y\rangle \ge m_A\|A_h(x) - A_h(y)\|^2, \quad \forall x, y \in X. \tag{1.6}$$

Then, we present these results combined with finite-dimensional approximations of the space. Finally, an illustrative example is given.

Assume that the dual mapping U^s satisfies the following conditions

$$\langle U^s(x) - U^s(y), x-y\rangle \ge m_s\|x-y\|^s, \quad m_s > 0, \tag{1.7}$$

$$\|U^s(x) - U^s(y)\| \le C(R)\|x-y\|^\nu, \quad 0 < \nu \le 1, \tag{1.8}$$

where $C(R)$, $R > 0$, is a positive increasing function on $R = \max\{\|x\|, \|y\|\}$. It is well-known that when $X = L^2[a,b]$ is a Hilbert space, then $U^s = I$, $s = 2$, $m_s = 1$, $\nu = 1$ and $C(R) = 1$, where I denotes the identity operator in the setting space (see [1]).

Finally, we use the symbols $\rightharpoonup$ and $\to$ to denote the weak convergence and convergence in norm, respectively, and the notation $a \sim b$ means that $a = O(b)$ and $b = O(a)$.

2. Main result

Theorem 2.1. *If for $h, \delta, \varepsilon > 0$ conditions (1.4), (1.5) hold and*

(i) A_h *is an inverse-strongly monotone operator from* X *into* X^**, Fréchet differentiable at some neighborhood of* $x_0 \in S_0$ *and satisfies that*

$$\|A_h(x) - A_h(x_0) - A_h'(x_0)(x - x_0)\| \le \tilde{\tau}\|A_h(x) - A_h(x_0)\|; \tag{2.1}$$

(ii) *there exists elements* z_h *such that* $\{z_h\}$ *is bounded, and*

$$A_h'(x_0)^* z_h = U^s(x_0 - x_*);$$

Then, if α *is chosen such that* $\alpha \sim (h + \delta + \varepsilon)^\eta$, $0 < \eta < 1$, *we have*

$$\|x^\tau_{\alpha(h,\delta,\varepsilon)} - x_0\| = O((h + \delta + \varepsilon)^{\mu_1}), \quad \mu_1 = \min\left\{\frac{1-\eta}{s}, \frac{\eta}{2s}\right\}.$$

Proof. It follows from (1.1), (1.3) that

$$\begin{aligned}\langle A_h(x^\tau_\alpha) - A_h(x_0), x^\tau_\alpha - x_0\rangle &+ \alpha\langle U^s(x^\tau_\alpha - x_*) - U^s(x_0 - x_*), x^\tau_\alpha - x_0\rangle \\ &\le \alpha\langle U^s(x_0 - x_*), x_0 - x^\tau_\alpha\rangle + \langle A_h(x_0) - A(x_0), x_0 - x^\tau_\alpha\rangle \\ &+ \langle f - f_\delta, x_0 - x^\tau_\alpha\rangle + \varphi_\varepsilon(x_0) - \varphi(x_0) + \varphi(x^\tau_\alpha) - \varphi_\varepsilon(x^\tau_\alpha).\end{aligned} \tag{2.2}$$

Using the monotone property of A_h and (1.4), (1.5), (1.7), the inequality (2.2) becomes

$$\begin{aligned}\|x^\tau_\alpha - x_0\|^s \le &\langle U^s(x_0 - x_*), x_0 - x^\tau_\alpha\rangle \\ &+ \frac{hg(\|x_0\|) + \delta}{\alpha}\|x_0 - x^\tau_\alpha\| \\ &+ \frac{\varepsilon}{\alpha}[d(\|x_0\|) + d(\|x^\tau_\alpha\|)].\end{aligned} \tag{2.3}$$

Hence, the boundedness of the sequence $\{x^\tau_\alpha\}$ follows from (2.3) and the properties of $g(t), d(t)$ and α. On the other hand, basing on (2.2), the property of U^s and the inverse-strongly monotone property of A_h, we get

$$\begin{aligned}\|A_h(x^\tau_\alpha) - A_h(x_0)\|^2 \le m_A^{-1}\Big\{&[hg(\|x_0\|) + \delta + \alpha\|x_0 - x_*\|^{s-1}]\|x_0 - x^\tau_\alpha\| \\ &+ \varepsilon[d(\|x_0\|) + d(\|x^\tau_\alpha\|)]\Big\}.\end{aligned}$$

