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GENERALIZED FIXED POINT THEOREMS IN A b-METRIC SPACE

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Abstract. A brief comparison of various contractive conditions in a b-metric space is made, and two generalized fixed point theorems are established. One for a Nesic type contraction, and the other involving a generalized class of auxiliary functions. Also, contractive fixed points in a b-metric space are obtained for some contractive conditions.

1. Introduction

Let X be a nonempty set and $d : X \times X \to [0, \infty)$ be a mapping satisfying the conditions:

(m1) $d(x, x) = 0$ for all $x \in X$,
(m2) $d(x, y) = 0$ implies that $x = y$ for all $x, y \in X$,
(m3) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(m4) $d(x, y) \le d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

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Then the pair (X, d) is called a metric space with metric d. A quasi-metric space [27] satisfies the conditions (m1), (m2) and (m4), while for a metric-like space [2] (or dislocated metric space [11]), (m2), (m3) and (m4) hold. Replacing the triangle inequality (m4) with a generalized one, we have the following notion of a b-metric space:

Definition 1.1. Let $s \geq 1$, X be a nonempty set and $\rho_s : X \times X \to [0, \infty)$ be a mapping satisfying the conditions:

(b1) $\rho_s(x, x) = 0$ for all $x, y \in X$,
(b2) $\rho_s(x, y) = 0$ implies that $x = y$ for all $x, y \in X$,
(b3) $\rho_s(x, y) = \rho_s(y, x)$ for all $x, y \in X$,
(b4) $\rho_s(x, y) \leq s[\rho_s(x, z) + \rho_s(z, y)]$ for all $x, y \in X$.

Then ρ_s is called a b-metric on X, and (X, ρ_s) denotes a b-metric space.

The notion of b-metric space was introduced by Bakthin [5] in 1989. Later, in 1993, two generalizations of Banach's contraction mapping theorem were obtained by Czerwik [8] in b-metric space with $s = 2$. Every metric space is a b-metric space with $s = 1$. A b-metric ρ_s is not continuous (See [28]), though a metric d is known to be continuous. A space X, satisfying (b1), (b2) and (b4) is known as a quasi b-metric space, which was introduced by Shah and Hussain [26] in 2012. While, Alghamdi et al. [1] introduced a b-metric-like space (or dislocated b-metric space [11]), as a generalization of b-metric space, by dropping (b1) in b-metric space. Further, replacing (b4) with the following stronger form:

(bk) For $k = 1, 2, 3, \ldots$ and all $x, y_1, y_2, \ldots, y_k, y \in X$,

$$\rho_s(x, y) \leq s[\rho_s(x, y_1) + \rho_s(y_1, y_2) + \cdots + \rho_s(y_k, y)]. \tag{1.1}$$

Khamsi [12] in 2010, defined a metric-type space (X, ρ_s, s), with continuity of ρ_s, and y_1, y_2, ..., y_k need not be distinct. However, for a fixed k, if each of y_1, y_2, ..., y_k is distinct from x and y in (1.1), we obtain the recent notion of a $b_k(s)$-metric space, due to Mitrovic and Radenovic [16]. When $k = 1$, a $b_k(s)$-metric space reduces to a b-metric space, and $k = 2$ gives a rectangular b-metric space, introduced by George et al [10] in 2015.

Let (X, ρ_s) be a b-metric space. The family of all b-balls in X, given by

$$B_{\rho_s}(x, r) = \{y \in X : \rho_s(x, y) < r\}, \tag{1.2}$$

forms a base topology, called the b-metric topology $\tau(\rho_s)$ on X. A sequence $\{x_n\} \subset X$ is said to be b-convergent with limit $p \in X$, if it converges to p in $\tau(\rho_s)$. While, $\{x_n\} \subset X$ is said to be b-Cauchy, if $\lim_{n,m\to\infty} \rho_s(x_n, x_m) = 0$. A b-metric space X is said to be b-complete, if every b-Cauchy sequence in X is b-convergent in it. Also, a b-convergent sequence has a unique limit, and is necessarily b-Cauchy.

2. A brief comparison of contraction conditions

In 2013, Kir and Kiziltunc [13] established the following Banach's and Kannan contraction mapping theorems in a b-metric space, respectively:

Theorem 2.1. *Suppose that (X, ρ_s) is a complete b-metric space with constant s, and f is a self-map on X satisfying the condition*

$$\rho_s(fx, fy) \leq \alpha \rho_s(x, y) \text{ for all } x, y \in X, \tag{2.1}$$

where $0 \leq \alpha < 1/s$. Then f has a unique fixed point p.

Theorem 2.2. *Let (X, ρ_s) be a complete b-metric space with constant s, and $f : X \to X$ satisfy the condition*

$$\rho_s(fx, fy) \leq \beta[\rho_s(x, fx) + \rho_s(y, fy)] \text{ for all } x, y \in X, \tag{2.2}$$

where $0 \leq \beta < 1/2$. Then f has a unique fixed point p.

Remark 2.3. If $\alpha = 0$ and $\beta = 0$, with $y = fx$, (2.1) and (2.2) imply that $\rho_s(fx, f^2x) = 0$ for each $x \in X$. That is, the fixed point of f is not unique in the sense that each $y = fx$ is a fixed point. Therefore, the contraction constants in Theorem 2.1 and Theorem 2.2 should be positive.

