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GENERALIZATION OF FIXED POINT THEOREMS WITH RESPECT TO Ω-DISTANCE

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Abstract. In this paper, we introduce the new concept of Ω-distance on an S-metric space and prove a fixed point theorem for a self-map. This is a generalization of well-known results which are proved by Guran [7].

1. Introduction

It is well known that the Banach contraction principle is a fundamental result in fixed point theory. After this classical result, many authors have extended, generalized and improved this theorem in different ways (see for details, [1], [2], [3], [4]). Also recently, fixed and common fixed point results in different types of spaces have been developed. For example, ultra metric spaces [16], fuzzy metric spaces [8] and uniform spaces [15].

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In this paper we recall the definitions of S-metric space and give some properties of it (see [5, 10]). After than, we prove a fixed point theorem for single-valued operators in terms of a Ω-distance.

We begin by briefly recalling some basic definitions and results for S-metric spaces that will be needed in the sequel. For more details please see in [6, 9, 11].

Definition 1.1. ([5, 10]) Let X be a (nonempty) set. An S-metric on X is a function $S : X^3 \longrightarrow [0, \infty)$ that satisfies the following conditions: for each $x, y, z, a \in X$,

(1) $S(x, y, z) \geq 0$,
(2) $S(x, y, z) = 0$ if and only if $x = y = z$,
(3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$, for all $x, y, z, a \in X$.

The pair (X, S) is called an S-metric space.

Immediate examples of such S-metric spaces are:

Example 1.2. ([12, 14])

(i) Let $X = \mathbb{R}^n$ and $\| . \|$ a norm on X. Then $S(x, y, z) = \| y + z - 2x \| + \| y - z \|$ is an S-metric on X.
(ii) Let X be a nonempty set, d be an ordinary metric on X. Then $S(x, y, z) = d(x, z) + d(y, z)$ is an S-metric on X. This S-metric is called the usual S-metric on X.

Definition 1.3. ([13]) Let (X, S) be an S-metric space.

(1) A sequence $\{x_n\} \subset X$ converges to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $S(x_n, x_n, x) < \varepsilon$. We write $x_n \to x$ for brevity.
(2) A sequence $\{x_n\} \subset X$ is Cauchy if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $S(x_n, x_n, x_m) < \varepsilon$.
(3) The S-metric space (X, S) is complete if every Cauchy sequence is convergent.

Definition 1.4. ([12]) Let (X, S) be an S-metric space. For $r > 0$ and $x \in X$ we define the open ball $B_s(x, r)$ and closed ball $B_s[x, r]$ with center x and radius r as follows respectively:

$$\begin{aligned} B_s(x, r) &= \{y \in X : S(y, y, x) < r\}, \\ B_s[x, r] &= \{y \in X : S(y, y, x) \leq r\}. \end{aligned}$$

Example 1.5. ([12]) Let $X = \mathbb{R}$ and $S(x, y, z) = |y + z - 2x| + |y - z|$ for all $x, y, z \in \mathbb{R}$. Then

$$\begin{aligned} B_s(1, 2) &= \{y \in \mathbb{R} : S(y, y, 1) < 2\} = \{y \in \mathbb{R} : |y - 1| < 1\} \\ &= \{y \in \mathbb{R} : 0 < y < 2\} = (0, 2). \end{aligned}$$

Lemma 1.6. ([13]) *Let (X, S) be an S-metric space. If $r > 0$ and $x \in X$, then the ball $B_s(x, r)$ is open subset of X.*

Lemma 1.7. ([12, 13, 14]) *In an S-metric space, we have $S(x, x, y) = S(y, y, x)$.*

Lemma 1.8. ([14]) *Let (X, S) be an S-metric space. If sequence $\{x_n\}$ converges to x, then x is unique.*

Lemma 1.9. ([14]) *Let (X, S) be an S-metric space. If sequence $\{x_n\}$ is convergent to x, then $\{x_n\}$ is a Cauchy sequence.*

Lemma 1.10. ([12, 13, 14]) *Let (X, S) be an S-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y)$.*

2. Fixed point theorems in Ω-distance

Now, we introduce the new concept of Ω-distance on an S-metric space.