Hence,

$$\|A_h(x^\tau_\alpha) - A_h(x_0)\| = O(\sqrt{h + \delta + \varepsilon + \alpha}).$$

Further, by virtue of conditions (i), (ii) and the last inequality we obtain

$$\begin{aligned}\langle U^s(x_0 - x_*), x_0 - x^\tau_\alpha\rangle &= \langle z_h, A_h'(x_0)(x_0 - x^\tau_\alpha)\rangle \\ &\le \|z_h\|(\tilde{\tau} + 1)\|A_h(x^\tau_\alpha) - A_h(x_0)\| \\ &\le \|z_h\|(\tilde{\tau} + 1)O(\sqrt{h + \delta + \varepsilon + \alpha}).\end{aligned}$$

Consequently, (2.3) has the form

$$\|x_\alpha^\tau - x_0\|^s \le \frac{hg(\|x_0\|)+\delta}{\alpha}\|x_0 - x_\alpha^\tau\| + \|z_h\|(\tilde{\tau}+1)O(\sqrt{h+\delta+\varepsilon+\alpha}) + \frac{\varepsilon}{\alpha}[d(\|x_0\|)+d(\|x_\alpha^\tau\|)].$$

When α is chosen such that $\alpha \sim (h+\delta+\varepsilon)^\eta$, $0<\eta<1$, then from the last inequality we have

$$\|x^\tau_{\alpha(h,\delta,\varepsilon)} - x_0\|^s = O\big((h+\delta+\varepsilon)^{1-\eta}\big)\|x_0 - x^\tau_{\alpha(h,\delta,\varepsilon)}\| + O\big((h+\delta+\varepsilon)^{\eta/2}\big) + O\big((h+\delta+\varepsilon)^{1-\eta}\big).$$

Therefore,

$$\|x^\tau_{\alpha(h,\delta,\varepsilon)} - x_0\| = O((h+\delta+\varepsilon)^{\mu_1}).$$

□

Remarks.

1. Note that condition (2.1) was proposed in [7] for studying convergence analysis of the Landweber iteration method for a class of nonlinear operator. The use of this condition to estimate the convergence rates of the regularized solutions of ill-posed variational inequalities was considered in [3].
2. In the works [6, 11] the given conditions are required for exact operator A, when studying nonlinear ill-posed problems. Therefore, they contain some negative aspects in solving ill-posed problems, when:
   (i) the exact operator A is not always known priori;
   (ii) the exact operator A is well-know, but it is not differentiable and;
   (iii) the well-know approximated operators A_h are not differentiable.
   In all those cases, we can also approximate them by differentiable operators (see example).

Now we consider the question of finite-dimensional approximations. Let X_n be a sequence of finite-dimensional subspaces of X: $X_n \subset X_{n+1}$, $\forall n$ and P_n a linear projection from X onto X_n such that $P_n x \to x$, $\forall x \in X$ as $n \to \infty$. Assume that P_n is uniformly bounded on X. Without loss of generality, we suppose that $\|P_n\| = 1$ (see [15]). Then the inequality

$$\langle A_h^n(x^\tau_{\alpha,n}) + \alpha U^{sn}(x^\tau_{\alpha,n} - x^n_*) - f^n_\delta, x^n - x^\tau_{\alpha,n}\rangle + \varphi_\varepsilon(x^n) - \varphi_\varepsilon(x^\tau_{\alpha,n}) \ge 0, \quad \forall x^n \in X_n, \tag{2.4}$$

where

$$A_h^n = P_n^* A_h P_n,\ U^{sn} = P_n^* U^s P_n,\ x^n = P_n x,\ f_\delta^n = P_n^* f_\delta$$

and P_n^* is the conjugate of P_n, has an unique solution $x_{\alpha,n}^\tau$ for every fixed $\alpha > 0$, $\tau > 0$ and n.

We are now in a position to prove the following result.