Given below is a result also proved in [13]:

Theorem 2.4. *Suppose that (X, ρ_s) is a complete b-metric space with constant s, and f is a self-map on X satisfying the condition*

$$\rho_s(fx, fy) \leq \gamma[\rho_s(x, fy) + \rho_s(y, fx)] \text{ for all } x, y \in X, \tag{2.3}$$

where γ is a real number such that $0 < \gamma s < 1/2$. Then f has a unique fixed point p.

Sarwar and Rahman [17] proved the following theorem:

Theorem 2.5. *Let f be a self-map on a complete b-metric space (X, ρ_s) with coefficient $s \geq 1$ such that*

$$\rho_s(fx, fy) \leq a\rho_s(x, y) + b\rho_s(x, fx) + c\rho_s(y, fy) \text{ for all } x, y \in X, \tag{2.4}$$

where a, b and c are non-negative real numbers, not all being zero, such that $s(a + b) + c < 1$. Then f has a unique fixed point.

Mishra et al. [15] proved the following theorem:

Theorem 2.6. *Let f be a self-map on a complete b-metric space (X, ρ_s) with coefficient $s \geq 1$ such that*

$$\begin{aligned} \rho_s(fx, fy) \leq{}& a\rho_s(x, y) + b\rho_s(x, fx) + c\rho_s(y, fy) \\ &+ e\rho_s(x, fy) + h\rho_s(y, fx) \text{ for all } x, y \in X, \end{aligned} \tag{2.5}$$

where a, b, c, e and h are non-negative real numbers such that

$$0 < s(a+b+c+e+h) < 1/2.$$

Then f has a unique fixed point.

Remark 2.7. When $s = 1$, the contraction conditions (2.1), (2.2), (2.3), (2.4) and (2.5) reduce to Banach, Kannan, Chatterjea, Reich, and Hardy-Roger's contractions respectively. However, the constants in Hardy-Roger's contraction on metric space are such that $0 < a+b+c+e+h < 1$. Also, a Banach's contraction is uniformly continuous on X, while Kannan contraction is discontinuous. Rhoades [24], in his comparative study of various contraction conditions in metric spaces, established that Banach and Kannan contractions are independent of each other. Kannan implies Reich, but not the converse and Chatterjea implies Hardy-Roger's contraction, but not the converse.

Remark 2.8. With $b = c = 0$, (2.4) reduces to a Banach's contraction (2.1). Hence Theorem 2.1 follows from Theorem 2.5 as a special case. While, writing $a = 0$ and $c = b$ in (2.4), we obtain (2.2), but the choice of the contraction constant is restricted to $0 < b < 1/(s+1)$, since $s \geq 1$. Thus Theorem 2.2 follows from Theorem 2.5 as a special case with this restriction.

Remark 2.9. Writing $b = c = e = h = 0$, (2.5) reduces to a Banach contraction (2.1), under the restricted choice $0 < a < 1/2s$. Thus Theorem 2.1 follows from Theorem 2.6 as a special case under the restriction that $0 < a < 1/2s$. While writing $a = e = h = 0$ and $c = b$, (2.5) reduces to (2.2). Hence Theorem 2.2 follows from Theorem 2.6 as a special case under the restriction $0 < b < 1/4s$, where β is replaced with b. While, with $a = b = c = 0$ and $h = e$ in (2.5), we get (2.3), but the contraction constant is restricted to $0 < e < 1/4s$. In other words, Theorem 2.4 follows from Theorem 2.6 under the restriction $0 < e < 1/4s$, where γ is replaced with e.

Definition 2.10. ([6, 7]) Let (X, ρ) be a metric space. A self-map f on X is said to be a weak contraction, if

$$\rho(fx, fy) \leq c\rho(x, y) + \mu\rho(x, fy) \text{ for all } x, y \in X \tag{2.6}$$

where $0 < c < 1$ and $\mu \geq 0$.

In view of symmetry (b2), (2.6) implicitly includes its dual form

$$\rho(fx, fy) \leq c\rho(x, y) + \mu\rho(y, fx) \text{ for all } x, y \in X. \tag{2.7}$$

Therefore, in order to check whether f is a weak contraction on X, it is necessary that both (2.6) and (2.7) hold. When $\mu = 0$, weak contraction reduces to a Banach's contraction. However, a Banach's contraction need not be a weak contraction [7]. Weak contraction in a b-metric space (X, ρ_s) is defined in a similar way as in metric space (X, d), by just replacing the metric

d with b-metric ρ_s. It may be noted that a Banach's contraction is also a weak contraction in a b-metric space. Further, a Kannan contraction on a metric space (X, ρ), with the choice

$$\rho(fx, fy) \leq k[\rho(x, fx) + \rho(y, fy)] \text{ for all } x, y \in X, \tag{2.8}$$

is a weak contraction [6], where $c = k/(1-k)$ and $\mu = 2k/(1-k)$. However, a Kannan contraction (2.2) on a b-metric space (X, ρ_s) will be a weak contraction [13], only when $0 < \beta s < 1/2$. On the other hand, a Chatterjea contraction (2.3) on a b-metric space (X, ρ_s) will also be a weak contraction [13] with $c = \gamma/(1-\gamma)$ and $\mu = 2\gamma/(1-\gamma)$. For $s = 1$, (2.3) on a b-metric space (X, ρ_s) reduces to a Chatterjea contraction on a metric space (X, ρ) with the choice

$$\rho(fx, fy) \leq \nu[\rho(x, fy) + \rho(y, fx)] \text{ for all } x, y \in X,\ 0 < \nu < 1/2, \tag{2.9}$$

which, in accordance with [6], is a weak contraction on X. It may be noted that Chatterjea contraction (2.3) on a b-metric space (X, ρ_s) uses the stronger condition that $0 < s\gamma < 1/2$ than $0 < \gamma < 1/2$.