Definition 2.1. Let (X, S) be an S-metric space. Then a function $\Omega : X \times X \times X \longrightarrow [0, \infty)$ is called an Ω-distance on X if the followings are satisfied:

(1) $\Omega(x, y, z) \leq \Omega(x, x, a) + \Omega(y, y, a) + \Omega(z, z, a)$ for all $x, y, z, a \in X$,
(2) for each $x \in X$, $\Omega(x, x, .) : X \longrightarrow [0, \infty)$ is a lower semi-continuous,
(3) for each $\epsilon > 0$, there is $\delta > 0$ such that $\Omega(x, x, a) \leq \delta$, $\Omega(y, y, a) \leq \delta$ and $\Omega(z, z, a) \leq \delta$ imply $S(x, y, z) \leq \epsilon$.

Let us give some examples of Ω-distance.

Example 2.2. Let (X, S) be an S-metric space. Then we know that the metric S is an Ω-distance. If set $\Omega = S$, then (1) is obvious. For (2), let $\{x_n\}$ be a sequence in X such that $\lim\limits_{n\to\infty} x_n = u$. Then we have

$$S(x, x, x_n) = S(x_n, x_n, x) \leq 2S(x_n, x_n, u) + S(x, x, u).$$

Taking $n \to \infty$ of above inequality, we have

$$\lim_{n\to\infty} S(x, x, x_n) \leq S(x, x, u).$$

Similarly,

$$S(x, x, u) = S(u, u, x) \leq 2S(u, u, x_n) + S(x, x, x_n).$$

Taking $n \to \infty$ of above inequality, we have

$$S(x, x, u) \leq \lim_{n\to\infty} S(x, x, x_n).$$

That is, $\lim_{n\to\infty} S(x,x,x_n) = S(x,x,u)$. We show (3). Let $\epsilon > 0$ be given and be choose $\delta \leq \frac{\epsilon}{3}$. Then, we have

$$\begin{aligned} S(x,y,z) &\leq S(x,x,a) + S(y,y,a) + S(z,z,a) \\ &= \Omega(x,x,a) + \Omega(y,y,a) + \Omega(z,z,a) \\ &\leq \epsilon. \end{aligned}$$

Example 2.3. Let $X = \mathbb{R}$ be real number and (X,S) be an S-metric space with $S(x,y,z) = |x-z| + |y-z|$. Then function $\Omega : X \times X \times X \longrightarrow [0,\infty)$ defined by $\Omega(x,y,z) = |x| + |y|$ for every $x,y,z \in X$ is a Ω-distance on X. The condition (1) is obvious. For (2), let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} x_n = u$. Then we have

$$2|x| = \Omega(x,x,x_n) \leq \liminf_{n\to\infty} \Omega(x,x,x_n) = 2|x|.$$

We show (3). Let $\epsilon > 0$ be given, if choose $\delta \leq \frac{\epsilon}{3}$. Then, we have

$$\begin{aligned} S(x,y,z) &= |x-z| + |y-z| \leq |x| + |y| + 2|z| \\ &\leq \Omega(x,x,a) + \Omega(y,y,a) + \Omega(z,z,a) \\ &\leq 3\delta \\ &\leq \epsilon. \end{aligned}$$

Example 2.4. Let X be a normed linear space with norm $||\cdot||$ and (X,S) be an S-metric space with $S(x,y,z) = ||x-z|| + ||y-z||$. Then function $\Omega : X \times X \times X \longrightarrow [0,\infty)$ defined by $\Omega(x,y,z) = ||x-z||$ for every $x,y,z \in X$ is an Ω-distance on X. The condition (1) is obvious. For (2), let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} x_n = u$. Since $||x-u|| = \lim_{n\to\infty} ||x-x_n||$, we have

$$\begin{aligned} \Omega(x,x,u) &= ||x-u|| \\ &= \lim_{n\to\infty} ||x-x_n|| = \lim_{n\to\infty} \Omega(x,x,x_n) \\ &\leq \liminf_{n\to\infty} \Omega(x,x,x_n) = ||x-u||. \end{aligned}$$

Let $\epsilon > 0$ be given, if choose $\delta \leq \frac{\epsilon}{4}$. Then, we have

$$\begin{aligned} S(x,y,z) &= ||x-z|| + ||y-z|| \\ &\leq ||x-a|| + ||z-a|| + ||y-a|| + ||z-a|| \\ &\leq \Omega(x,x,a) + \Omega(y,y,a) + 2\Omega(z,z,a) \\ &\leq 4\delta \\ &\leq \epsilon. \end{aligned}$$

Example 2.5. Let $X = [0,\infty)$ and (X,S) be an S-metric space with $S(x,y,z) = |x-z| + |x+y-2z|$. Then function $\Omega : X \times X \times X \longrightarrow [0,\infty)$ defined by