Theorem 2.2. *The sequence $x_{\alpha,n}^\tau$ converges the solutions x_α^τ of (1.3), as $n \to \infty$.*

Proof. It follows from (1.7) and (2.4) that

$$\begin{aligned}\alpha m_s\|x_{\alpha,n}^\tau - P_n x_\alpha^\tau\|^s &\le \alpha\langle U^s(x_{\alpha,n}^\tau - x_*^n) - U^s(P_n x_\alpha^\tau - x_*^n), x_{\alpha,n}^\tau - P_n x_\alpha^\tau\rangle \\ &\le \langle A_h^n(x_{\alpha,n}^\tau) - f_\delta^n, P_n x_\alpha^\tau - x_{\alpha,n}^\tau\rangle + \varphi_\varepsilon(P_n x_\alpha^\tau) - \varphi_\varepsilon(x_{\alpha,n}^\tau) \\ &\quad + \alpha\langle U^s(P_n x_\alpha^\tau - x_*^n), P_n x_\alpha^\tau - x_{\alpha,n}^\tau\rangle.\end{aligned}$$

Using the monotonicity of A_h and the projective property of P_n, the last inequality has the form

$$\begin{aligned}\alpha m_s\|x_{\alpha,n}^\tau - P_n x_\alpha^\tau\|^s \le\ &\langle A_h(P_n x_\alpha^\tau) - A(P_n x_\alpha^\tau) + A(P_n x_\alpha^\tau) \\ &- f_\delta, P_n x_\alpha^\tau - x_{\alpha,n}^\tau\rangle \\ &+ \varphi_\varepsilon(P_n x_\alpha^\tau) - \varphi_\varepsilon(x_{\alpha,n}^\tau) \\ &+ \alpha\langle U^s(P_n x_\alpha^\tau - x_*^n), P_n x_\alpha^\tau - x_{\alpha,n}^\tau\rangle.\end{aligned} \tag{2.5}$$

We invoke (1.4), (1.5) and (2.5) to deduce that

$$\begin{aligned}\alpha m_s\|x_{\alpha,n}^\tau - P_n x_\alpha^\tau\|^s \le\ &\big(hg(\|P_n x_\alpha^\tau\|) + \|A(P_n x_\alpha^\tau)\| + \|f_\delta\| + C_0\big) \\ &\times \|P_n x_\alpha^\tau - x_{\alpha,n}^\tau\| \\ &+ \alpha\langle U^s(P_n x_\alpha^\tau - x_*^n), P_n x_\alpha^\tau - x_{\alpha,n}^\tau\rangle.\end{aligned} \tag{2.6}$$

Obviously, the inequality (2.6) gives the boundedness of the sequence $x_{\alpha,n}^\tau$. Without loss of generality, we suppose that $x_{\alpha,n}^\tau \rightharpoonup \bar{x}_\alpha^\tau \in X$ as $n \to \infty$. It follows from (2.4) that

$$\langle A_h^n(x_{\alpha,n}^\tau) + \alpha U^s(x_{\alpha,n}^\tau - x_*^n) - f_\delta^n, P_n x - x_{\alpha,n}^\tau\rangle + \varphi_\varepsilon(P_n x) - \varphi_\varepsilon(x_{\alpha,n}^\tau) \ge 0, \quad \forall x \in X.$$

In this inequality, by letting $n \to \infty$ and using properties of A_h, φ_ε, P_n and the weak convergence of the sequence $\{x_{\alpha,n}^\tau\}$, we get

$$\langle A_h(\bar{x}_\alpha^\tau) + \alpha U^s(\bar{x}_\alpha^\tau - x_*) - f_\delta, x - \bar{x}_\alpha^\tau\rangle + \varphi_\varepsilon(x) - \varphi_\varepsilon(\bar{x}_\alpha^\tau) \ge 0, \quad \forall x \in X.$$

Since the problem (1.3) has a unique solution, so $\bar{x}_\alpha^\tau = x_\alpha^\tau$ and all the sequences $\{x_{\alpha,n}^\tau\}$ converge weakly to x_α^τ. It follows from (2.6) that the sequence $\{x_{\alpha,n}^\tau\}$ converges strongly to x_α^τ as $n \to \infty$. □

Now we set

$$\gamma_n(x) = \|(I - P_n)x\|, \quad x \in X.$$

The convergence of $x_{\alpha,n}^\tau$ to x_0 is determined by the following theorem.