3. Contractive fixed point

Let f be a self-map on a metric space (X, d) and $x_0 \in X$. The orbit $O_f(x_0)$ at x_0 is the sequence of f-iterates $x_0, fx_0, ..., f^n x_0, ...$ A fixed point p of f is known to be a contractive fixed point, if every $O_f(x_0)$ converges to p. Existence of contractive fixed points in metric spaces was investigated by Edelstein [9], Leader and Hoyle [14] and Reich [23]. Contractive fixed points in G-metric spaces were investigated in [3], [19], [20], [21], [22] and [25].

Definition 3.1. Let f be a self-map on a b-metric space (X, ρ_s) and $x_0 \in X$. A fixed point p of f is said to be a b-contractive fixed point, if for every $x_0 \in X$, the f-orbit $O_f(x_0) = \langle x_0, fx_0, ..., f^n x_0, ...\rangle$ converges to p.

Since a convergent sequence in metric space has a unique limit, a contractive fixed point is also a unique fixed point. However, a unique fixed point need not be a contractive fixed point in a b-metric space as shown in the following example:

Example 3.2. Let $X = [0, 1]$ and $\rho_s(x, y) = |x - y|^2$ for all $x, y \in X$. Then (X, ρ_s) is a b-metric space with coefficient $s = 2$, which is not a metric space. Define $f : X \to X$ by

$$fx = \begin{cases} 1/2, & x < 1/2 \\ 1, & x \geq 1/2\cdot \end{cases} \tag{3.1}$$

We see that $x = 1$ is the unique fixed point of f. But for $0 \leq x_0 < 1/2$, the f-orbit $O_f(x_0) = \langle \frac{1}{2}, \frac{1}{2}, ...\rangle \to 1/2$, while for $x_0 \geq 1/2$, $O_f(x_0) = \langle 1, 1, ...\rangle \to 1$. In other words, 1 is not a contractive fixed point of f.

Now, we prove that unique fixed point is also a b-contractive fixed point for certain contraction conditions in a b-metric space.

Theorem 3.3. *Let (X, ρ_s) be a b-metric space with constant s, and p be a unique fixed point of a Kannan contraction f on X satisfying the condition* (2.2)*, with an additional condition that $0 < \beta s < 1/2$. Then p is a b-contractive fixed point.*

Proof. Let $x_0 \in X$ be arbitrary. Writing $x = f^{n-1}x_0$ and $y = p$ in (2.2), then from (b3), we see that

$$\begin{aligned}\rho_s(f^n x_0, p) &= \rho_s(f^n x_0, f^n p)\\ &\le \beta[\rho_s(f^{n-1}x_0, f^n x_0) + \rho_s(f^{n-1}p, f^n p)]\\ &\le \beta s[\rho_s(f^{n-1}x_0, p) + \rho_s(p, f^n x_0)]\end{aligned}$$

or

$$\rho_s(f^n x_0, p) \le \left(\tfrac{\beta s}{1-\beta s}\right) \rho_s(f^{n-1}x_0, p).$$

Hence, by induction on n, (3.2) gives

$$\rho_s(f^n x_0, p) \le \left(\tfrac{\beta s}{1-\beta s}\right)^n \rho_s(x_0, p). \tag{3.2}$$

Hence, it follows from (b3) and (2.2) that

$$\begin{aligned}\rho_s(x_0, p) &\le s[\rho_s(x_0, fx_0) + \rho_s(fx_0, fp)] \qquad (3.3)\\ &\le s\rho_s(x_0, fx_0) + \beta s[\rho_s(x_0, fx_0) + \rho_s(p, fp)]\\ &= s(1+\beta)\rho_s(x_0, fx_0).\end{aligned}$$

Substituting (3.3) in (3.2), we get

$$\rho_s(f^n x_0, p) \le \left(\tfrac{\beta s}{1-\beta s}\right)^n s(1+\beta)\rho_s(x_0, fx_0) \tag{3.4}$$

for $x_0 \in X, n = 1, 2, ...$ Since $[\beta s/(1-\beta s)]^n \to 0$ as $n \to \infty$, (3.4), implies that $\rho_s(f^n x_0, p) \to 0$ as $n \to \infty$ for all $x_0 \in X$. In other words, p is a b-contractive fixed point. □

Theorem 3.4. *Let (X, ρ_s) be a b-metric space with constant s, and p be a unique fixed point of a Chatterjea contraction f on X satisfying the condition* (2.3)*. Then p is a b-contractive fixed point.*