$\Omega(x,y,z)=\max\{x,y\}$ for every $x,y,z\in X$ is an Ω-distance on X. The conditions (1) and (2) are obvious. Let $\epsilon>0$ be given, if choose $\delta\le\frac{\epsilon}{6}$. Then, we have

$$\begin{aligned} S(x,y,z) &= |x-z|+|x+y-2z| \\ &\le 2|x|+|y|+3|z| = 2x+y+3z \\ &\le 2\Omega(x,x,a)+\Omega(y,y,a)+3\Omega(z,z,a) \\ &\le 6\delta \\ &\le \epsilon. \end{aligned}$$

Example 2.6. Let $X=[0,\infty)$ and (X,S) be an S-metric space with $S(x,y,z)$ $=|x-z|+|y-z|$. Then function $\Omega: X\times X\times X\longrightarrow[0,\infty)$ defined by $\Omega(x,y,z)=\max\{x,y,z\}$ for every $x,y,z\in X$ is an Ω-distance on X. In fact, the condition (1) is obvious. For (2), let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty}x_n=u$. Then we have

$$\begin{aligned} \max\{x,x_n\} &= \Omega(x,x,x_n) \\ &\le \liminf_{n\to\infty}\Omega(x,x,x_n) \\ &= \max\{x,u\}=\Omega(x,x,u). \end{aligned}$$

Let $\epsilon>0$ be given, if choose $\delta\le\frac{\epsilon}{4}$. Then, we have

$$\begin{aligned} S(x,y,z) &= |x-z|+|y-z| \\ &\le x+y+2z \\ &\le \Omega(x,x,a)+\Omega(y,y,a)+2\Omega(z,z,a) \\ &\le 4\delta \\ &\le \epsilon. \end{aligned}$$

Definition 2.7. An Ω-distance of S-metric space (X,S) is called symmetric if $\Omega(x,x,y)=\Omega(y,y,x)$ for all $x,y\in X$.

We can easily show that the Ω-distances in Example 2.4 and Example 2.6 are symmetric.

The following lemma plays an important role to prove fixed point theorems.

Lemma 2.8. *Let (X,S) be an S-metric space and let Ω be an Ω-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences in $[0,\infty)$ converging to 0. Then, for every $x,y,z\in X$ we have the following statements:*

(1) *If $\Omega(x,x,x_n)\le\alpha_n$, $\Omega(y,y,x_n)\le\beta_n$ and $\Omega(z,z,x_n)\le\gamma_n$ for any $n\in\mathbb{N}$, then $x=y=z$. In particular, $\Omega(x,x,x_n)\le\alpha_n$ and $\Omega(y,y,x_n)\le\beta_n$, then $x=y$,*

(2) *If* $\Omega(y_n, y_n, x_n) \leq \alpha_n$ *and* $\Omega(y, y, x_n) \leq \beta_n$ *for any* $n \in \mathbb{N}$*, then* $S(y_n, y_n, y) \longrightarrow 0$*, that is,* $\{y_n\}$ *converges to* y*,*
(3) *If* $\Omega(y_n, y_n, x_n) \leq \alpha_n$ *and* $\Omega(y_m, y_m, x_n) \leq \beta_n$ *for any* $n, m \in \mathbb{N}$ *with* $m > n$*, then* $\{y_n\}$ *is a Cauchy sequence,*
(4) *If* $\Omega(x_n, x_n, a) \leq \alpha_n$ *for every* $n \in \mathbb{N}$ *and* $a \in X$*, then* $\{x_n\}$ *is a Cauchy sequence.*

Proof. To prove (1), let $\epsilon > 0$ be given. From the definition of Ω-distance, we can choose $n_0 \in \mathbb{N}$ such that $\alpha_n \leq \delta$, $\beta_n \leq \delta$ and $\gamma_n \leq \delta$ for every $n \geq n_0$. Then for any $n \geq n_0$ we have, $\Omega(x, x, x_n) \leq \alpha_n \leq \delta$, $\Omega(y, y, x_n) \leq \beta_n \leq \delta$ and $\Omega(z, z, x_n) \leq \gamma_n \leq \delta$. Hence $S(x, y, z) \leq \epsilon$. This implies that $x = y = z$.

To prove (2), let $\epsilon > 0$ be given. From the definition of Ω-distance, we can choose $n_0 \in \mathbb{N}$ such that $\alpha_n \leq \delta$ and $\beta_n \leq \delta$ for every $n \geq n_0$. Then for any $n \geq n_0$ we have, $\Omega(y_n, y_n, x_n) \leq \alpha_n \leq \delta$ and $\Omega(y, y, x_n) \leq \beta_n \leq \delta$. Hence $S(y_n, y_n, y) \leq \epsilon$. This implies that $\{y_n\}$ converges to y.