Theorem 2.3. *If $h/\alpha, \delta/\alpha, \varepsilon/\alpha$ and $\gamma_n(x)/\alpha \to 0$ as $\alpha \to 0$ and $n \to \infty$, then the sequence $\{x^\tau_{\alpha,n}\}$ converges to $x_0 \in S_0$.*

Proof. For $x \in S_0, x^n = P_n x$. In the same way as in the proof of Theorem 2.2, we have

$$\begin{aligned} m_s\|x^\tau_{\alpha,n} - x^n\|^s \le \frac{1}{\alpha}\Big[&\langle A_h(x^n) - A(x^n) + A(x^n) - A(x) + A(x) - f \\ &+ f - f_\delta, x^n - x^\tau_{\alpha,n}\rangle + \varphi_\varepsilon(x^n) - \varphi_\varepsilon(x^\tau_{\alpha,n})\Big] + \langle U^s(x^n - x^n_*), x^n - x^\tau_{\alpha,n}\rangle. \end{aligned} \tag{2.7}$$

On the other hand, we invoke the monotonicity of A to deduce that

$$\|A(x^n) - A(x)\| \le \tilde{C}_0\gamma_n(x),$$

where $\tilde{C}_0$ is a positive constant depending only on x. Therefore, using this inequality, $x \in S_0$ and (1.4), (1.5), it follows from (2.7) that

$$\begin{aligned} m_s\|x^\tau_{\alpha,n} - x^n\|^s \le \frac{1}{\alpha}\Big[&\langle A_h(x^n) - A(x^n) + A(x^n) - A(x) + f - f_\delta, x^n - x^\tau_{\alpha,n}\rangle \\ &+ \langle A(x) - f, x - x^\tau_{\alpha,n}\rangle + \varphi(x) - \varphi(x^\tau_{\alpha,n}) \\ &+ \langle A(x) - f, x^n - x\rangle + \varphi_\varepsilon(x^n) - \varphi_\varepsilon(x) \\ &+ \varphi_\varepsilon(x) - \varphi(x) - \varphi_\varepsilon(x^\tau_{\alpha,n}) + \varphi(x^\tau_{\alpha,n})\Big] \\ &+ \langle U^s(x^n - x^n_*), x^n - x^\tau_{\alpha,n}\rangle, \end{aligned}$$

which implies that

$$\begin{aligned} m_s\|x^\tau_{\alpha,n} - x^n\|^s \le\ & \frac{hg(\|x^n\|) + \tilde{C}_0\gamma_n(x) + \delta}{\alpha}\|x^n - x^\tau_{\alpha,n}\| \\ &+ \frac{\varepsilon}{\alpha}\big(d(\|x^\tau_{\alpha,n}\|) + d(\|x\|)\big) \\ &+ \frac{(C_0 + \|Ax - f\|)\gamma_n(x)}{\alpha} \\ &+ \langle U^s(x^n - x^n_*), x^n - x^\tau_{\alpha,n}\rangle. \end{aligned} \tag{2.8}$$

Hence, without loss of generality, we suppose that $x^\tau_{\alpha,n} \rightharpoonup x_1 \in X$ as h/α, δ/α, ε/α, $\gamma_n(x)/\alpha \to 0$ and $n \to \infty$. By (2.4) and the properties of A_h, P_n it implies that

$$\langle A_h(x^\tau_{\alpha,n}) - f_\delta, x^n - x^\tau_{\alpha,n}\rangle + \alpha\langle U^s(x^\tau_{\alpha,n} - x^n_*), x^n - x^\tau_{\alpha,n}\rangle + \varphi_\varepsilon(x^n) \ge \varphi_\varepsilon(x^\tau_{\alpha,n}), \quad \forall x^n \in X^n.$$

After passing $h, \delta, \varepsilon, \alpha \to 0$ and $n \to +\infty$ in this inequality, we obtain

$$\langle A(x_1) - f, x - x_1\rangle + \varphi(x) - \varphi(x_1) \ge 0, \quad \forall x \in X.$$

Thus, $x_1 \in S_0$.