Proof. Let $x_0 \in X$ be arbitrary. Writing $x = f^{n-1}x_0$ and $y = p$ in (2.3), then from (b3),

$$\begin{aligned}\rho_s(f^n x_0, p) &= \rho_s(f^n x_0, f^n p)\\ &\le \gamma[\rho_s(f^{n-1}x_0, f^n p) + \rho_s(f^{n-1}p, f^n x_0)]\end{aligned}$$

or

$$\rho_s(f^n x_0, p) \le \left(\tfrac{\gamma}{1-\gamma}\right) \rho_s(f^{n-1} x_0, p).$$

Hence, by induction on n, (3.5) gives

$$\rho_s(f^n x_0, p) \le \left(\tfrac{\gamma}{1-\gamma}\right)^n \rho_s(x_0, p). \tag{3.5}$$

But again, by (b3) followed by (2.3), it follows that

$$\begin{aligned}\rho_s(x_0, p) &\le s[\rho_s(x_0, fx_0) + \rho_s(fx_0, fp)] \\ &\le s\rho_s(x_0, fx_0) + s\gamma[\rho_s(x_0, p) + \rho_s(p, fx_0)] \\ &\le \left(\tfrac{s}{1-s\gamma}\right) \rho_s(x_0, fx_0) + \left(\tfrac{s\gamma}{1-s\gamma}\right) \rho_s(fx_0, p).\end{aligned} \tag{3.6}$$

Now,

$$\begin{aligned}\rho_s(fx_0, p) &= \rho_s(fx_0, fp) \\ &\le \gamma[\rho_s(x_0, fx_0) + \rho_s(fx_0, fp)],\end{aligned}$$

that is,

$$\rho_s(fx_0, p) \le \left(\tfrac{\gamma}{1-\gamma}\right) \rho_s(x_0, fx_0). \tag{3.7}$$

Substituting (3.6) and (3.7) in (3.5), we get

$$\rho_s(f^n x_0, p) \le \left(\tfrac{\gamma}{1-\gamma}\right)^n \left[\tfrac{s}{1-s\gamma} + \tfrac{s\gamma}{1-s\gamma} \cdot \tfrac{\gamma}{1-\gamma}\right] \rho_s(x_0, fx_0) \tag{3.8}$$

for all $x_0 \in X,\ n = 1, 2, 3, \ldots$. Proceeding the limit as $n \to \infty$ in (3.8), and using the choice of γ, we get $\rho_s(f^n x_0, p) \to 0$ as $n \to \infty$ for all $x_0 \in X$. In other words, p is a b-contractive fixed point. □

Theorem 3.5. *Let (X, ρ_s) be a b-metric space with constant s, and p be a unique fixed point of a self-map f on X, satisfying the condition* (2.4)*. Then p is a b-contractive fixed point.*

Proof. Let $x_0 \in X$ be arbitrary. Writing $x = f^{n-1}x_0$ and $y = p$ in (2.4), then using (b3), we get

$$\begin{aligned}\rho_s(f^n x_0, p) &= \rho_s(ff^{n-1}x_0, fp) \\ &\le a\rho_s(f^{n-1}x_0, p) + b\rho_s(f^{n-1}x_0, ff^{n-1}x_0) + c\rho_s(p, fp) \\ &\le a\rho_s(f^{n-1}x_0, p) + bs[\rho_s(f^{n-1}x_0, f^{n-1}p) + \rho_s(f^{n-1}p, f^n x_0)] \\ &\le \left(\tfrac{a}{1-bs}\right) \rho_s(f^{n-1}x_0, p).\end{aligned}$$

Hence, by induction on n,

$$\rho_s(f^n x_0, p) \le \left(\tfrac{a}{1-bs}\right)^n \rho_s(x_0, p). \tag{3.9}$$

Hence, it follows from (b3) and (2.4) that

$$\begin{aligned}\rho_s(x_0,p) &\le s[\rho_s(x_0,fx_0)+\rho_s(fx_0,fp)]\\ &\le s\rho_s(x_0,fx_0)+s[a\rho_s(x_0,p)+b\rho_s(x_0,fx_0)+c\rho_s(p,fp)]\\ &= \left(\tfrac{(1+b)s}{1-as}\right)\rho_s(x_0,fx_0).\end{aligned}$$

With this, (3.9) becomes

$$\rho_s(f^n x_0,p) \le \left(\tfrac{(1+b)s}{1-as}\right)\left(\tfrac{a}{1-bs}\right)^n \rho_s(x_0,fx_0) \tag{3.10}$$

for $x_0 \in X,\ n=1,2,3,\ldots$. Proceeding the limit as $n\to\infty$ in (3.10), it follows that $\rho_s(f^n x_0,p)\to 0$ as $n\to\infty$ for all $x_0\in X$. In other words, p is a b-contractive fixed point. □

4. Generalized fixed point theorems

A self-map f on a b-metric space (X,ρ_s) is called a Nesic type b-contraction, if it satisfies the condition:

$$\begin{aligned}[1+\mu\rho_s(x,y)]\rho_s(fx,fy) \le \mu[\rho_s(x,fx)\rho_s(y,fy)+\rho_s(y,fx)\rho_s(x,fy)]&\\ +\lambda\max\Bigg\{\rho_s(x,y),\rho_s(x,fx),\rho_s(y,fy),&\\ \frac{\rho_s(x,fy)+\rho_s(y,fx)}{2}\Bigg\}&\end{aligned} \tag{4.1}$$

for all $x,y\in X$, where $\mu\ge 0$ and $0<\lambda<1/s$.