In order to prove that the statement (3) holds. Let $\epsilon > 0$ be given. As in the proof of (2), choose $\delta > 0$. Then for any $n, m \geq n_0$,

$$\Omega(y_n, y_n, x_n) \leq \alpha_n \leq \delta \text{ and } \Omega(y_m, y_m, x_n) \leq \beta_n \leq \delta.$$

Hence $S(y_n, y_n, y_m) \leq \epsilon$. This implies that $\{y_n\}$ is a Cauchy sequence. As in the proof of (3), we can prove (4). Because, for any $n, m \geq n_0$, we can choose $\delta > 0$ such that

$$\Omega(x_n, x_n, a) \leq \alpha_n \leq \delta \text{ and } \Omega(x_m, x_m, a) \leq \alpha_n \leq \delta.$$

Hence $S(x_n, x_n, x_m) \leq \epsilon$. This implies that $\{x_n\}$ is a Cauchy sequence. □

In the first part of the section, we introduce and prove the following fixed point theorem.

Theorem 2.9. *Let* (X, S) *be a complete S-metric space and* Ω *be a symmetric* Ω*-distance on* X*. Let* $T : X \longrightarrow X$ *be a mapping such that*

$$\Omega(Tx, Ty, Tz) \leq k \max \left\{ \begin{array}{l} \Omega(x, y, z), \Omega(x, x, Tx), \Omega(y, y, Ty), \Omega(z, z, Tz), \\ \frac{1}{7}(\Omega(x, x, Ty) + \Omega(y, y, Tz) + \Omega(z, z, Tx)) \end{array} \right\}$$

holds for each $x, y, z \in X$ *and* $0 \leq k < 1$*. Suppose that if* $u \neq Tu$*,*

$$\inf\{\Omega(Tx, Tx, u) : x \in X\} > 0.$$

Then T *has a fixed point. Moreover, if* $\Omega(x, x, x) \leq \Omega(x, x, y)$ *for every* $x, y \in X$*, then* T *has a unique fixed point.*

Proof. Let $x_0 \in X$ and $x_{n+1} = Tx_n$ for each $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$ for which $x_{n+1} = x_n$, then x_n is a fixed point of T. In the following, we assume $x_{n+1} \neq x_n$ for each $n \in \mathbb{N}$.

First we shall prove that $\lim_{n\to\infty}\Omega(x_n, x_n, x_{n+1}) = 0$. For $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\Omega(x_n, x_n, x_{n+1}) \\
&= \Omega(Tx_{n-1}, Tx_{n-1}, Tx_n) \\
&\le k \max\left\{\begin{array}{l} \Omega(x_{n-1}, x_{n-1}, x_n), \Omega(x_{n-1}, x_{n-1}, Tx_{n-1}), \\ \Omega(x_{n-1}, x_{n-1}, Tx_{n-1}), \Omega(x_n, x_n, Tx_n) \end{array}\right\} \\
&= k \max\left\{\begin{array}{l} \Omega(x_{n-1}, x_{n-1}, x_n), \Omega(x_{n-1}, x_{n-1}, x_n), \\ \Omega(x_{n-1}, x_{n-1}, x_n), \Omega(x_n, x_n, x_{n+1}), \\ \frac{1}{7}(\Omega(x_{n-1}, x_{n-1}, Tx_{n-1}) + \Omega(x_{n-1}, x_{n-1}, Tx_n) \\ +\Omega(x_n, x_n, Tx_{n-1})) \end{array}\right\} \\
&\le k \max\left\{\begin{array}{l} \Omega(x_{n-1}, x_{n-1}, x_n), \Omega(x_n, x_n, x_{n+1}), \\ \frac{1}{7}(\Omega(x_{n-1}, x_{n-1}, x_n) + \Omega(x_{n-1}, x_{n-1}, x_{n+1}) \\ +\Omega(x_n, x_n, x_n)) \end{array}\right\}.
\end{aligned}$$