Now, replacing x^n in (2.8) by $x_1^n = P_n x_1$ we see that the sequence $\{x_{\alpha,n}^\tau\}$ converges strongly to x_1 and

$$\langle U^s(x - x_*), x - x_1\rangle \geq 0, \quad \forall x \in S_0.$$

Replacing x by $tx_1 + (1-t)x,\ t \in (0,1)$ in the last inequality, dividing by $(1-t)$ and letting t to 1, we get

$$\langle U^s(x_1 - x_*), x - x_1\rangle \geq 0, \quad \forall x \in S_0,$$

which leads to the following

$$\langle U^s(x_1 - x_*), x - x_*\rangle \geq \langle U^s(x_1 - x_*), x_1 - x_*\rangle = \|x_1 - x_*\|^s,\ \forall x \in S_0.$$

Hence, $\|x_1 - x_*\| \leq \|x - x_*\|$, $\forall x \in S_0$. Because of the convexity and the closedness of S_0, and the strictly convexity of X, we conclude that $x_1 = x_0$. The proof is complete. □

Set

$$\gamma_n = max\{\gamma_n(x_0), \gamma_n(x_*)\}.$$

Now, we consider the convergence rate of $\{x_{\alpha,n}^\tau\}$.

Theorem 2.4. *Assume that*

(i) *Conditions (i) and (ii) of Theorem 2.1 hold;*

(ii) *$A_h(X_n)$ are contained in X_n for sufficiently large n and small h.*

Then, for $\alpha \sim (h + \delta + \varepsilon + \gamma_n)^{\eta_1}$, $0 < \eta_1 < 1$,

$$\|x_{\alpha,n}^\tau - x_0\| = O((h + \delta + \varepsilon + \gamma_n)^{\mu_2} + \gamma_n^{\mu_3}),$$
$$\mu_2 = \min\left\{\frac{1-\eta_1}{s}, \frac{\eta_1}{2s}\right\},\ \mu_3 = \min\left\{\frac{1}{s}, \frac{\nu}{s-1}\right\}.$$

Proof. Replacing x^n by $x_0^n = P_n x_0$ in (2.8) we obtain

$$\begin{aligned} m_s\|x_{\alpha,n}^\tau - x_0^n\|^s \leq & \frac{hg(\|x_0^n\|) + \tilde{C}_0\gamma_n + \delta}{\alpha}\|x_0^n - x_{\alpha,n}^\tau\| \\ & + \frac{\varepsilon}{\alpha}\big(d(\|x_{\alpha,n}^\tau\|) + d(\|x_0\|)\big) \\ & + \frac{(C_0 + \|Ax_0 - f\|)\gamma_n}{\alpha} \\ & + \langle U^s(x_0 - x_*), x_0^n - x_{\alpha,n}^\tau\rangle \\ & + \langle U^s(x_0^n - x_*^n) - U^s(x_0 - x_*), x_0^n - x_{\alpha,n}^\tau\rangle. \end{aligned} \tag{2.9}$$

It follows from (1.7), (1.8) and condition (i) that

$$\langle U^s(x_0^n - x_*^n) - U^s(x_0 - x_*), x_0^n - x_{\alpha,n}^\tau\rangle \leq C(\tilde{R})2^\nu\gamma_n^\nu\|x_0^n - x_{\alpha,n}^\tau\|, \tag{2.10}$$

where $\tilde{R} > \|x_0 - x_*\|$, and

$$\begin{aligned}\langle U^s(x_0 - x_*), x_0^n - x_{\alpha,n}^\tau\rangle &= \langle U^s(x_0 - x_*), x_0^n - x_0\rangle \\ &\quad + \langle z_h, A_h'(x_0)(x_0 - x_{\alpha,n}^\tau)\rangle \\ &\le \|x_0 - x_*\|^{s-1}\gamma_n \\ &\quad + \|z_h\|(1+\tilde{\tau})\|A_h(x_0) - A_h(x_{\alpha,n}^\tau)\|.\end{aligned} \tag{2.11}$$