Theorem 4.1. *Let f be a Nesic type b-contraction on a complete b-metric space (X,ρ_s). Then f has a unique fixed point.*

Proof. Let $x_0\in X$ be arbitrary. Define $\{x_n\}\subset X$ by

$$x_n = fx_{n-1} \text{ for } n\ge 1. \tag{4.2}$$

Now writing $x=x_{n-1}$ and $y=x_n$ in (4.1), we know that

$$\begin{aligned}&[1+\mu\rho_s(x_{n-1},x_n)]\rho_s(fx_{n-1},fx_n)\\ &\le \mu\big[\rho_s(x_{n-1},fx_{n-1})\rho_s(x_n,fx_n)+\rho_s(x_n,fx_{n-1})\rho_s(x_{n-1},fx_n)\big]\\ &\quad +\lambda\max\big\{\rho_s(x_{n-1},x_n),\rho_s(x_{n-1},fx_{n-1}),\rho_s(x_{n-1},fx_{n-1}),\\ &\qquad \tfrac{1}{2}[\rho_s(x_{n-1},fx_n)+\rho_s\rho_s(x_n,fx_{n-1})]\big\}\end{aligned}$$

or

$$\rho_s(x_n,x_{n+1}) \le \lambda\max\{\rho_s(x_{n-1},x_n),\tfrac{1}{2}[\rho_s(x_{n-1},x_n)+\rho_s(x_n,x_{n+1})]\}$$

so that

$$\rho_s(x_n, x_{n+1}) \leq \lambda \rho_s(x_{n-1}, x_n) \text{ for } \ n \geq 1.$$

By induction on n, it follows that

$$\rho_s(x_n, x_{n+1}) \leq \lambda^n \rho_s(x_0, x_1) \text{ for } \ n \geq 1. \tag{4.3}$$

Now for $m > n$, by the repeated use of (b4), we know that

$$\begin{aligned}
\rho_s(x_n, x_m) &\leq s\rho_s(x_n, x_{n+1}) + s^2 \rho_s(x_{n+1}, x_{n+2}) \\
&\quad + \cdots + s^{m-n} \rho_s(x_{m-1}, x_m) \\
&\leq (\underbrace{s\lambda^n + s^2\lambda^{n+1} + \cdots + s^{m-n}\lambda^{n+(m-n-1)}}_{m-n \text{ terms}}) \rho_s(x_0, x_1) \\
&= s\lambda^n (\underbrace{1 + s\lambda + \cdots + (s\lambda)^{m-n-1}}_{m-n \text{ terms}}) \rho_s(x_0, x_1) \\
&\leq (s\lambda^n) \left(\tfrac{1-(s\lambda)^{m-n}}{1-s\lambda}\right) \rho_s(x_0, x_1) \\
&\leq \left(\tfrac{s}{1-s\lambda}\right) \lambda^n \rho_s(x_0, x_1).
\end{aligned} \tag{4.4}$$

Note that $s/(1 - s\lambda)$ is positive and finite, and $\lambda < 1$. Therefore, in the limit as $n \to \infty$, we see that $\rho_s(x_n, x_m) \to 0$. Thus $\{x_n\}$ is a b-Cauchy sequence in X.

Since X is b-complete, we can find a point $p \in X$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} f x_{n-1} = p. \tag{4.5}$$

Now, writing $x = x_{n-1}$ and $y = p$, the inequality (4.1) gives

$$\begin{aligned}
&[1 + \mu\rho_s(x_{n-1}, p)]\rho_s(fx_{n-1}, fp) \\
&\leq \mu[\rho_s(x_{n-1}, fx_{n-1})\rho_s(p, fp) + \rho_s(p, fx_{n-1})\rho_s(p, fx_{n-1})] \\
&\quad + \lambda \max\{\rho_s(x_{n-1}, p), \rho_s(x_{n-1}, fx_{n-1}), \rho_s(p, fp), \\
&\qquad \tfrac{1}{2}[\rho_s(x_{n-1}, fp) + \rho_s(p, fx_{n-1})]\}.
\end{aligned}$$

Taking the limit as $n \to \infty$ and using (4.5), we get that

$$\begin{aligned}
[1 + \mu.0]\rho_s(p, fp) &\leq \mu[0 \cdot \rho_s(p, fp) + 0] \\
&\quad + \lambda \max\{0, 0, \rho_s(p, fp), \tfrac{1}{2}[\rho_s(0, fp) + 0]\}.
\end{aligned}$$

It implies that $\rho_s(p, fp) \leq \lambda\rho_s(p, fp)$ or $\rho_s(p, fp) = 0$. That is, $fp = p$.