Since Ω is symmetric, by Definition 2.1 we have

$$\begin{aligned}
&\Omega(x_n, x_n, x_{n+1}) \\
&\le k \max\left\{\begin{array}{l} \Omega(x_{n-1}, x_{n-1}, x_n), \Omega(x_n, x_n, x_{n+1}), \\ \frac{1}{7}(\Omega(x_{n-1}, x_{n-1}, x_n) + \Omega(x_{n-1}, x_{n-1}, x_{n+1}) + \Omega(x_n, x_n, x_n)) \end{array}\right\} \\
&\le k \max\left\{\begin{array}{l} \Omega(x_{n-1}, x_{n-1}, x_n), \Omega(x_n, x_n, x_{n+1}), \\ \frac{1}{7}(\Omega(x_{n-1}, x_{n-1}, x_n) + 2\Omega(x_{n-1}, x_{n-1}, x_n) \\ +\Omega(x_n, x_n, x_{n+1}) + 3\Omega(x_n, x_n, x_{n+1})) \end{array}\right\}.
\end{aligned}$$

Now, if

$$\max\{\Omega(x_{n-1}, x_{n-1}, x_n), \Omega(x_n, x_n, x_{n+1})\} = \Omega(x_n, x_n, x_{n+1}),$$

by above inequality, it follows that $\Omega(x_n, x_n, x_{n+1}) < \Omega(x_n, x_n, x_{n+1})$ which is a contradiction. Therefore,

$$\Omega(x_n, x_n, x_{n+1}) \le \Omega(x_{n-1}, x_{n-1}, x_n).$$

Then, we have, for any $n \in \mathbb{N}$,

$$\begin{aligned}
\Omega(x_n, x_n, x_{n+1}) &\le k\Omega(x_{n-1}, x_{n-1}, x_n) \\
&\le k^2\Omega(x_{n-2}, x_{n-2}, x_{n-1}) \\
&\;\vdots \\
&\le k^n\Omega(x_0, x_0, x_1) = \alpha_n. \qquad (2.1)
\end{aligned}$$

So, if $m > n$, then

$$\begin{aligned}
\Omega(x_m, x_m, x_{n+1}) &= \Omega(x_{n+1}, x_{n+1}, x_m) \\
&\leq 2\sum_{i=n+1}^{m-2} \Omega(x_i, x_i, x_{i+1}) + \Omega(x_{m-1}, x_{m-1}, x_m) \\
&\leq 2\sum_{i=n+1}^{m-2} k^i\ \Omega(x_0, x_0, x_1) + k^{m-1}\ \Omega(x_0, x_0, x_1) \\
&\leq 2k^{n+1}\ \Omega(x_0, x_0, x_1)[1 + k + k^2 + \cdots] \\
&\leq \frac{2k^{n+1}}{1-k}\ \Omega(x_0, x_0, x_1) \\
&= \beta_n. \qquad (2.2)
\end{aligned}$$

That is, for every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$\Omega(x_n, x_n, x_m) < \epsilon,\ \forall n, m \geq n_0.$$

Thus

$$\liminf_{n\to\infty}\ \Omega(x_n, x_n, x_m) \leq \liminf_{n\to\infty}\ \epsilon = \epsilon,\ \forall n, m \geq n_0.$$

Also, by (2.1), (2.2) and by Lemma 2.8, $\{x_n\}$ is a Cauchy sequence. Since (X, S) is complete, $\{x_n\}$ converges to some point $z \in X$. Suppose $z \neq Tz$ and let $n_0 \in \mathbb{N}$ be fixed. Then, since $\{x_n\}$ converges to z and $\Omega(x_n, x_n, z)$ is lower semicontinuous at z in X, we have

$$\Omega(x_n, x_n, z) \leq \liminf_{m\to\infty}\ \Omega(x_n, x_n, x_m) \leq \epsilon,\ \forall n, m \geq n_0.$$

On the other hand, we have

$$\begin{aligned}
0 < \inf\{\Omega(Tx, Tx, z) : x \in X\} &\leq \inf\{\Omega(Tx_n, Tx_n, z) : n \geq n_0\} \\
&\leq \inf\{\liminf_{n\to\infty}\ \Omega(Tx_n, Tx_n, x_m\} \leq \epsilon,
\end{aligned}$$

which contradicts the hypotheses. Therefore, $z = Tz$ and hence z is a fixed point of T. We shall deal now with the uniqueness of the fixed point of T. Suppose that there are u and v in X fixed points of the mapping T. By hypotheses, since $\max\{\Omega(v, v, v), \Omega(u, u, u)\} \leq \Omega(v, v, u)$. It follows that