Now, we estimate the value $\|A_h(x_{\alpha,n}^\tau) - A_h(x_0)\|$. By replacing x^n by x_0^n in (2.4), using the projective property of P_n, we get

$$\begin{aligned}&\langle A_h(x_{\alpha,n}^\tau) - A_h(x_0^n) + A_h(x_0^n) - A_h(x_0) + A_h(x_0) - A(x_0) + A(x_0) - f + f \\ &- f_\delta, x_0^n - x_{\alpha,n}^\tau\rangle + \alpha\langle U^s(x_{\alpha,n}^\tau - x_*^n), x_0^n - x_{\alpha,n}^\tau\rangle + \varphi_\varepsilon(x_0^n) - \varphi_\varepsilon(x_{\alpha,n}^\tau) \ge 0,\end{aligned}$$

which leads to the following

$$\begin{aligned}\langle A_h(x_{\alpha,n}^\tau) - A_h(x_0^n), x_{\alpha,n}^\tau - x_0^n\rangle &\le \langle A_h(x_0^n) - A_h(x_0) \\ &\quad + A_h(x_0) - A(x_0) + f - f_\delta, x_0^n - x_{\alpha,n}^\tau\rangle \\ &\quad + \alpha\langle U^s(x_{\alpha,n}^\tau - x_*^n), x_0^n - x_{\alpha,n}^\tau\rangle \\ &\quad + \langle A(x_0) - f, x_0^n - x_0 + x_0 - x_{\alpha,n}^\tau\rangle \\ &\quad + \varphi_\varepsilon(x_0^n) - \varphi_\varepsilon(x_{\alpha,n}^\tau).\end{aligned}$$

Using the inverse-strongly monotone property of A_h, (1.4) and (1.5) we have

$$\begin{aligned}m_A\|A_h(x_{\alpha,n}^\tau) - A_h(x_0^n)\|^2 &\le \Big[\widetilde{C}_1\gamma_n + hg(\|x_0\|) + \delta + \alpha\|x_{\alpha,n}^\tau - x_*^n\|^{s-1}\Big] \\ &\quad \times \|x_0^n - x_{\alpha,n}^\tau\| + (C_0 + \|A(x_0) - f\|)\gamma_n \\ &\quad + \varepsilon\big(d(\|x_{\alpha,n}^\tau\|) + d(\|x_0\|)\big),\end{aligned}$$

where $\widetilde{C}_1$ is a positive constant depending only on x_0. Thus,

$$\|A_h(x_{\alpha,n}^\tau) - A_h(x_0^n)\| = O(\sqrt{h + \delta + \varepsilon + \alpha + \gamma_n}).$$

Moreover, since

$$\|A_h(x_{\alpha,n}^\tau) - A_h(x_0)\| \le \|A_h(x_{\alpha,n}^\tau) - A_h(x_0^n)\| + \|A_h(x_0^n) - A_h(x_0)\|,$$

it follows readily that

$$\|A_h(x_{\alpha,n}^\tau) - A_h(x_0)\| \le O(\sqrt{h + \delta + \varepsilon + \alpha + \gamma_n}) + \widetilde{C}_1\gamma_n.$$

Combining (2.10), (2.11) and the last inequality, it follows from (2.9) that

$$\begin{aligned} m_s\|x^\tau_{\alpha,n} - x^n_0\|^s \le & \Big[\frac{\delta + hg(\|x^n_0\|) + \tilde{C}_0\gamma_n}{\alpha} + C(\tilde{R})2^\nu\gamma_n^\nu\Big]\|x^n_0 - x^\tau_{\alpha,n}\| \\ & + \tilde{R}^{s-1}\gamma_n + \frac{\varepsilon}{\alpha}(d(\|x^\tau_{\alpha,n}\|) + d(\|x_0\|)) \\ & + \frac{(C_0 + \|Ax_0 - f\|)\gamma_n}{\alpha} \\ & + \|z_h\|(1+\tilde{\tau})\big[O\big(\sqrt{h+\delta+\varepsilon+\alpha+\gamma_n}\big) + \widetilde{C}_1\gamma_n\big]. \end{aligned} \tag{2.12}$$