To establish the uniqueness of the fixed point, let $q \neq p$ be also a fixed point of f. Then with $x = p$ and $y = q$ in (4.1),

$$\begin{aligned}
&[1 + \mu\rho_s(p, q)]\rho_s(fp, fq)\\
&\leq \mu[\rho_s(p, fp)\rho_s(q, fq) + \rho_s(q, fp)\rho_s(q, fp)]\\
&\quad + \lambda \max\{\rho_s(p, q), \rho_s(p, fp), \rho_s(q, fq), \tfrac{1}{2}[\rho_s(p, fq) + \rho_s(q, fp]\}
\end{aligned}$$

which on simplifying, yields $0 < \rho_s(p, q) \leq \lambda\rho_s(p, q) < \rho_s(p, q)$. This is a contradiction. Hence $p = q$, and the fixed point is unique. □

Setting $\mu = 0$ in Theorem 4.1, we have the following corollary.

Corollary 4.2. *Let (X, ρ_s) be a complete b-metric space and $f : X \to X$ be a generalized b-contraction such that*

$$\rho_s(fx, fy) \leq \lambda \max\left\{\rho_s(x, y), \rho_s(x, fx), \rho_s(y, fy), \tfrac{\rho_s(x,fy)+\rho_s(y,fx)}{2}\right\}, \qquad (4.6)$$

for all $x, y \in X$, where $0 < \lambda < 1/s$. Then f has a unique fixed point, which is also a b-contractive fixed point.

Proof. For arbitrary $x_0 \in X$, let $x = f^{n-1}x_0$ and $y = p$ in (4.6). Then

$$\begin{aligned}
\rho_s(f^n x_0, p) &= \rho_s(ff^{n-1}x_0, fp)\\
&\leq \lambda \max\left\{\rho_s(f^{n-1}x_0, p), \rho_s(f^{n-1}x_0, ff^{n-1}x_0), \rho_s(p, fp),\right.\\
&\qquad\qquad \left.\tfrac{1}{2}[\rho_s(f^{n-1}x_0, fp) + \rho_s(p, ff^{n-1}x_0)]\right\}\\
&= \lambda M_n, \qquad (4.7)
\end{aligned}$$

where

$$M_n = \max\left\{\rho_s(f^{n-1}x_0, p), \rho_s(f^{n-1}x_0, f^n x_0)\right\} \text{ for } n = 1, 2, ...$$

Suppose that $M_n = \rho_s(f^{n-1}x_0, p)$. Then (4.7) becomes

$$\rho_s(f^n x_0, p) \leq \lambda\rho_s(f^{n-1}x_0, p) \leq \lambda s\rho_s(f^{n-1}x_0, p). \qquad (4.8)$$

On the other hand, let $M_n = \rho_s(f^{n-1}x_0, f^n x_0)$. Then by (4.6) and (b3), we know that

$$\begin{aligned}
\rho_s(f^{n-1}x_0, f^n x_0) &\leq \lambda \max\{\rho_s(f^{n-2}x_0, f^{n-1}x_0), \rho_s(f^{n-2}x_0, ff^{n-2}x_0),\\
&\qquad \rho_s(f^{n-1}x_0, ff^{n-1}x_0),\\
&\qquad \tfrac{1}{2}[\rho_s(f^{n-2}x_0, ff^{n-1}x_0) + \rho_s(f^{n-1}x_0, ff^{n-2}x_0)]\}\\
&\leq \lambda \max\{\rho_s(f^{n-2}x_0, f^{n-1}x_0), \rho_s(f^{n-1}x_0, f^n x_0),\\
&\qquad \tfrac{s}{2}[\rho_s(f^{n-2}x_0, f^{n-1}x_0) + \rho_s(f^{n-1}x_0, f^n x_0)]\}\\
&= \lambda s \max\{\rho_s(f^{n-2}x_0, f^{n-1}x_0), \rho_s(f^{n-1}x_0, f^n x_0)\}\\
&= \lambda s\rho_s(f^{n-2}x_0, f^{n-1}x_0). \qquad (4.9)
\end{aligned}$$

Inserting (4.8) and (4.9) in (4.7), we get

$$\rho_s(f^n x_0, p) \le \lambda s \max\{\rho_s(\rho_s(f^{n-1}x_0, p), \rho_s(f^{n-2}x_0, f^{n-1}x_0)\}. \tag{4.10}$$

Hence, by induction on n,

$$\rho_s(f^n x_0, p) \le (\lambda s)^{n-1} \max\{\rho_s(fx_0, p), \rho_s(x_0, fx_0)\} \text{ for all } n \ge 1. \tag{4.11}$$

Taking the limit as $n \to \infty$ in this, it follows that $\rho_s(f^n x_0, p) \to 0$ as $n \to \infty$ for all $x_0 \in X$. In other words, p is a b-contractive fixed point. □

Remark 4.3. It is not difficult to show that (2.5) implies (4.6). Hence, Theorem 2.6 follows from Corollary 4.2, and the unique fixed point of self-map f satisfying (2.5) is a b-contractive fixed point.

Taking $s = 1$ in Theorem 4.1, we get the following corollary.