$$\begin{aligned}
\Omega(v, v, u) &= \Omega(Tv, Tv, Tu) \\
&\leq k\max\left\{\begin{array}{l} \Omega(v, v, u), \Omega(v, v, Tv), \Omega(v, v, Tv), \Omega(u, u, Tu), \\ \frac{1}{7}(\Omega(v, v, Tv) + \Omega(v, v, Tu) + \Omega(u, u, Tv)) \end{array}\right\} \\
&= k\max\left\{\begin{array}{l} \Omega(v, v, u), \Omega(v, v, v), \Omega(v, v, v), \Omega(u, u, u), \\ \frac{1}{7}(\Omega(v, v, v) + \Omega(v, v, u) + \Omega(u, u, v)) \end{array}\right\} \\
&< \Omega(v, v, u),
\end{aligned}$$

which is possible only for $\Omega(v,v,u) = 0$. Similarly, it can be proved that $\Omega(u,u,v) = 0$. According to the definition of an Ω-distance, $S(v,v,u) = 0$ this imply that $u = v$. Hence, T has a unique fixed point. □

Example 2.10. Let $X = \{0,1,2,\cdots\}$ and $S(x,y,z) = |x-y| + |x+y-2z|$. Then it is clear that (X,S) is a complete S-metric spaces. Let $\Omega : X^3 \to [0,\infty)$, defined by $\Omega(x,y,z) = \max\{x,y,z\}$ for all $x,y,z \in X$. Then for all $x,y,z \in X$ we have

(1) $\Omega(x,y,z) \leq \Omega(x,x,a) + \Omega(y,y,a) + \Omega(z,z,a)$ for all $a \in X$,
(2) for each $x \in X$, $\Omega(x,x,.) : X \longrightarrow [0,\infty)$ is a lower semi-continuous,
(3) for each $\epsilon > 0$, we can choose $\delta \leq \frac{\epsilon}{6}$ such that $\Omega(x,x,a) \leq \delta$, $\Omega(y,y,a) \leq \delta$ and $\Omega(z,z,a) \leq \delta$ imply

$$\begin{array}{rcl} |x-y| + |x+y-2z| & = & S(x,y,z) \\ & \leq & 2x + 2y + 2z \\ & < & 2\Omega(x,x,a) + 2\Omega(y,y,a) + 2\Omega(z,z,a) \\ & < & 6\delta \\ & < & \epsilon, \end{array}$$

(4) $\Omega(x,x,y) \leq \Omega(y,y,x)$.

If define

$$T(x,y) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{x}{2} & \text{if } x = 2n, \\ \frac{x-1}{2} & \text{if } x = 2n+1, \end{cases}$$

then it is clear that $T0 = 0$. But in otherwise, that is $Tu \neq u$ for any $u \in X$, we have

$$\begin{array}{l} \inf\{\Omega(Tx,Tx,u) \mid x \in X\} \\ = \inf \left\{ \begin{array}{cc} \Omega(\frac{x}{2},\frac{x}{2},u) = \max\{\frac{x}{2},u\} & \text{if } x = 2n, \\ \Omega(\frac{x-1}{2},\frac{x-1}{2},u) = \max\{\frac{x-1}{2},u\} & \text{if } x = 2n+1, \end{array} \right\} \\ \geq u > 0. \end{array}$$

On the other hand for $k = \frac{1}{2}$, we have three cases:
Case 1. If $x,y,z \in X$ are even, then

$$\begin{array}{rcl} \Omega(x,y,z) & = & \Omega(\frac{x}{2},\frac{y}{2},\frac{z}{2}) \\ & = & \frac{1}{2}\max\{x,y,z\} \\ & \leq & k\Omega(x,y,z). \end{array}$$

Case 2. If $x, y, z \in X$ are odd, then

$$\begin{aligned}\Omega(x,y,z) &= \Omega(\frac{x-1}{2}, \frac{y-1}{2}, \frac{z-1}{2}) \\ &= \frac{1}{2}\max\{x,y,z\} - \frac{1}{2} \\ &\leq \frac{1}{2}\max\{x,y,z\}.\end{aligned}$$

Case 3. If $x, y \in X$ are even and z is an odd, then

$$\begin{aligned}\Omega(x,y,z) &= \Omega(\frac{x-1}{2}, \frac{y-1}{2}, \frac{z}{2}) \\ &= \max\{\frac{x-1}{2}, \frac{y-1}{2}, \frac{z}{2}\} \\ &\leq \frac{1}{2}\max\{x,y,z\}.\end{aligned}$$

These shows that all conditions of Theorem 2.9 for $k = \frac{1}{2}$ are satisfied and so T has a unique fixed point $x = 0$ in X.

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