If α is chosen such that $\alpha \sim (h+\delta+\varepsilon+\gamma_n)^{\eta_1}$, then from (2.12) we obtain the inequality

$$\begin{aligned} \|x^\tau_{\alpha,n} - x^n_0\|^s \le & \overline{C}_1\Big[(h+\delta+\varepsilon+\gamma_n)^{1-\eta_1} + \gamma_n^\nu\Big]\|x^n_0 - x^\tau_{\alpha,n}\| + \overline{C}_2\gamma_n \\ & + \overline{C}_3(h+\delta+\varepsilon+\gamma_n)^{1-\eta_1} + \overline{C}_4(h+\delta+\varepsilon+\gamma_n)^{\eta_1/2}, \end{aligned}$$

$\overline{C}_i$, $i = 1,2,3,4$ are positive constants. Thus,

$$\|x^\tau_{\alpha,n} - x^n_0\| = O\big((h+\delta+\varepsilon+\gamma_n)^{\mu_2} + \gamma_n^{\mu_3}\big).$$

Hence,

$$\|x^\tau_{\alpha,n} - x_0\| = O\big((h+\delta+\varepsilon+\gamma_n)^{\mu_2} + \gamma_n^{\mu_3}\big),$$

which completes the proof. □

3. Numerical examples

We now apply the obtained results from the previous sections to solve the following optimization problem:

$$\min_{x\in X}\{F(x) + \varphi(x)\} \tag{3.1}$$

where F is Gâteaux differentiable with the Gâteaux derivative A, φ is a weakly lower semicontinuous and proper convex functional on X. So x_0 is a solution of Problem (3.1) if and only if x_0 is a solution of Problem (1.1) (see [5]).

We consider the case when X is a real Hilbert space and $F(x) = \frac{1}{2}\langle Ax, x\rangle$, with A being a self-adjoint linear bounded operator on X such that $\langle Ax, x\rangle \ge 0$, $\forall x \in X$. φ is a nonsmooth function and is approximated by a sequence of smooth functions φ_ε. So the method (1.3) in this case can be written in the form

$$A_h(x^\tau_\alpha) + \alpha(x^\tau_\alpha - x_*) + \varphi'_\varepsilon(x^\tau_\alpha) = f_\delta. \tag{3.2}$$

The computational results here are obtained by using MATLAB. We shall give an example.

Consider the case where $H = L^2[0,1]$, with

• $A : L^2[0,1] \to L^2[0,1]$ is defined by $(Ax)(t) = \int_0^1 k(t,s)x(s)ds$, where

$$k(t,s) = \begin{cases} \dfrac{(1-s)^2 st^2}{2} - \dfrac{(1-s)^2 t^3(1+2s)}{6} + \\ \qquad + \dfrac{(t-s)^3}{6}, & \text{if} \quad t \geq s, \\ \dfrac{s^2(1-s)(1-t)^2}{2} + \dfrac{s^2(1-t)^3(2s-3)}{6} + \\ \qquad + \dfrac{(s-t)^3}{6}, & \text{if} \quad t < s, \end{cases}$$

are kernel functionals defined on the square $\{0 \leq t, s \leq 1\}$.

$$(A_h x)(t) = \int_0^1 k_h(t,s)x(s)ds,$$

is an approximation of A, where $k_h(t,s) = k(t,s) + hts,\ h \to +0$. So, A_h is an inverse-strongly monotone operator and Fréchet differentiable with the Fréchet derivative A_h.

• The function $\varphi : L^2[0,1] \to \mathbb{R} \cup \{+\infty\}$ is defined by $\varphi(x) = \psi\big(\frac{1}{2}\langle Ax, x\rangle\big)$, with $\psi : \mathbb{R} \to \mathbb{R}$ is chosen as follows

$$\psi(t) = \begin{cases} 0 & , \quad t \leq a_0, \\ c(t-a_0) & , \quad t > a_0,\ c, a_0 > 0. \end{cases}$$