Corollary 4.4 (Theorem 1, Nesic [18])**.** *Let (X, d) be a complete metric space and $f : X \to X$ be a Nesic contraction on X such that*

$$\begin{aligned} &[1 + \mu d(x, y)]d(fx, fy) \\ &\le \mu[d(x, fx)d(y, fy) + d(y, fx)d(x, fy)] \\ &\quad + \lambda \max\left\{d(x, y), d(x, fx), d(y, fy), \tfrac{d(x,fy)+d(y,fx)}{2}\right\} \end{aligned} \tag{4.12}$$

for all $x, y \in X$, where $\mu \ge 0$ and $0 < \lambda < 1$. Then f has a unique fixed point.

5. Generalized class of auxiliary functions

Given $\alpha > 0$, we consider the following generalized class of auxiliary functions:

$$\Phi_\alpha = \{\phi : [0, \infty) \to [0, \infty) : \phi(0) = 0, \phi(\alpha t) < t \ \text{ for } \ t > 0\}. \tag{5.1}$$

When $\alpha = 1$, the class Φ_α reduces to the class of contractive modulii Ω, whose members have the choice

$$\omega(0) = 0 \text{ and } \omega(t) < t \quad \text{for} \quad t > 0. \tag{5.2}$$

Theorem 5.1. *Suppose that (X, ρ_s) is a complete b-metric space and f is a self-map on X satisfying the condition*

$$\rho_s(fx, fy) \le \phi\big(\max\left\{\rho_s(x, y), \rho_s(x, fx), \rho_s(y, fy), \rho_s(x, fy), \rho_s(y, fx)\right\}\big) \tag{5.3}$$

for all $x, y \in X$, where $\phi \in \Phi_2$ is nondecreasing and upper semicontinuous. Then f has a unique fixed point p.

Proof. Let $x_0 \in X$ be arbitrary. Define $\{x_n\} \subset X$ by

$$x_n = fx_{n-1} \quad \text{for} \quad n \ge 1. \tag{5.4}$$

Writing with $x = x_{n-1}$ and $y = x_n$ in (5.3), we get

$$\rho_s(fx_{n-1}, fx_n) \le \phi\Big(\max\big\{\rho_s(x_{n-1}, x_n), \rho_s(x_{n-1}, fx_{n-1}), \rho_s(x_n, fx_n), \rho_s(x_{n-1}, fx_n), \underbrace{\rho_s(x_n, fx_{n-1})}_{=0}\big\}\Big)$$

or

$$\rho_s(x_n, x_{n+1}) \le \phi\Big(\max\big\{\rho_s(x_{n-1}, x_n), \rho_s(x_n, x_{n+1}), \rho_s(x_{n-1}, x_{n+1})\big\}\Big) \le \phi\Big(s\left[\rho_s(x_{n-1}, x_n) + \rho_s(x_n, x_{n+1})\right]\Big).$$

Define

$$t_n = \rho_s(x_{n-1}, x_n) \text{ for } n \ge 1. \tag{5.5}$$

Then the above inequality is written as

$$t_{n+1} \le \phi\big(s[t_n + t_{n+1}]\big) \text{ for } n \ge 1. \tag{5.6}$$

If $t_m < t_{m+1}$ for some $m \ge 1$, then $t_{m+1} > 0$. Since ϕ is nondecreasing, it follows from (5.6) that

$$t_{m+1} \le \phi\big(s[t_{m+1} + t_{m+1}]\big) < t_{m+1},$$

which is a contradiction. Thus $\{t_n\}_{n=1}^{\infty}$ is a nonincreasing sequence of nonnegative real numbers, that is

$$t_n \ge t_{n+1} \text{ for } n \ge 1. \tag{5.7}$$

and hence it converges to some $t \ge 0$. Now using (5.7) in (5.6), we get

$$t_{n+1} \le \phi(2st_n) \text{ for } n \ge 1.$$

Taking the limit superior as $n \to \infty$ in this and then using the upper semicontinuity of ϕ, we obtain that

$$t \le \phi(2st). \tag{5.8}$$

If $t > 0$ in (5.8), then the choice of ϕ implies that $t \le \phi(2st) < t$, which is a contradiction. Therefore, we have

$$t = 0. \tag{5.9}$$

We now prove that $\{x_n\}$ is a b-Cauchy sequence in X. If possible suppose that $\{x_n\}$ is not b-Cauchy. Then for some $\epsilon > 0$, we choose sequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of positive integers such that for $m_k > n_k > k$,

$$\rho_s(x_{m_k}, x_{n_k}) \ge \epsilon \text{ for } k = 1, 2, 3, ... \tag{5.10}$$

Suppose that m_k is the smallest integer exceeding n_k satisfying (5.10). That is

$$\rho_s(x_{m_k}, x_{n_k}) \ge \epsilon. \tag{5.11}$$

Now by (b3), we see that

$$\epsilon \leq \rho_s(x_{m_k}, x_{n_k}) \leq s[\rho_s(x_{m_k}, x_{m_k-1}) + \rho_s(x_{m_k-1}, x_{n_k})]. \tag{5.12}$$

But from (5.9) and (5.11) respectively, we see that

$$\lim_{k\to\infty} \rho_s(x_{m_k-1}, x_{m_k}) = 0 \text{ and } \lim_{k\to\infty} \rho_s(x_{m_k-1}, x_{m_k}) < \epsilon. \tag{5.13}$$

Using (5.13) in (5.12), we get

$$\epsilon < \lim_{k\to\infty} \rho_s(x_{m_k}, x_{n_k}) < s\epsilon. \tag{5.14}$$

Also by (b3), we get

$$\rho_s(x_{n_k-1}, x_{m_k}) \leq s[\rho_s(x_{n_k-1}, x_{n_k}) + \rho_s(x_{n_k}, x_{m_k})].$$

As $k \to \infty$ this and (5.14) give

$$\lim_{k\to\infty} \rho_s(x_{n_k-1}, x_{m_k}) = 2\epsilon. \tag{5.15}$$

On the other hand, from (5.3) with $x = x_{m_k-1}$, $y = x_{n_k-1}$ and (5.10), we have

$$\begin{aligned}
\epsilon &\leq \rho_s(x_{m_k}, x_{n_k}) \\
&= \rho_s(fx_{m_k-1}, fx_{n_k-1}) \\
&\leq \phi\Big(\max\big\{\rho_s(x_{m_k-1}, x_{n_k-1}), \rho_s(x_{m_k-1}, fx_{m_k-1}), \rho_s(x_{n_k-1}, fx_{n_k-1}), \\
&\qquad\qquad \rho_s(x_{m_k-1}, fx_{n_k-1}), \rho_s(x_{n_k-1}, fx_{m_k-1})\big\}\Big) \\
&= \phi\Big(\max\big\{\rho_s(x_{m_k-1}, x_{n_k-1}), \rho_s(x_{m_k-1}, x_{m_k}), \rho_s(x_{n_k-1}, x_{n_k}), \\
&\qquad\qquad \rho_s(x_{m_k-1}, x_{n_k}), \rho_s(x_{n_k-1}, x_{m_k})\big\}\Big).
\end{aligned}$$

Taking the limit as $n \to \infty$ in this, and then using upper semicontinuity of ϕ, (5.11), (5.13), (5.14) and (5.15), we get

$$\epsilon \leq \phi\Big(\max\big\{\epsilon, 0, 0, \epsilon, 2\epsilon\big\}\Big) = \phi(2\epsilon) < \epsilon, \tag{5.16}$$

since ϕ is nondecreasing. This is a contradiction to the choice of ϕ. Hence $\{x_n\}$ must be a b-Cauchy sequence in X. Since X is b-complete, there exists a point $p \in X$ such that $\{x_n\}$ is b-convergent to p. That is

$$\lim_{n\to\infty} x_{n-1} = \lim_{n\to\infty} x_n = p. \tag{5.17}$$

We now establish that p is a fixed point of f. In fact, writing $x = x_{n-1}$ and $y = p$ in (5.10),

$$\begin{aligned}\rho_s(x_n, fp) &= \rho_s(fx_{n-1}, fp)\\ &\le \phi\big(\max\big\{\rho_s(x_n, p), \rho_s(x_{n-1}, fx_{n-1}), \rho_s(p, fp),\\ &\qquad \rho_s(x_{n-1}, fp), \rho_s(p, fx_{n-1})\big\}\big)\\ &= \phi\big(\max\big\{\rho_s(x_n, p), \rho_s(x_{n-1}, x_n), \rho_s(p, fp),\\ &\qquad \rho_s(x_{n-1}, fp), \rho_s(p, x_n)\big\}\big).\end{aligned}$$

Taking the limit as $n \to \infty$ in this and then using (5.17), we get

$$\begin{aligned}\rho_s(p, fp) &\le \phi\big(\max\big\{0, 0, \rho_s(p, fp), \rho_s(p, fp), 0\big\}\big)\\ &= \phi(\rho_s(p, fp)).\end{aligned} \tag{5.18}$$

If $p \neq fp$, then $\rho_s(p, fp) > 0$. Since ϕ is nondecreasing, (5.18) implies that

$$0 < \rho_s(p, fp) \le \phi(\rho_s(p, fp)) < \rho_s(p, fp),$$

which is a contradiction. Hence $p = fp$.

To ensure the uniqueness of the fixed point, we suppose that p and q are fixed points of f with $p \neq q$. Let $x = p$ and $y = q$ in (5.10). Then

$$\begin{aligned}\rho_s(p, q) &= \rho_s(fp, fq)\\ &\le \phi\big(\max\big\{\rho_s(p, q), \rho_s(p, fp), \rho_s(q, fq), \rho_s(p, fq), \rho_s(q, fp)\big\}\big)\\ &= \phi\big(\max\big\{\rho_s(p, q), 0, 0, \rho_s(p, q), \rho_s(q, p)\big\}\big) = \phi\big(\rho_s(p, q)\big),\end{aligned}$$

which implies that $p = q$. □

With an argument, similar to that of Theorem 5.1, the following result can be established:

Theorem 5.2. *Suppose that (X, ρ_s) is a complete b-metric space and f is a self-map on X satisfying the condition:*

$$\begin{aligned}&\rho_s(fx, fy)\\ &\le \phi\big(\max\big\{\rho_s(x, fx) + \rho_s(y, fy), \rho_s(x, fy) + \rho_s(y, fx) + \rho_s(y, fy)\big\}\big)\end{aligned} \tag{5.19}$$

for all $x, y \in X$, where $\phi \in \Phi_{2s+1}$ is nondecreasing and upper semicontinuous. Then f has a unique fixed point p.

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