The function $\varphi_\varepsilon(x) = \psi_\varepsilon\big(\frac{1}{2}\langle Ax, x\rangle\big)$ is an approximation of $\varphi(x)$ with

$$\psi_\varepsilon(t) = \begin{cases} 0 & , \quad t \leq a_0, \\ \dfrac{c(t-a_0)^2}{2\varepsilon} & , \quad a_0 < t \leq a_0 + \varepsilon \\ c(t - a_0 - \dfrac{\varepsilon}{2}) & , \quad t > a_0 + \varepsilon. \end{cases}$$

Obviously, $\varphi'_\varepsilon(x) = \psi'_\varepsilon\big(\frac{1}{2}\langle Ax, x\rangle\big)Ax$ is an monotone operator from $L^2[0,1]$ to $L^2[0,1]$.

• $f_\delta(t) = \delta,\ t \in [0,1]$ is an approximation of $f = \theta \in L^2[0,1]$.

We compute the regularized solutions $x^\tau_{\alpha,n}$ by approximating $L^2[0,1]$ by the sequence of linear spaces H_n which is a set of all linear combinations of $\{\phi_1, \phi_2, ..., \phi_n\}$ defined on uniform grid of $n+1$ points in $[0,1]$:

$$\phi_j(t) = \begin{cases} 1 & , \quad t \in (t_{j-1}, t_j], \\ 0 & , \quad t \notin (t_{j-1}, t_j]. \end{cases}$$

Hence $P_n x(t) = \sum_{j=1}^{n} x(t_j)\phi_j(t)$, with $\|P_n\| = 1$ and $\|(I - P_n)x^0\| = O(n^{-1})$, $\forall x \in L^2[0,1]$ (see [13]). Then, the finite-dimensional regularized equation (3.2) is of the form

$$B_{h_1}\tilde{x} + \varphi'^{n}_{i\varepsilon}(\tilde{x}) = f^n_\delta, \qquad (3.3)$$

where

$$B_{h_1} = \begin{pmatrix} h_1k_h(t_1,t_1)+\alpha & h_1k_h(t_1,t_2) & ... & h_1k_h(t_1,t_n) \\ h_1k_h(t_2,t_1) & h_1k_h(t_2,t_2)+\alpha & ... & h_1k_h(t_2,t_n) \\ ... & ... & ... & ... \\ h_1k_h(t_n,t_1) & h_1k_h(t_n,t_2) & ... & h_1k_h(t_n,t_n)+\alpha \end{pmatrix}$$

and $\varphi'^{n}_{\varepsilon}(\tilde{x}) = (\varphi'_\varepsilon(\tilde{x}_1), ..., \varphi'_\varepsilon(\tilde{x}_n))^T$, $f^n_\delta = (\delta, ..., \delta)^T$, $\tilde{x} = (\tilde{x}_1, ..., \tilde{x}_n)^T$, $\tilde{x}_j \sim x(t_j)$, $j = 1, ..., n$, $h_1 = \frac{1}{n}$. Applying Theorem 2.4 for $\alpha \sim (h + \delta + \varepsilon + \gamma_n)^{\eta_1}$, $0 < \eta_1 < 1$, we should obtain the convergence rates $r^\tau_{\alpha,n} = \|x^\tau_{\alpha,n} - x^0\|$. Taking account of the iterative method in [14] for finding approximation solutions, we get the tables of computational results with $c = \frac{1}{4}$, $a_0 = \frac{10^{-3}}{3}$, $\delta = h = \varepsilon = \frac{1}{n}$.

n	α	$r^\tau_{\alpha,n}$
40	0.085499	0.050812
80	0.053861	0.029435
100	0.046416	0.024636
500	0.015874	0.007128

Table 2.1: $\eta_1 = \frac{2}{3}$

n	α	$r^\tau_{\alpha,n}$
40	0.15811	0.043055
80	0.1118	0.0252
100	0.1	0.021186
500	0.044721	0.006395

Table 2.2: $\eta_1 = \frac{1}{2}$

Remarks. From Table 2.1 and 2.2 we can see that:

1. For sufficiently small h, δ, ε, the approximate solutions $x^\tau_{\alpha,n}$ are closed to the exact solution of the original problem;
2. The convergence rate of regularized solutions depends on the choice of values of α depending on h, δ, ε.

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